$$f : X \longrightarrow Y$$

due spazi vettoriali

DEF. $f : X \to Y$ si dice LINEARE se verifica

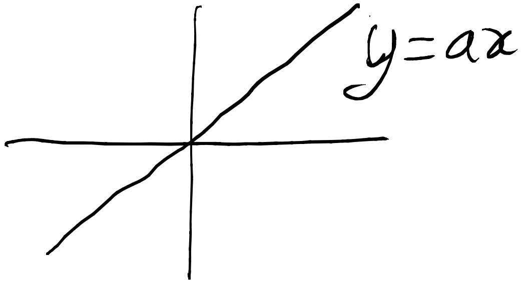
1) $f(x+x') = \underbrace{f(x)}_{\in Y} + \underbrace{f(x')}_{\in Y}$ $\quad \forall x, x' \in X$

2) $\underbrace{f(\lambda x)}_{\in Y} = \lambda \underbrace{f(x)}_{\in Y}$ $\quad \forall \lambda \in \mathbb{R} \ \forall x \in X$

Osservazione se f è lineare allora $\boxed{f(\alpha x + \beta x') = \alpha f(x) + \beta f(x')}$

$$\boxed{f\left(\sum_{1}^{n} \alpha_i x_i\right) = \sum_{1}^{n} \alpha_i f(x_i)}$$

$\boxed{f(x)=ax}$ $\qquad a,x\in\mathbb{R} \qquad f:\mathbb{R}\to\mathbb{R}$



$\Downarrow$

$$f(x+y)=a(x+y)=ax+ay=f(x)+f(y)$$

$$f(\lambda x)=a(\lambda x)\underset{\substack{\text{assoc.}\\ \text{in }\mathbb{R}\\ \text{del prod.}}}{=}(a\lambda)x\underset{\substack{\text{comm}\\ \text{del}\\ \text{prodotto}}}{\equiv}(\lambda a)x\underset{\text{assoc.}}{=}\lambda(ax)\equiv\lambda f(x)$$

Ogni funzione lineare su $\mathbb{R}$ in $\mathbb{R}$ è del tipo precedente.

Se $f:\mathbb{R}\to\mathbb{R}$ è lineare $\quad f(x)=f(x\cdot 1)\underset{\substack{\text{linear.}\\ 2)}}{=}x\,\underset{\substack{\| \\ a}}{\boxed{f(1)}}$

X = funzioni derivabili su $\mathbb{R}$

$$\left.\begin{aligned}(f+g)(x)&=f(x)+g(x)\\ (\lambda f)(x)&=\lambda f(x)\end{aligned}\right\}\ \begin{aligned}&\forall x\in\mathbb{R}\\ &\forall\lambda\in\mathbb{R}\end{aligned}$$

$\forall f,g\in X$

$$\begin{cases}(f+g)'=f'+g' \leftarrow & A(f)=f'\\ (\lambda f)'=\lambda f' \leftarrow & A:X\to Y\end{cases}$$

$$\boxed{f: \underline{\mathbb{R}^2} \to \underline{\mathbb{R}} \quad \boxed{f \text{ lineare}}}$$

$$e_1 = (1,0) \qquad e_2 = (0,1) \qquad \{e_1, e_2\}$$

$$\longrightarrow \quad \underline{\underline{x}} \equiv (x_1, x_2) = x_1(1,0) + x_2(0,1) = \underline{\underline{x_1 e_1 + x_2 e_2}}$$

$$f(x) = f(x_1 e_1 + x_2 e_2) \underset{\substack{\text{linearità} \\ \text{di } f}}{=\!=} x_1 \underbrace{f(e_1)}_{\substack{a_1 \\ \in\mathbb{R}}} + x_2 \underbrace{f(e_2)}_{\substack{a_2 \\ \in\mathbb{R}}} \equiv \underbrace{a_1 x_1 + a_2 x_2} =$$

$$= \boxed{\underset{\substack{\equiv \\ a}}{(a_1, a_2)}} \underset{\substack{= \\ x}}{(x_1, x_2)} = ax$$

Il prodotto è un prodotto scalare dei vettori a e x

$$\boxed{\begin{matrix} f(x,y) = ax + by \\ (a,b)(x,y) \end{matrix}}$$

polinomi omogenei di I grado nelle incognite $x_1 \ldots x_n$

$$f(x) = \underbrace{\big(f(e_1), f(e_2)\big)}_{\substack{\equiv \\ a}} \underbrace{(x_1, x_2)}_{x} = ax$$

$f : \mathbb{R} \to \mathbb{R}^n$ lineare

$$\overbrace{f(x)}^{\in \mathbb{R}^n} = f(x \cdot 1) \underset{\text{linearità}}{=\!=} x \underbrace{\overbrace{f(1)}^{\in \mathbb{R}^n}}_{a} = \overset{\in \mathbb{R}^n}{a} \underset{\in \mathbb{R}}{x}$$

$x \in \mathbb{R}$, $x \cdot 1$: $\in \mathbb{R}$

prodotto dello scalare x per il vettore a

$$a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\underline{f(x)} = \underbrace{x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{x\,a} = \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix}$$

$f : \mathbb{R}^n \to \mathbb{R}^m$, $n, m > 1$ $\quad \mathbb{R}^n \ni x = (x_1, x_2, \ldots, x_n)$

$e_1 \ldots e_n$ base canonica di $\mathbb{R}^n$

$$f(x) = f\Big(\underbrace{\sum_1^n x_i e_i}_{=x}\Big) \underset{\text{linearità}}{=\!=} \sum_1^n \underset{\in \mathbb{R}}{x_i} \underbrace{f(e_i)}_{a_i \in \mathbb{R}^m} = \boxed{\sum_1^n \underset{\in \mathbb{R}}{x_i} \underset{\in \mathbb{R}^m}{a_i}}$$

$a = (a_1, \ldots, a_n)$, $a_i \in \mathbb{R}^m$

$x = \sum_1^n x_i e_i$

$$x = (x_1, x_2, \ldots, x_n) = x_1 \overset{e_1}{(1, 0 \ldots 0)} + x_2 \overset{e_2}{(0, 1, 0 \ldots 0)} + \ldots + x_n \overset{e_n}{(0 \ldots 0, 1)}$$

? $\underline{\underset{\in \mathbb{R}^n}{a x}}$.

f è lineare

1° $f:\mathbb{R}\to\mathbb{R}$ $\quad \exists a\in\mathbb{R} : f(x)=ax$ $\quad$ prodotto di numeri

2° $f:\mathbb{R}^n\to\mathbb{R}$ $\quad \exists a\in\mathbb{R}^n : f(x)=ax$ $\quad$ prodotto scalare

3° $f:\mathbb{R}\to\mathbb{R}^m$ $\quad \exists a\in\mathbb{R}^m : f(x)=ax$ $\quad$ prodotto scalare per vettore

4° $f:\mathbb{R}^n\to\mathbb{R}^m$ $\quad \exists a\in ? : f(x)=\underset{?}{a\cdot x}$ $\quad$ prodotto matrice per vettore

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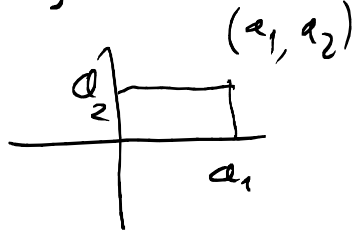
Prodotto cartesiano di insiemi

A, B insiemi $\quad A\times B=\{(a,b) : a\in A \ \ b\in B\}$

$\mathbb{R}^2=\mathbb{R}\times\mathbb{R}=\{(x,y) : x\in\mathbb{R} \ \ y\in\mathbb{R}\}$



$\boxed{+ : \underset{\text{coppie di vettori}}{X\times X}\longrightarrow \underset{\text{vettori}}{X}}$

multiplo scalare di un vettore : $\underset{\text{coppie scalare-vettore}}{\mathbb{R}\times X}\longrightarrow X \ni$ prodotto $\to$ vettore

$f : \mathbb{R} \to \mathbb{R}^2 \qquad x \in \mathbb{R} \longrightarrow f(x) \in \mathbb{R}^2$

$+(x, y) \qquad \begin{array}{l} x \in X \\ y \in X \end{array} \qquad \underbrace{+(x, y)}_{\underline{x+y}} \in X$

$$\underline{\underline{\mathbb{N} \times \mathbb{R}}} = \{ (n, x) : n \in \mathbb{N} \quad x \in \mathbb{R} \} \ni \begin{array}{l} (2, \pi) \\ (7, \sqrt{2}) \end{array}$$

$$\times : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \qquad \underset{\in \mathbb{R} \times \mathbb{R}^3}{\left(3, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right)} \longrightarrow 3 \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \underset{\in \mathbb{R}^3}{\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}}$$

$\cdot : \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{\text{coppie di vettori}} \longrightarrow \mathbb{R}$ $\qquad \cdot =$ prodotto scalare in $\mathbb{R}^n$

$$xy = \sum_{1}^{n} x_i y_i \in \mathbb{R}$$

$x_i, y_i \in \mathbb{R}$

$u, v \in \mathbb{R}^n$

$$\alpha(\overset{\in \mathbb{R}^n}{u}, \overset{\in \mathbb{R}^n}{v}) = uv$$

$$\boxed{\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}}$$

(u, v)

che, alla coppia u, v di vettori associa il loro prodotto scalare ($\in \mathbb{R}$)

$\alpha : X \times X \to \mathbb{R}$ si dirà **bilineare** se sono **LINEARI** le due funzioni

$\rightarrow \quad u \longrightarrow \alpha(u, \bar{v}) \quad \forall \bar{v} \in X$ fissato

$\boxed{v \longrightarrow \alpha(\bar{u}, v) \quad \forall \bar{u} \in X \text{ fissato}}$

$$\boxed{\alpha(u,v) = uv = \sum_{1}^{n} u_i v_i}$$

fissato $\bar{u}$ $\quad \alpha(\bar{u}, v) = \sum_{1}^{n} \bar{u}_i v_i = \bar{u} v$

$$f(v+w) = \underline{\alpha(\bar{u}, v+w)} = \bar{u}(v+w) = \bar{u}v + \bar{u}w = \overset{f(v)+f(w)}{\underline{\alpha(\bar{u}, v) + \alpha(\bar{u}, w)}}$$

$$f(\lambda v) = \underline{\alpha(\bar{u}, \lambda v)} = \bar{u}(\lambda v) = \lambda \bar{u} v = \underbrace{\lambda \alpha(\bar{u}, v)}_{\lambda f(v)}$$

$$(u+v)(u+v) = (u+v)u + (u+v)v = \underline{uu + vu + uv + vv}$$

$X_1 \times X_2 \times \dots \times X_n$ $\quad$ X_i insiemi

$$|||$$

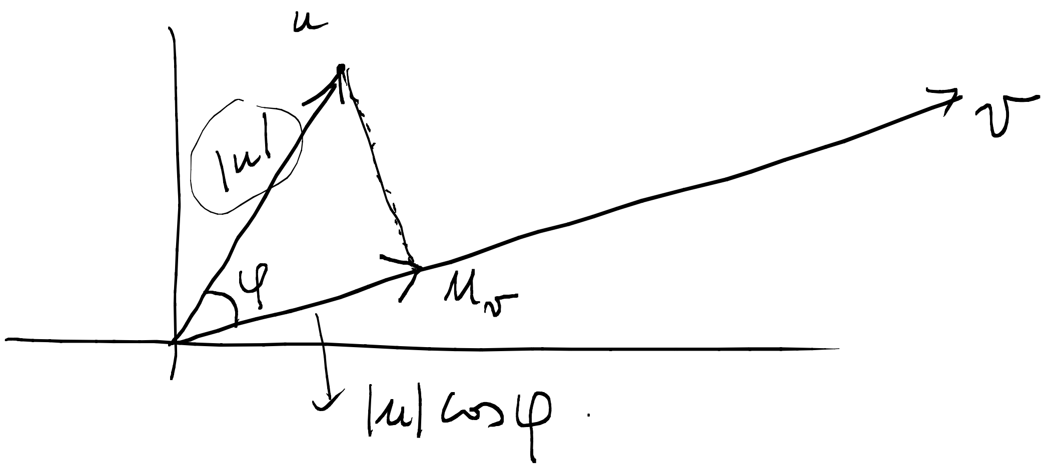
$$\{ \underbrace{(x_1, x_2, \dots, x_n)} : \underline{x_i \in X_i} \ \forall i \}$$

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ copie di } \mathbb{R}} \ni (x_1, x_2, \dots, x_n) \qquad x_i \in \mathbb{R} \ \forall i$$

X_1 = quadrati $\quad$ X_2 = triangoli $\quad$ X_3 = pentagoni

$X_1 \times X_2 \times X_3 = \{$ (un quadrato, un triangolo, un pentagono) arbitrari $\}$

PROIEZIONE (ORTOGONALE)



$v \neq 0 \quad u \neq 0$

u_v proiezione di u su v

Siccome u_v è diretto come v deve essere un multiplo del suo versore

lunghezza di u_v — versore di v

$$u_v = \cancel{|u|}\frac{uv}{\cancel{|u|}|v|}\,\frac{v}{|v|} = \frac{uv}{|v|^2}\,v$$

ipotenusa

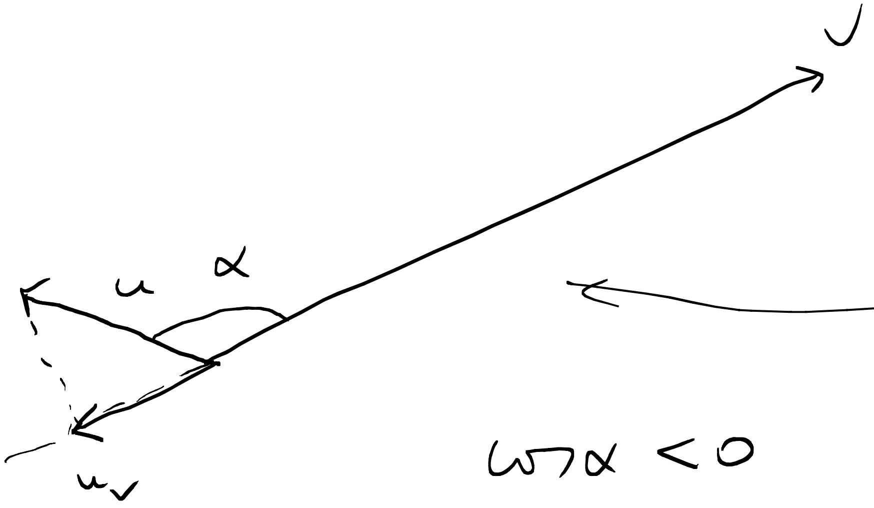
coseno dell'angolo adiacente

$$\cos\widehat{uv} = \frac{uv}{|u||v|}$$

φ

DEF $\boxed{u_v = \dfrac{uv}{|v|^2}\,v}$

$u = |u|\dfrac{u}{|u|}$



$$u_v = \frac{u v}{|v|^2} v$$

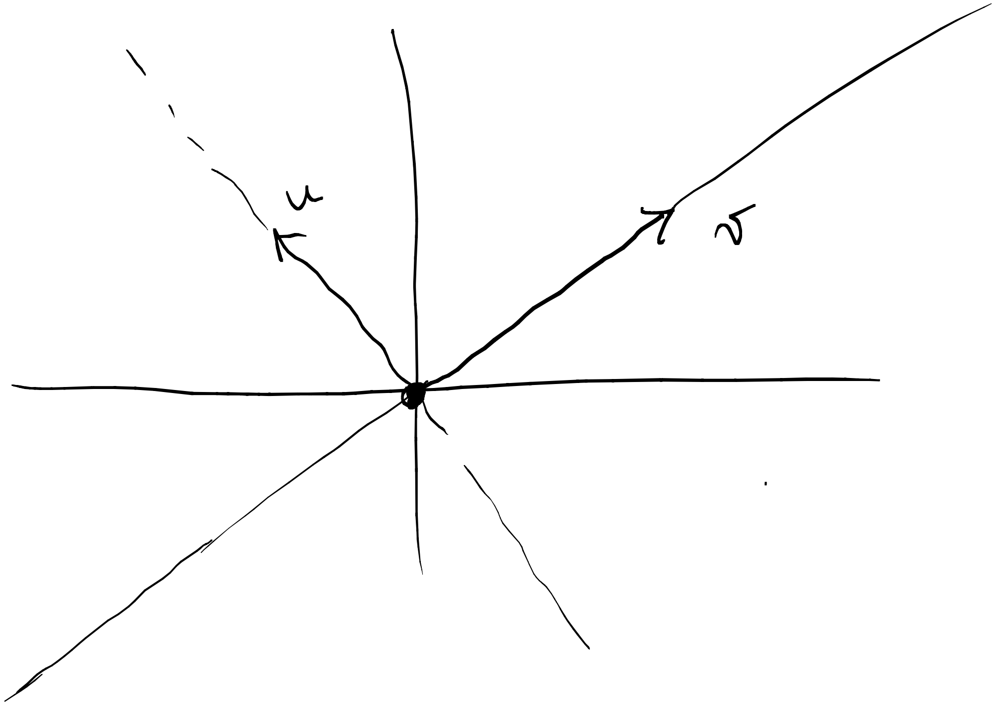
<0

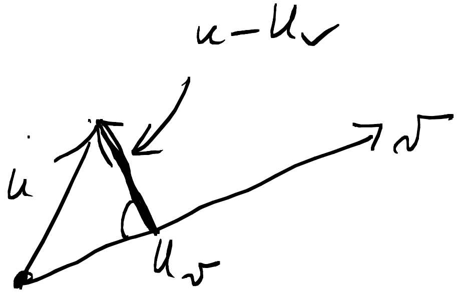
$u_v \equiv$ proiezione di u su v

$\cos\alpha < 0$

$$\overbrace{(1,1,2)}^{u}{}_{\underbrace{(1,1,1)}_{v}} = \frac{1\cdot 1+1\cdot 1+2\cdot 1}{1^2+1^2+1^2}(1,1,1) = \frac{4}{3}(1,1,1) = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$$

$$(1,1,-2)_{(1,1,1)} = \frac{1+1-2}{3}(1,1,1) = 0(1,1,0) = (0,0,0)$$





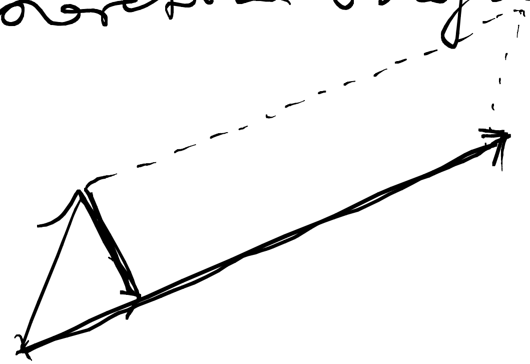
$$(u-u_v)v = uv - (u_v)v = uv - \underbrace{\left(\frac{uv}{|v|^2}v\right)}_{u_v}v$$

$(\lambda v)v =$
$= \lambda (vv)$

$$= uv - \frac{uv}{\cancel{|v|^2}}\cancel{(vv)} = 0$$

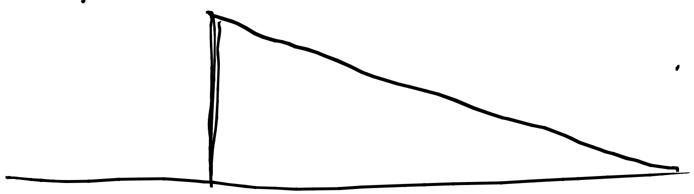
$\boxed{\text{Th. della proiezione}}$

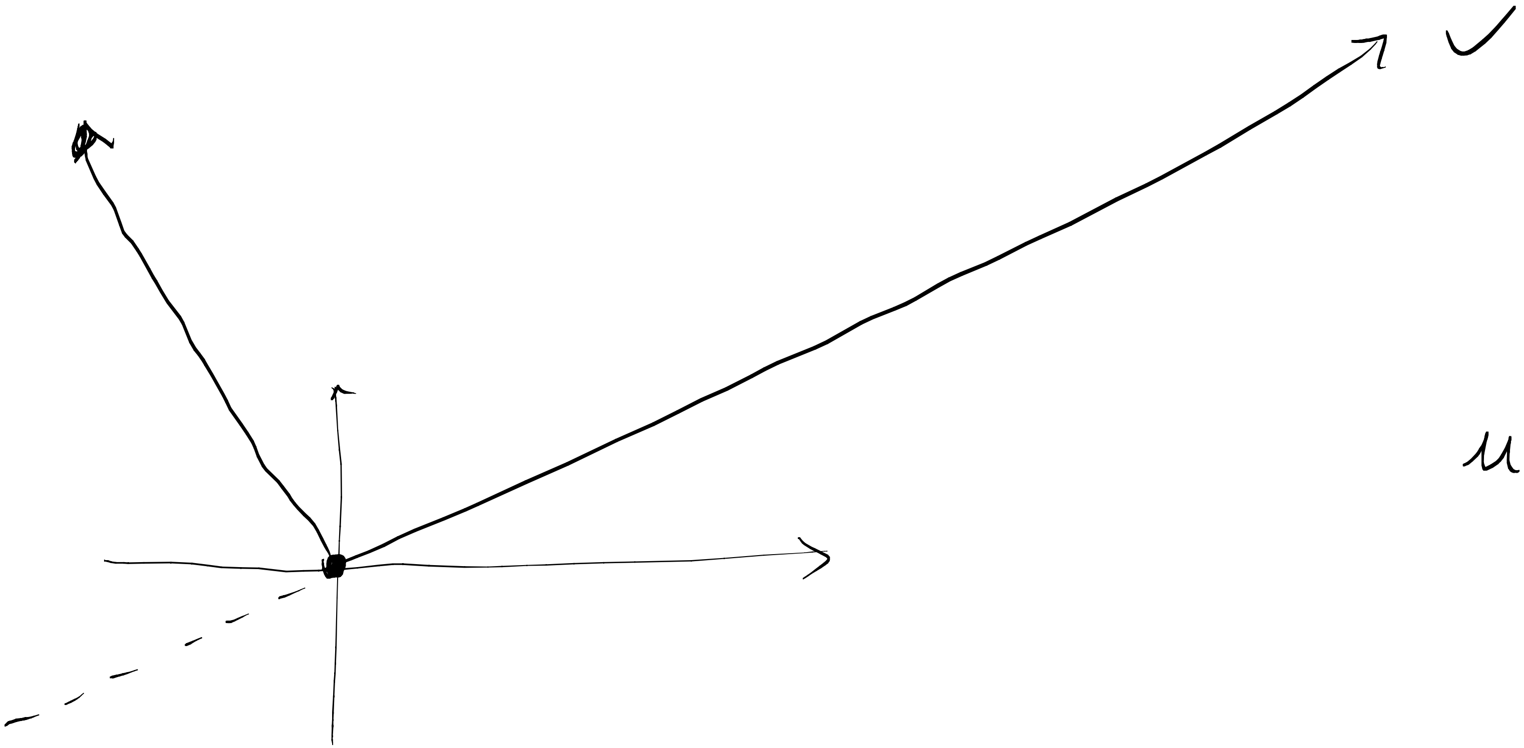
Dunque il "resto" $u - u_v$ della proiezione ortogonale è ortogonale al vettore v



$$|u|^2 = |\underbrace{u-u_v}_{\perp v} + \underbrace{u_v}_{/\!/ v}|^2 = |u-u_v|^2 + |u_v|^2 \geq |u_v|^2$$

Pitagora ≥ 0





$$u - \overset{\circ}{u}{}^{\parallel}_{v} = u$$