Note Title 10/6/2020

$f : X \longrightarrow Y$ si dice LINEARE

1) $f(x+y) = f(x) + f(y)$ (Additività)

2) $f(\lambda x) = \lambda f(x)$ (Omogeneità)

$$(f(x)+g(x))' = f'(x) + g'(x)$$

$A(f) = f'$

$A(f+g) = (f+g)' = f' + g' =$

$= A(f) + A(g)$

$f(x) = x^2$

$f(x+y) = (x+y)^2 =$

$= x^2 + y^2 + 2xy$

$\underbrace{x^2}_{f(x)} + \underbrace{y^2}_{f(y)}$ ✗

$f(\lambda x) = (\lambda x)^2 = \lambda^2 x^2 = \lambda^2 f(x)$

$$\boxed{p(a) = 3x + 4y + 5z} \qquad p: \mathbb{R}^3 \to \mathbb{R}$$

$$a \equiv (x, y, z)$$

$$p(\lambda a) = p(\lambda x, \lambda y, \lambda z) =$$

$$= 3\lambda x + 4\lambda y + 5\lambda z = \lambda \underbrace{(3x + 4y + 5z)}_{p(a)}$$

$a \equiv (x, y, z)$ $b \equiv (u, v, w)$

$$p(a+b) = p(x+u,\ y+v,\ z+w) = 3(x+u) + 4(y+v) + 5(z+w) =$$

$$= 3x + 4y + 5z + 3u + 4v + 5w = p(a) + p(b)$$

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fissato u a piacere in $\mathbb{R}^n$ poniamo $f(v) = uv \quad \forall v \in \mathbb{R}^n$

$$\boxed{\left.\begin{aligned} f(v+w) &= u(v+w) = uv + uw = f(v) + f(w) \\ f(\lambda v) &= u(\lambda v) = \lambda(uv) = \lambda f(v) \end{aligned}\right\} f \text{ è LINEARE}}$$

$$g(v) = vu = uv$$

$$(u, v) \longrightarrow uv$$

bilineare vuol dire che
SE SI FISSA UNO DEI FATTORI
IL PRODOTTO E' LINEARE
NEL SECONDO FATTORE

$$fg = \int_a^b f(x)\, g(x)\, dx$$

Z dimensione finita

X, Y sottospazi di Z

$$\boxed{\underline{\dim(X+Y)} + \dim(X\cap Y) = \underline{\dim X + \dim Y}}$$

CASO 1

$\dim X\cap Y = \{0\} \Leftrightarrow X\cap Y = \{0\} \Rightarrow X+Y = X\oplus Y$

$\Rightarrow \underline{\dim(X\oplus Y) = \dim X + \dim Y}$ e la tesi segue.

Si usa il $\underline{\text{th}}$: se la somma è diretta, la dimensione della somma è la somma delle dimensioni.

$\dim(X \cap Y) > 0$ $\qquad$ $X \cap Y \subset X$

$\qquad\qquad\qquad\qquad\quad$ $X \cap Y \subset Y$

- sia $w_1 \dots w_k$ una base di $X \cap Y$ (che, dunque, ha dim. k)
- fissiamo $w_1 \dots w_k\, x_{k+1} \dots x_n$ una base di X $\quad(\dim X = n)$

  $w_1 \dots w_k\, y_{k+1} \dots y_m$ una base di Y $\quad(\dim Y = m)$

Occorre provare

TS $\quad \dim(X+Y) = n+m-k$

DICO CHE $\{w_1, \dots, w_k, x_{k+1}, \dots, x_n, y_{k+1} \dots y_m\}$ è una base per $X+Y$.

$$X+Y = \{x+y : \underline{x\in X}, y\in Y\}$$

$$\exists\ \alpha_1,\alpha_2,\dots,\alpha_k,\beta_{k+1},\dots,\beta_n : \quad x = \sum_1^k \alpha_i w_i + \sum_{k+1}^n \beta_j x_j$$

$$\exists\ \gamma_1,\dots,\gamma_k,\delta_{k+1},\dots,\delta_m : \quad y = \sum_1^k \gamma_i w_i + \sum_{k+1}^m \delta_h y_h$$

$$x+y = \sum_1^k(\alpha_i+\gamma_i)w_i + \sum_{k+1}^n \beta_j x_j + \sum_{k+1}^m \delta_h y_h \in$$

$$\in \langle \underbrace{w_1,\dots,w_k}_{k}, \underbrace{x_{k+1}\dots x_n}_{n-k}, \underbrace{y_{k+1},\dots,y_m}_{m-k}\rangle$$

I vettori del sistema di generatori per X+Y sono indipendenti

$$\boxed{\underbrace{\sum_{1}^{k} \alpha_i w_i}_{w} + \underbrace{\sum_{k+1}^{n} \beta_j x_j}_{x} + \underbrace{\sum_{k+1}^{m} \gamma_h y_h}_{y} = 0} \overset{?}{\Longrightarrow} \alpha_i = \beta_j = \gamma_h = 0 \;\; ? \;\; \forall i, j, h.$$

$$(*) \quad w + x + y = 0 \Rightarrow \underbrace{w+x}_{\in X} = \underbrace{-y}_{\in Y} \Rightarrow \boxed{y \in X \cap Y}$$

($-y$ sta in Y ed è uguale al vettore di X $w+x$)

Per il lemma di ripartizione delle basi

$X = \langle w_1 \dots w_k \rangle \oplus \langle x_{k+1}, \dots, x_n \rangle$

$$\underbrace{(w+y)}_{\in X \cap Y} + \underbrace{x}_{\in \langle x_{k+1}, \dots x_n \rangle} = 0 \Longrightarrow w + y = 0 \text{ e } x = 0$$

$x=0 \Leftrightarrow \sum_{k+1}^{n} \beta_j x_j = 0$ per l'indip. di $x_{k+1} \dots x_n$

ne segue $\beta_{k+1} = \dots = \beta_n = 0$

sostituendo in $(*)$ ne segue

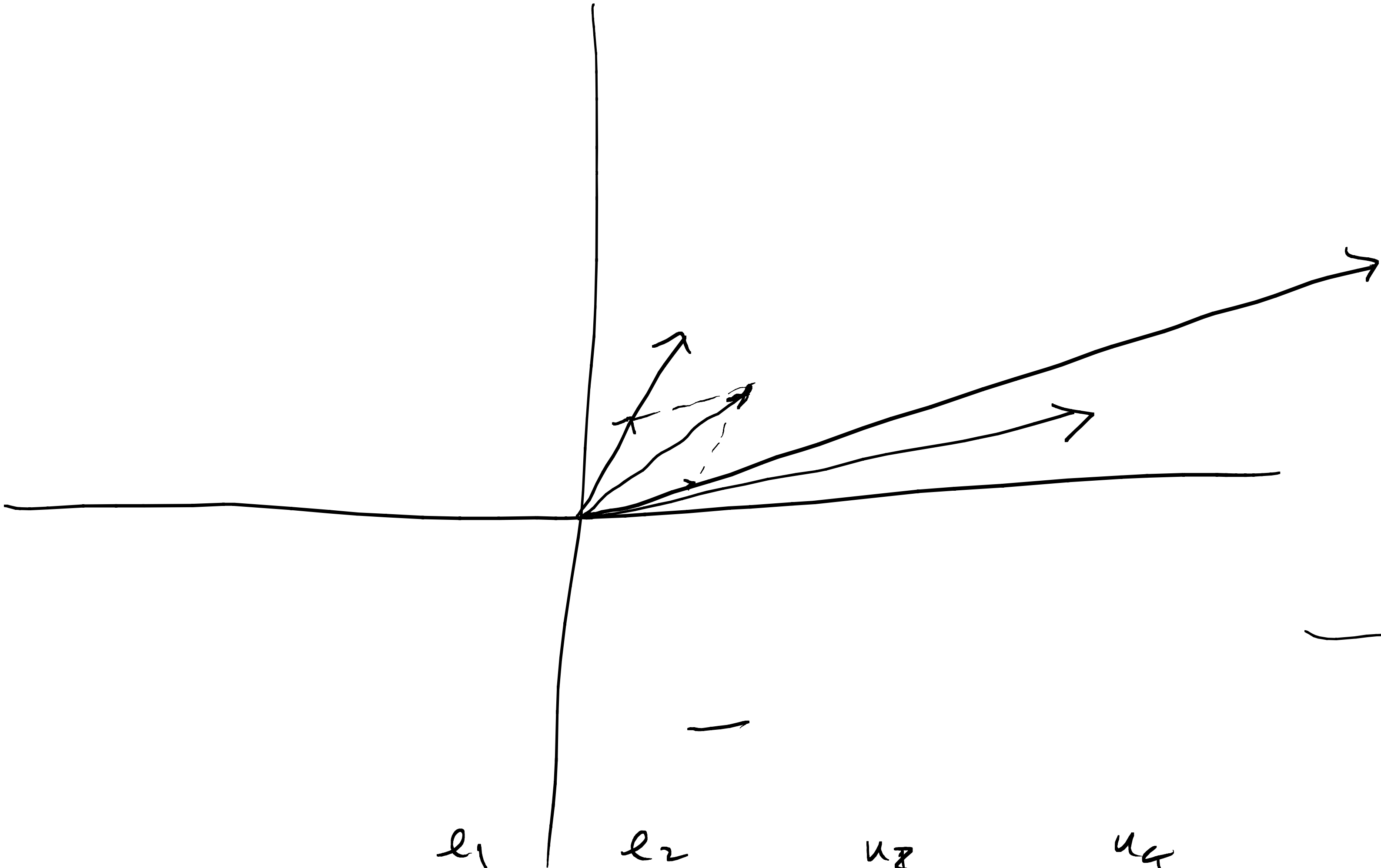
$$w+y=0 \Rightarrow \sum_1^k \alpha_i w_i + \sum_{k+1}^{m} \gamma_n y_n = 0$$

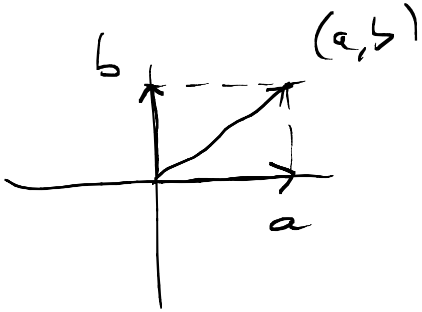
Dall'indipendenza di $w_1, \dots, w_k, y_{k+1} \dots y_m$ ne segue anche

$\alpha_1 = \dots = \alpha_k = 0$ e $\gamma_{k+1} = \dots = \gamma_m = 0$

$$\dim(X+Y) = k + (n-k) + (m-k) = n+m-k$$

$\square$





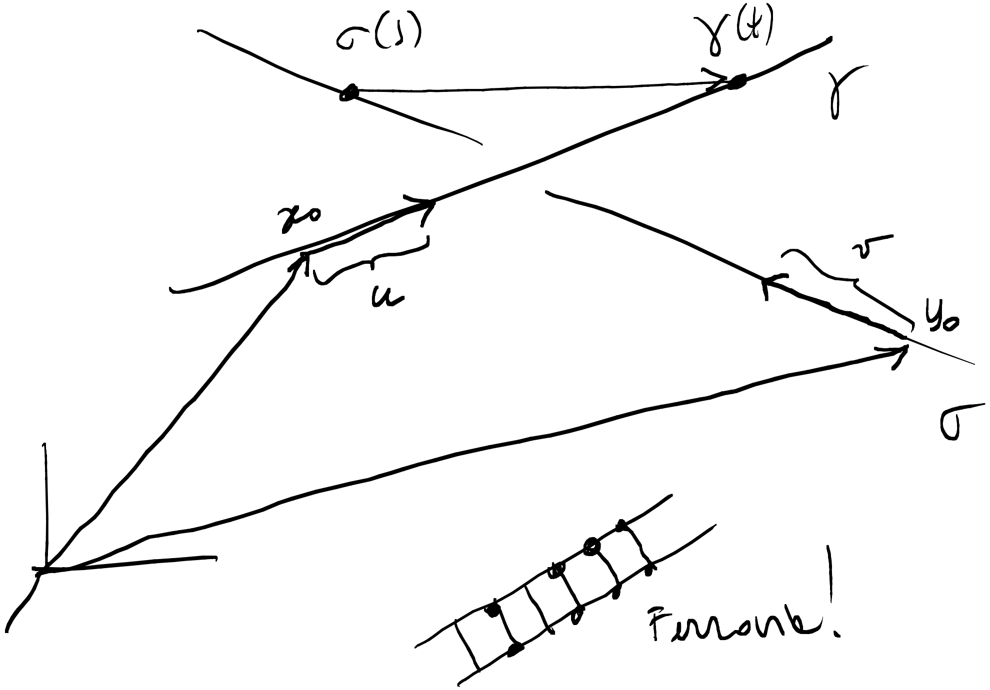
$$\mathbb{R}^2 = \Big\langle \underbrace{\overset{e_1}{(1,0)}, \overset{e_2}{(0,1)}}, \underbrace{\overset{u_3}{(1,1)}, \overset{u_4}{(-2,7)}, (\pi, e^{-1}), \dots, \overset{u_n}{(0,0)}} \Big\rangle$$

$$\forall\ (a,b) \overset{\in \mathbb{R}^2}{=} a e_1 + b e_2 + 0 u_3 + 0 u_4 + \dots + 0 u_n \in \langle e_1, e_2 \rangle \subseteq \langle e_1, e_2, \dots e_n \rangle$$

AL_1.1 / 1.2 sottospazi

$X \subseteq Y \quad X = Y \quad X + Y$

$X \cap Y$



$\gamma(t) = x_0 + tu \qquad t \in \mathbb{R}$

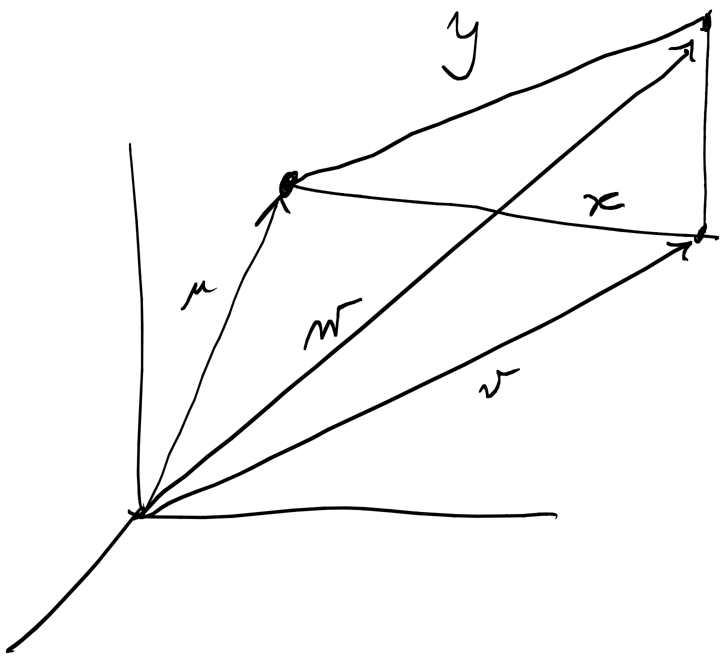
$\sigma(s) = y_0 + sv$

$u, v \in \mathbb{R}^n$

$u, v \neq 0$

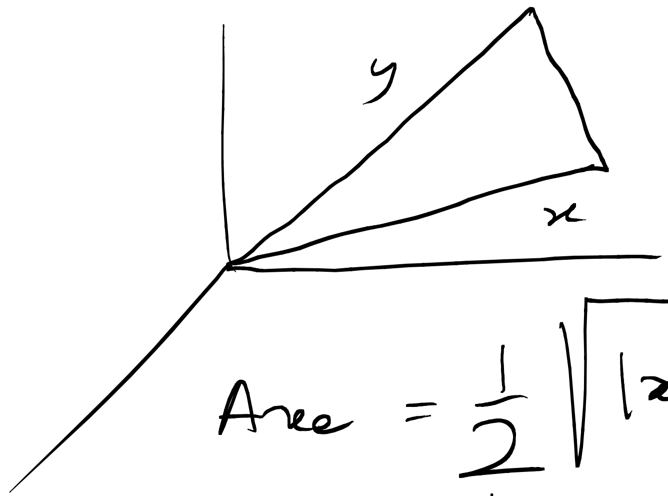
$$\boxed{\begin{cases} \left[\gamma(t) - \sigma(s)\right] u = 0 \\ \left[\quad\quad\quad\right] v = 0 \end{cases}}$$

Risolvere rispetto alle incognite t ed s



$$x = v - u$$

$$y = w - u$$



$$\underset{u}{(1,1,1)}, \quad \underset{v}{(1,2,1)}, \quad \underset{w}{(1,1,2)}$$

$$\text{Area} = \frac{1}{2}\sqrt{|x|^2|y|^2 - (xy)^2}$$

triangle

$$x = v - u = (0,1,0)$$

$$y = w - u = (0,0,1)$$

$$\text{Area} = \frac{1}{2}\sqrt{1\cdot 1 - (\underbrace{0\cdot 0 + 1\cdot 0 + 0\cdot 1}_{=0})^2} = \frac{1}{2}$$

$\boxed{\text{AL\_2.1}}$