LE OPERAZIONI IN $\mathbb{R}^n$

Note Title 10/1/2020

DEF Fissato n intero ($\in \mathbb{N}$)

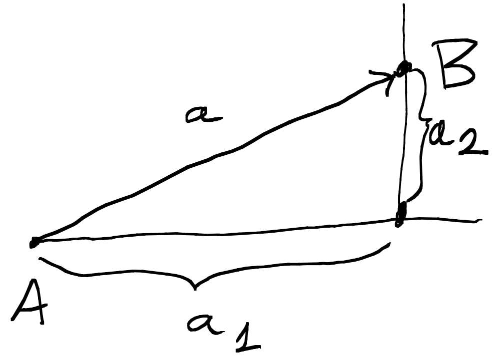
$$\mathbb{R}^n = \{ \underbrace{(a_1, a_2, \dots, a_n)}_{n\text{-upla}} : a_i \in \mathbb{R} \ \forall i = 1 \dots n \}$$

Vettore di $\mathbb{R}^n$ = n-uple ordinate di numeri reali $\quad a = (a_1, a_2, \dots, a_n)$

$\mathbb{R}^2$ = coppie ordinate di numeri reali $\quad (2,1) \neq (1,2)$

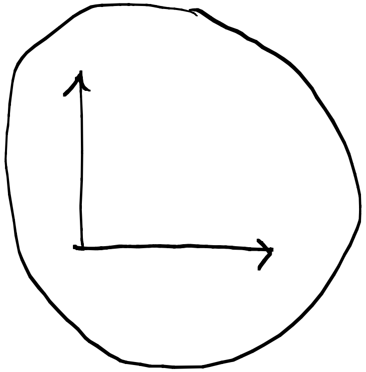
$(\sqrt{2}, 1), (0, -42) \dots$

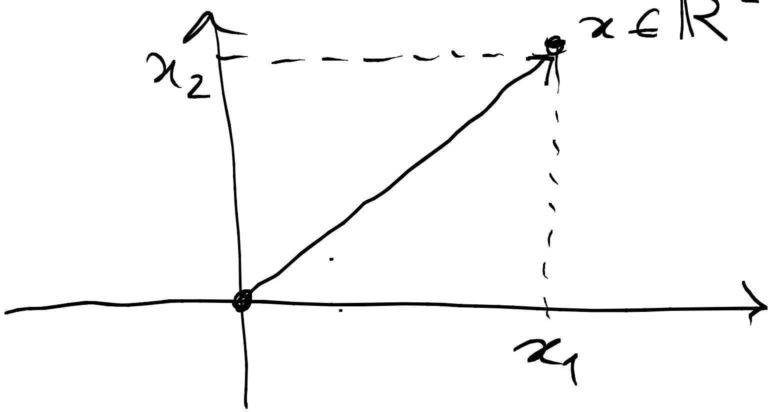
coordinate di punto rispetto ad un sistema di riferimento cartesiano ortogonale $\longrightarrow$ un elemento di $\mathbb{R}^2$

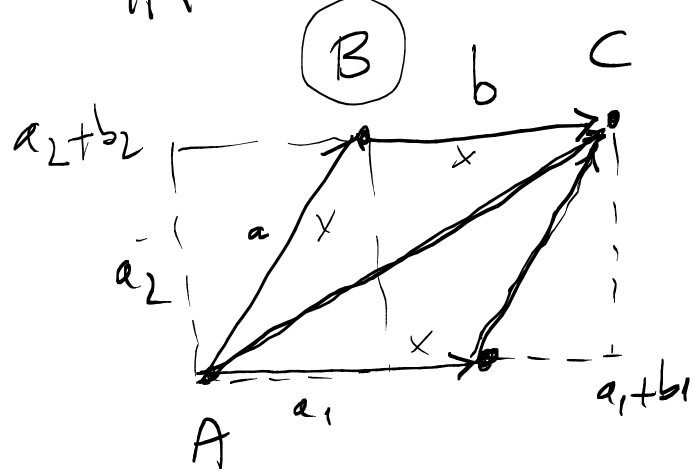


$\overrightarrow{AB}$ $\quad AB \simeq (a_1, a_2)$

$\mathbb{R}^n \simeq$ spostamenti nell'ambiente $\mathbb{R}^n$





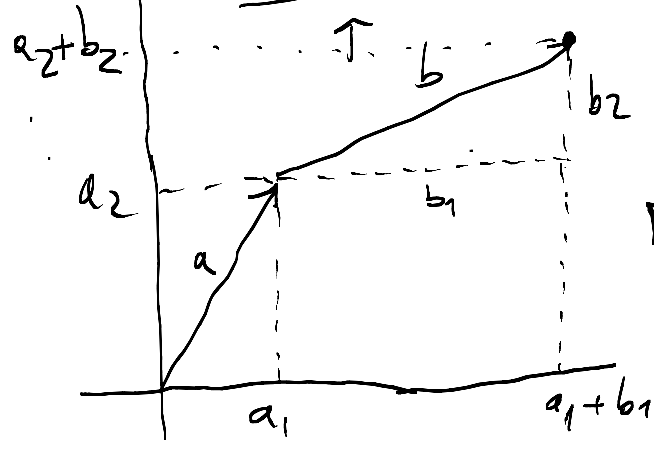


$$\forall\ a \equiv (a_1, \dots, a_n),\ b \equiv (b_1, \dots, b_n) \in \mathbb{R}^n$$

$$\longrightarrow \quad a+b \equiv (a_1+b_1,\ a_2+b_2, \dots, a_n+b_n)$$

algebrica

$\underline{a = (a_1, a_2)}$



geometrica

La somma possiede un elemento "neutro"

$$\exists\, 0 : a+0=a\,?$$

SCALARI = = NUMERI (reali o complessi)

sì $\boxed{0=(0,0,\dots,0)}$ ←

Un elemento di $\mathbb{R}^n$ possiede un opposto

$$\forall a\in\mathbb{R}^n \ \exists -a\in\mathbb{R}^n : \underbrace{a+(-a)}=0\,?$$

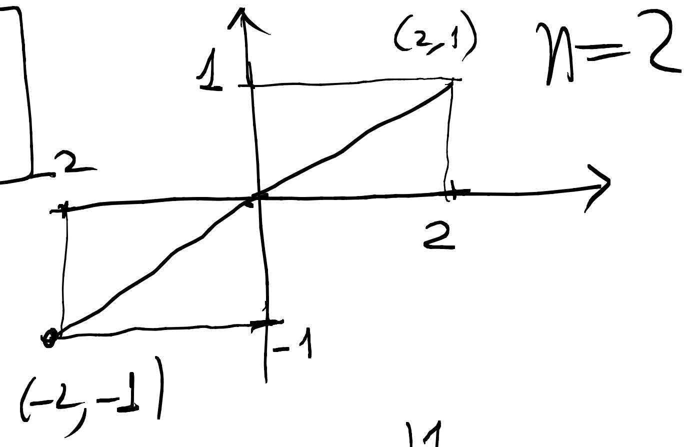
$$a+(-a)\equiv\left(a_1+(-a)_1, \dots, a_n+(-a)_n\right)$$

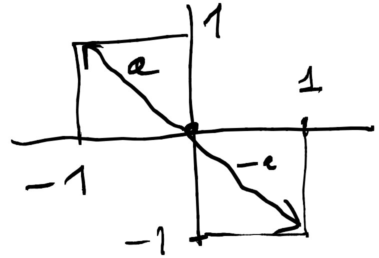
Perché si verifichi $a+(-a)=0$ occorre che:

$$\underbrace{a_1+\overbrace{(-a)_1}^{-a_1}}=0\ ,\ a_2+\overbrace{(-a)_2}^{-a_2}=0\ \dots\ a_n+\overbrace{(-a)_n}^{-a_n}=0 \Longrightarrow \boxed{(-a)_i=-a_i}\ \forall i=1..n$$

$$\boxed{-a \equiv (-a_1, -a_2, \cdots, -a_n)}$$

(geom. simmetrico rispetto all'origine





La somma ha 0 e opposto

$$a+b = b+a$$

$$(\cdots, a_i+b_i, \cdots) \qquad (\cdots, b_i+a_i, \cdots)$$

uguali

La SOMMA È COMMUTATIVA

È anche ASSOCIATIVA $\underbrace{(a+b)+c = a+(b+c)}$

$$[(a+b)+c]_i = \underbrace{(a+b)_i} + c_i = \underbrace{(a_i+b_i)+c_i = a_i+b_i+c_i}$$

$$\forall a, b \in \mathbb{R}^n \qquad \underbrace{a-b} \equiv a + \underbrace{(-b)} = (a_1-b_1, a_2-b_2, \dots, a_n-b_n)$$

$$b+(a-b) = (\dots\dots, b_i + (a-b)_i, \dots) = (\dots, \underbrace{b_i + a_i - b_i}_{a_i}, \dots)$$

$$(1,2,-3)+(2,0,1) = (1+2, 2+0, -3+1) = (3,2,-2)$$

$$(2,-1,-1,3)-(-1,0,2,0) = (2-(-1), -1-(0), -1-(2), 3-(0))$$

$$(3, -1, -3, 3)$$

$$(a+b)_i \underset{\substack{\text{def} \\ \text{somma}}}{\equiv} a_i + b_i \underset{\substack{\text{comm.} \\ \text{somma} \\ \text{di numeri}}}{=} b_i + a_i = (b+a)_i \quad \forall i = 1 \ldots n$$

$a+b = b+a$ perché hanno le stesse componenti

G + elemento neutro, opposto, associativa } G gruppo

insieme

$\uparrow$

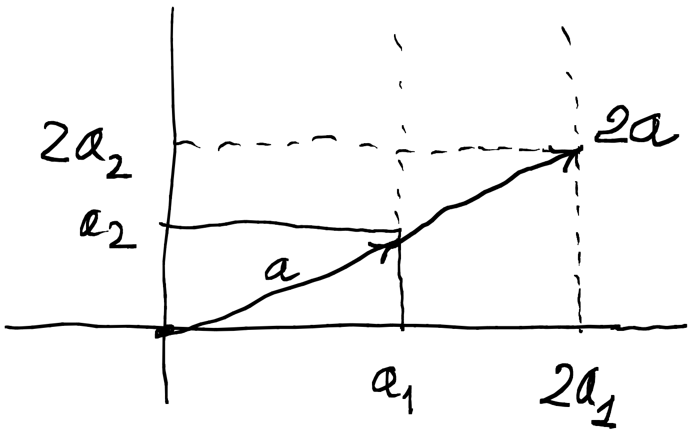
operazione

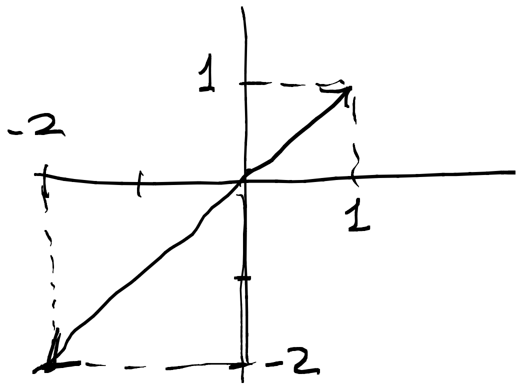
+ commutativa — G gruppo abeliano

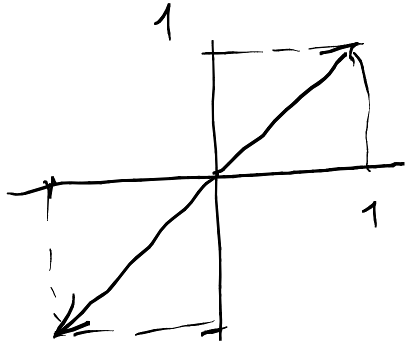
ABEL

$\mathbb{R}^n$ è un gruppo abeliano rispetto alla somma introdotta.

MULTIPLO SCALARE ≡ PRODOTTO DI UNO SCALARE PER UN VETTORE







$$\forall \lambda \in \mathbb{R} \quad \forall a \in \mathbb{R}^n$$

λ SCALARE
a VETTORE

$$\lambda a \equiv a \lambda = (\lambda a_1, \lambda a_2, \dots, \lambda a_n)$$

OVVERO

$$(\lambda a)_i = \lambda a_i \quad \forall i = 1..n$$

$$\pi(2, 0, -1) = (\pi \cdot 2, \pi \cdot 0, \pi(-1)) = (2\pi, 0, -\pi)$$

$$(2,0,-1)\pi = (2\pi, 0\pi, (-1)\pi) = (2\pi, 0, -\pi)$$

Proprietà del prodotto scalare per vettore

1) $1a = a$ $\quad \lambda = 1 \quad a = (a_1 \dots a_n) \Rightarrow (\lambda a)_i = 1 \cdot a_i = a_i$
$\forall i = 1 \dots n$

2) $\underline{\lambda(\mu a)} = \underline{(\lambda\mu)a}$ $\quad \forall \lambda, \mu \in \mathbb{R}$ (NUMERI = SCALARI) $\quad \forall a \in \mathbb{R}^n$ vettore

$$\rightarrow [\lambda(\mu a)]_i = \lambda(\mu a)_i = \lambda(\mu a_i) \underset{\text{prop. associativa in } \mathbb{R}}{=} \lambda\mu a_i = (\lambda\mu)a_i = ((\lambda\mu)a)_i$$

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3) $\lambda(a+b) = \lambda a + \lambda b \quad \forall \lambda \in \mathbb{R} \quad \forall a, b \in \mathbb{R}^n$

4) $(\lambda+\mu)a = \lambda a + \mu a \quad \forall \lambda, \mu \in \mathbb{R} \quad \forall a \in \mathbb{R}^n$

propr. distributiva del prodotto rispetto alla somma

Se un insieme qualunque ha definito una somma con le proprietà di un gruppo abeliano (+, 0, −, associativa, commut.)
e ha definito un prodotto per uno scalare "distributivo" "associativo" e verificante $1 \cdot a = a$

Si dice **SPAZIO VETTORIALE**

$$S = \{ \underline{\text{polinomi}} \} \qquad \text{Scalari} = \mathbb{R}$$

S è uno spazio vettoriale con le operazioni

$$\boxed{p+q} = \sum_{0}^{n} \alpha_i x^i + \sum_{0}^{m} \beta_i x^i = \underline{\sum_{0}^{\max(n,m)} (\alpha_i + \beta_i) x^i}$$

$$(1+x \underbrace{}_{+0\cdot x^2}) + (x^2+2x+1)$$

$$\|$$

$$(1+1) + (x+2x) + x^2$$

$$2\,p(x) = 2\sum_0^n \alpha_i x^i = \sum_0^n 2\alpha_i x^i \qquad 1 \cdot p(x) = p$$

$$p(x) = \sum_0^n \alpha_i x^i \qquad q(x) = \sum_0^n \beta_i x^i$$

$$(p+q)(x) \equiv \sum_0^{\max(n,m)} (\alpha_i + \beta_i) x^i$$

$$\lambda p(x) \equiv \sum_0^n (\lambda \alpha_i) x^i$$

$$(\lambda+\mu)\,p(x) \overset{\text{def}}{\equiv} \sum_0^n (\lambda+\mu)\alpha_i x^i = \sum_0^n \lambda\alpha_i x^i + \sum_0^n \mu\alpha_i x^i =$$

$$= \lambda p(x) + \mu p(x)$$

$$\alpha_i \in \mathbb{R}$$

$$\sum_{i=0}^n \alpha_i = \alpha_0 + \alpha_1 + \dots + \alpha_n$$

$$\boxed{\sum_0^n \alpha_i x^i = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n}$$