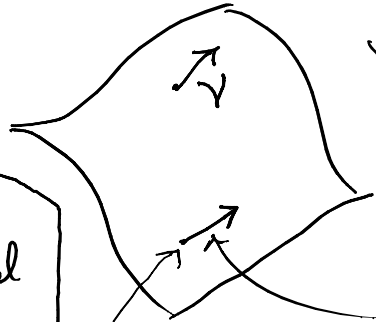
Teorema di Stokes

A campo def. in $\Omega \subseteq \mathbb{R}^3$ $\quad (A:\Omega \to \mathbb{R}^3)$

$A \in C^1(\Omega)$

$\phi : \Delta \to \Omega$

$\phi(\Delta) \subseteq \Omega$



$\phi(u,v)$

ν versore normale

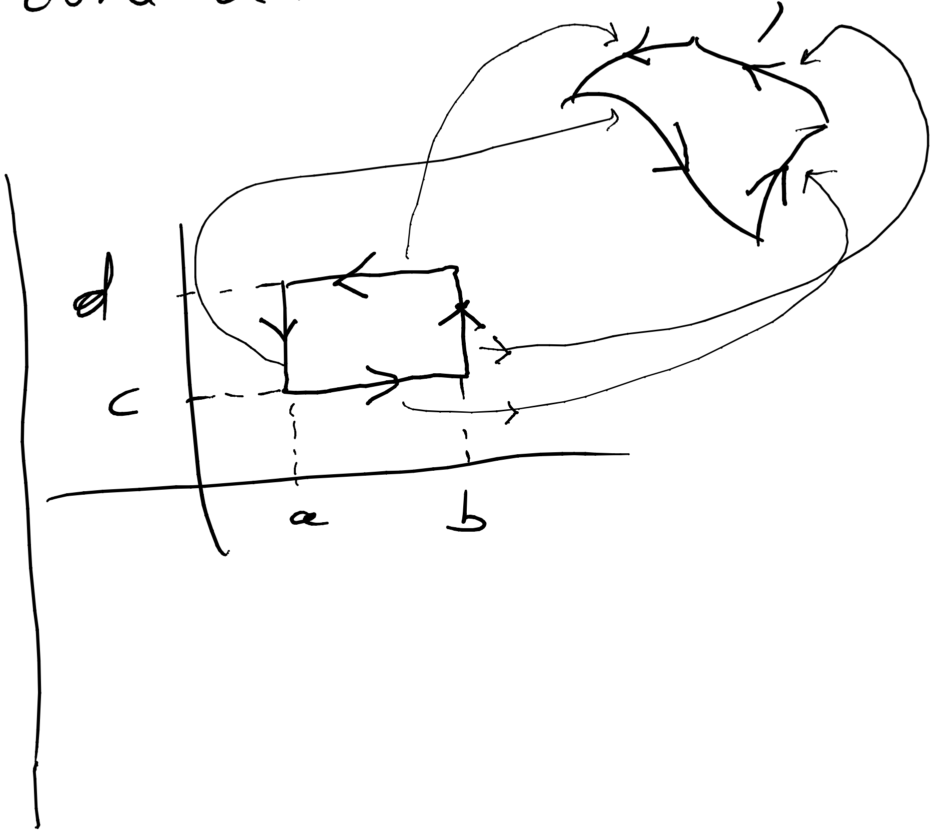
$$\nu(u,v) = \frac{\phi_u(u,v) \times \phi_v(u,v)}{|\phi_u \times \phi_v|}$$

$A(\phi(u,v))$

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \partial_y A_3 - \partial_z A_2 \\ -(\partial_x A_3 - \partial_z A_1) \\ \partial_x A_2 - \partial_y A_1 \end{pmatrix}$$

$$\underbrace{\int_\phi (\nabla \times A)\,\nu\, ds}_{\text{flusso del rotore}} = \underbrace{\int_{\partial\phi^+} A}_{\text{integrale del campo } A \text{ sulla curva } \partial\phi^+}$$

$\partial\phi^+$ è la curva parametrica ottenuta da ϕ considerandone la restrizione al bordo di Δ, PERCORSO IN SENSO ANTIORARIO

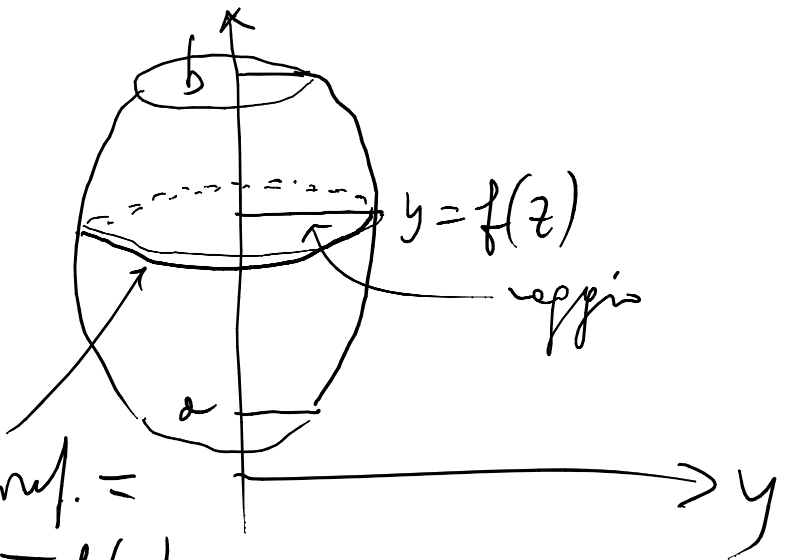


Es)

$$\Delta = [a,b] \times [c,d]$$

Se ν è orientato come $\phi_u \times \phi_v$ allora il bordo di Δ va orientato in senso ANTIORARIO

Superficie di solidi di rotazione



circonf. = $= 2\pi f(z)$

Volume $\pi f^2(z)\, dz$

Vol. totale $\int_a^b \pi f^2(z)\, dz$

$\rho = f(z)$

Rappres. parametrica del solido di rotazione

$$\rho = f(z) \qquad \begin{pmatrix} \theta \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} f(z)\cos\theta \\ f(z)\sin\theta \\ z \end{pmatrix} \longrightarrow \phi : \Delta \to \mathbb{R}^3$$

$\theta \in [0, 2\pi] \quad z \in [a, b] \quad \Delta = [0, 2\pi] \times [a, b] \to$

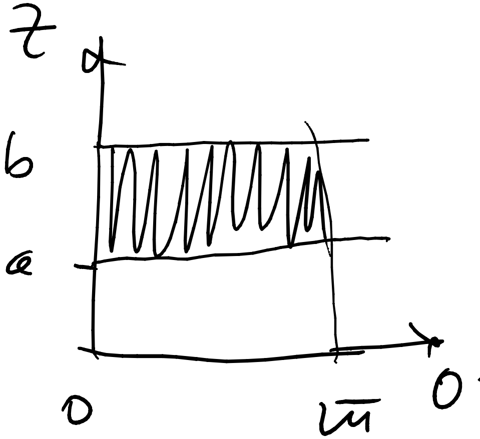
$$v(\theta, z) = \begin{pmatrix} -f(z)\sin\theta \\ f(z)\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} f'(z)\cos\theta \\ f'(z)\sin\theta \\ 1 \end{pmatrix} = \begin{pmatrix} f(z)\cos\theta \\ -(-f(z)\sin\theta) \\ -f(z)f'(z) \end{pmatrix}$$

vettore normale standard

$$|v| = \sqrt{f^2(z)\cos^2\theta + f^2(z)\sin^2\theta + f^2(z)f'^2(z)} = f(z)\sqrt{1+f'^2(z)}$$

$$\boxed{|v|\,d\theta\,dz = dS}$$

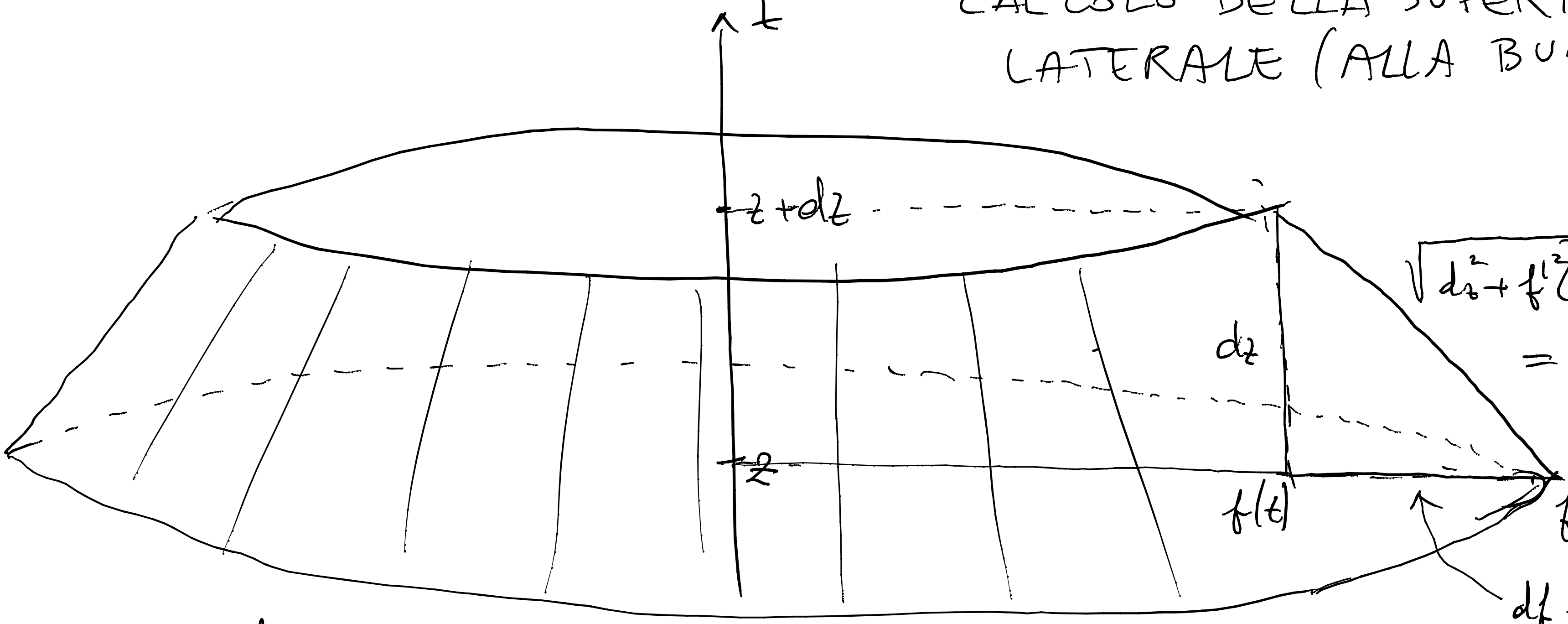
$f > 0$



$$\text{Area} = \int_0^{2\pi} d\theta \int_a^b dz\, f(z)\sqrt{1+f'^2(z)} = \ldots\; 2\pi f(z)$$

$$\text{Area} = \int_\Delta |v|\,d\theta\,dz = \ldots$$

CALCOLO DELLA SUPERFICIE LATERALE (ALLA BUONA!)



$$\sqrt{dz^2+f'^2(z)dz^2} = \sqrt{1+f'(z)^2}\,dz$$

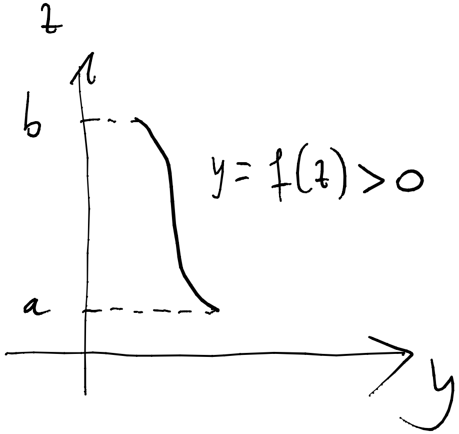
circonferenza di base $= 2\pi f(z)$

La porzione di superficie evidenziata ha "base" $2\pi \underbrace{f(z)}_{\text{raggio}}$ e "altezza" quella ricavata dal teorema di Pitagora applicato a

altezza effettiva della striscia relativa all'incremento verticale dz, $\sqrt{1+f'^2(z)}\,dz$

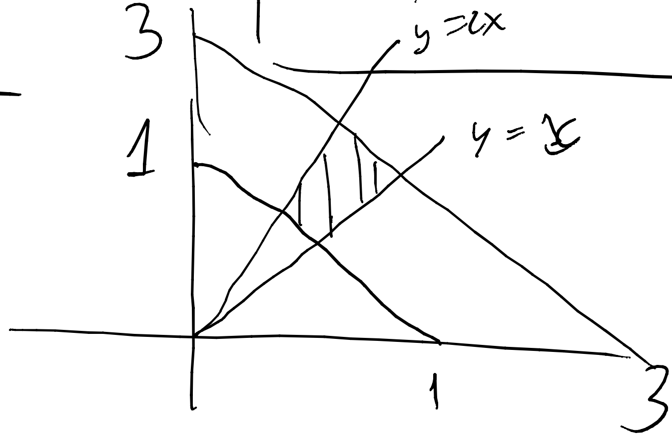
$$= 2\pi \int_a^b f(z)\sqrt{1+f'^2(z)}\,dz$$

PER I SOLIDI DI ROTAZIONE



$$\boxed{\begin{array}{l} \text{VOLUME} = \pi \int_a^b f^2(z)\,dz \\ \text{SUPERFICIE LATERALE} = 2\pi \int_a^b f(z)\sqrt{1+f'^2(z)}\,dz \end{array}}$$

$$\boxed{\int_\Omega \frac{dx\,dy}{xy} \qquad \Omega = \left\{ (x,y) \in \mathbb{R}^2 \;\; \begin{array}{l} 0 < x < y < 2x \\ 1 < x+y < 3 \end{array} \right\}}$$



$x + y =$

CAMBI DI VARIABILE

$$\boxed{1 < \frac{y}{x} < 2 \qquad\qquad 1 < x+y < 3}$$

u $\qquad\qquad\qquad\qquad\qquad$ v

$u = \dfrac{y}{x} \qquad v = x + y$

$y = xu \qquad v = x + ux = x(1+u) \Rightarrow \boxed{x = \dfrac{v}{1+u}}$

$$\boxed{y = \frac{v}{1+u}\, u}$$

$$T : (u, v) \to (x, y)$$

$$T' = \begin{pmatrix} -\dfrac{v}{(1+u)^2} & \dfrac{1}{1+u} \\[2ex] \dfrac{v(1+\cancel{u}) - \cancel{uv}}{(1+u)^2} & \dfrac{u}{1+u} \end{pmatrix}$$

$$\det T' =$$

$$= -\frac{uv}{(1+u)^3} - \frac{v}{(1+u)^3} = -\frac{v}{(1+u)^2} \neq 0$$

$$\int_T \frac{dx\,dy}{xy} = \int_1^2 du \int_1^3 dv\, \cancel{\frac{1+u}{v}}\, \frac{\cancel{1+u}}{uv}\, \cancel{\frac{v}{(1+u)^2}} = \int_1^2 du \int_1^3 dv\, \frac{1}{uv} =$$

$$= \int_1^2 du\, \frac{1}{u} \int_1^3 \frac{1}{v}\, dv = \boxed{\lg|v|\Big|_1^3 \cdot \lg|u|\Big|_1^2} = \lg 2 \lg 3$$

$$\boxed{\text{Calcolare l'area di } \Omega\ \left\{(x,y)\in\mathbb{R}^2 : y>0,\ x+y>0,\ x^2+y^2<3\sqrt{x^2+y^2}-3x\right\}}$$

Ω, in coordinate polari, diventa

buono in coord. polari

$$\begin{cases} \overbrace{\rho\sin\theta}^{y} > 0 \\ \underbrace{\rho(\cos\theta+\sin\theta)}_{x+y} > 0 \end{cases}$$

$$\Downarrow$$

$\rho>0 \quad \sin\theta>0 \quad \theta\in[0,\pi]$

$\cos\theta+\sin\theta>0$

$$\boxed{\cos\theta > -\sin\theta}$$

$$\rho^2 < 3\rho - 3\underbrace{\rho\cos\theta}_{x} \qquad \boxed{\rho>0}$$

$$\Downarrow$$

$$\rho < 3-3\cos\theta = 3(1-\cos\theta)$$

ρ appartiene all'intervallo (servirà cmq)

$$\boxed{0<\rho<3(1-\cos\theta)}$$

$\theta \in \left[\frac{\pi}{2}, \frac{3}{4}\pi\right]$ $|\cos| < |\sin|$

I e II membro sono negativi

se $\theta \in \left[\frac{3}{4}\pi, \pi\right]$

$|\cos| > |\sin|$

$\Rightarrow$ la dis. è falsa

$\cos\theta > -\sin\theta$ è verificata (in $[0,\pi]$) solo

per $\theta \in \left[0, \frac{3}{4}\pi\right]$

$$\int_0^{\frac{3}{4}\pi} d\theta \int_0^{3(1-\cos\theta)} 1 \cdot \rho \, d\rho = \int_0^{\frac{3}{4}\pi} \frac{1}{2}\left[9(1-\cos\theta)^2\right] d\theta$$

$9 - 18\cos\theta + 9\cos^2\theta$

funzione integranda per l'area

jacobiano coordinate polari

$\int \cos^2\theta$ formule di duplicazione

$$\Omega = \underbrace{\{x^2+y^2+4x \le 0\}}_{\Omega_1} \cup \underbrace{\{x>0\ ;\ y<|x|\}}_{\Omega_2}$$

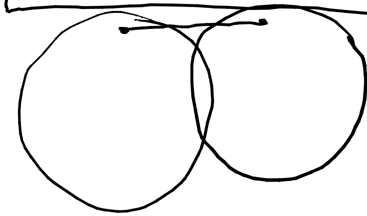
sommando e sottraendo 4

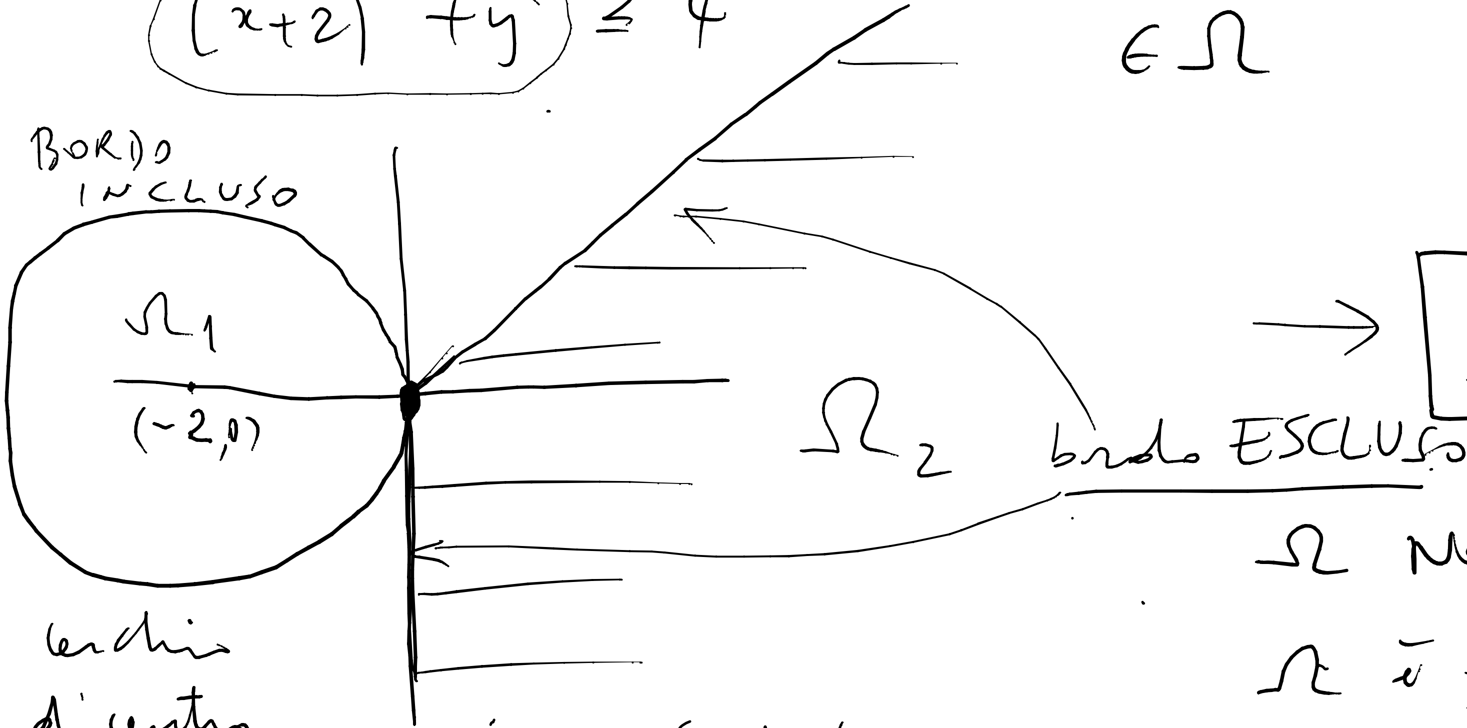
$$\underbrace{x^2+4x+4}+y^2 \le 4$$

$$(x+2)^2+y^2 \le 4$$

$$\boxed{(0,0)\in\Omega_1}$$
$$\Downarrow$$
$$\in\Omega$$

Ω è
- CONVESSO
- SEMPLICEMENTE CONNESSO
- CONNESSO





$$y<|x|$$

$$\longrightarrow \boxed{y<x}$$

cerchio di centro $(-2,0)$ e raggio 2, incluso il bordo ($\le$)

Ω NON E' CONVESSO

Ω è stella e quindi sempl. connesso

risp. a qualunque pto (x_0,y_0) $(0,0)$

$$\lim_{(0,0)} \frac{\sin(x^2-y^2)}{x^2+y^2}$$

$//$

0 se $x^2-y^2=0$

$$\frac{\sin(x^2-y^2)}{x^2-y^2} \cdot \frac{x^2-y^2}{x^2+y^2}$$

$x^2-y^2\neq 0$

$\downarrow$ 1

$\downarrow$ 0 - comunque NON COSTANTI

$=1$ se $y=0$

$=-1$ se $x=0$

LIMITE

NON ESISTE

INVERTIBILITA' LOCALE

$$\begin{cases} x = u^2 - v \\ y = u - v^2 \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 2u & -1 \\ 1 & -2v \end{pmatrix}$$

$$\det J = -4uv + 1$$

$\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{4}, 1\right)$

$\downarrow$

$uv = \frac{1}{6}$ $\boxed{\text{sì}}$

$uv = \frac{1}{4}$ NO

$\left(\frac{1}{2}, \frac{1}{2}\right)$

$uv = \frac{1}{4}$ NO

$$\boxed{uv = \frac{1}{4}}$$

NON SI PUO' APPLICARE IL TH. INV. LOCALE

N.A.

$$f(x,y) = |1-\cos xy|^{2/3} \quad \text{in } (0,0)$$

CONTINUITA'
DERIVABILITA'
DIFFERENZIABILITA'

1) $f(x,y)$ è continua in $(0,0)$ (composta di f. continue)

2) $f_x(0,0) = 0$ perché f è costante (e fa 0) sugli assi.

$f_y(0,0) = 0$

$$\exists\ \nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$w = (h,k)$

3)
$$\lim_{h,k\to 0} \frac{f(h,k) - f(0,0) - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}}{\underbrace{\sqrt{h^2+k^2}}_{|w|}} = \lim_{h,k\to 0} \frac{|1-\cos hk|^{2/3}}{\sqrt{h^2+k^2}} =$$

Ma a che fare con $1-\cos \square \to$ una cosa che va a zero. Limiti notevoli.

$hk = t$

$$\left(\frac{1}{2}\right)^{\frac{2}{3}} \leftarrow \frac{|1-\cos hk|^{2/3}}{(hk)^{2\cdot\frac{2}{3}}} \cdot \frac{(hk)^{\frac{4}{3}}}{\sqrt{h^2+k^2}}$$

$(hk)^{\frac{4}{3}} \to 2\cdot\frac{4}{3}$-omog. $\frac{8}{3}$-omog.

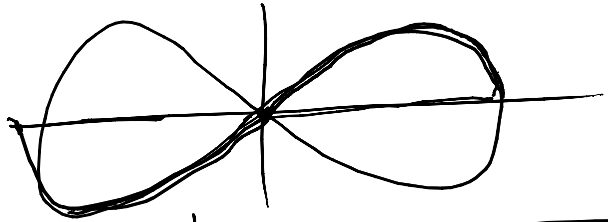
$\sqrt{h^2+k^2}$ — 1-omogenea sempre definita su $\partial B(0,1)$

$$\boxed{\lim_{t\to 0} \frac{1-\cos t}{t^2} = \frac{1}{2}}$$

$\frac{8}{3}-1>0$ la funzione è mai nulla su $\underline{h^2+k^2=1}$

$\Rightarrow$ Th. funzioni omog. grado >0 $\Rightarrow$ $\underline{\lim = 0}$

$\theta \in [0,\pi]$, $\rho = 1+\sin\theta$



$$\Lambda = \int_a^b \sqrt{\dot\rho^2 + \rho^2\dot\theta^2} \longrightarrow \begin{cases} \rho = 1+\sin t \\ \theta = t \end{cases}$$

$\boxed{\text{LUNGHEZZA IN COORDINATE POLARI}}$

$$\Lambda = \int_0^\pi \sqrt{\cos^2 t + (1+\sin t)^2 \cdot 1}\, dt = \int_0^\pi \sqrt{\cos^2 t + \sin^2 t + 2\sin t + 1}\, dt$$

$$= \int_0^\pi \sqrt{2}\sqrt{1+\sin t}\, dt \underset{u=\frac{\pi}{2}-t}{=}$$

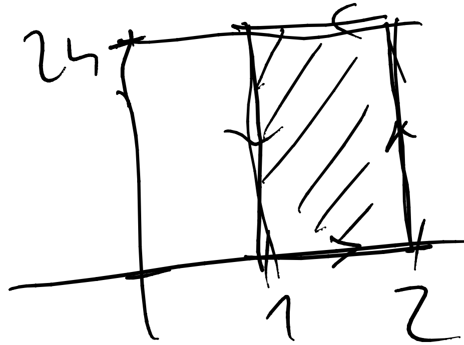
$$\boxed{\begin{aligned} &\cos t = \\ &= \cos 2\left(\tfrac{t}{2}\right) = 2\cos^2\tfrac{t}{2} - 1 \end{aligned}}$$

$$\boxed{f(x,y,z) = \sqrt{x^2+y^2} \quad \text{estesa a} \qquad \text{INTEGRALE SUPERFICIALE}}$$

$$\boxed{\phi(u,v) = \begin{pmatrix} u^2 \cos v \\ u^2 \sin v \\ v \end{pmatrix} \qquad u \in [1,2] \quad v \in [0, 2\pi]}$$

$$\Delta = [1,2] \times [0, 2\pi]$$

$$\boxed{\int_{\phi} f = \int_{\Delta} f(\phi(u,v)) \, |\phi_u \times \phi_v| \, du\, dv}$$



$$\int_1^2 du \int_0^{2\pi} dv \, \underbrace{\sqrt{u^4 \cos^2 v + u^4 \sin^2 v}}_{f(\phi(u,v))} \, \underbrace{2u\sqrt{1+u^4}}_{|\nu|}$$

$$\phi_u \to \begin{pmatrix} 2u \cos v \\ 2u \sin v \\ 0 \end{pmatrix} \times \begin{pmatrix} -u^2 \sin v \\ u^2 \cos v \\ 1 \end{pmatrix} \leftarrow \phi_v = \nu = \begin{pmatrix} 2u \sin v \\ -2u \cos v \\ 2u^3 \cos^2 v + 2u^3 \sin^2 v \end{pmatrix} = 2 \begin{pmatrix} u \sin v \\ -u \cos v \\ u^3 \end{pmatrix}$$

$$\boxed{|v| = 2\sqrt{u^2 \sin^2 v + u^2 \cos^2 v + u^6} = 2u\sqrt{1+u^4} \qquad u \in [1,2],\ u>0}$$

$$(*) = \int_1^2 du \int_0^{2\pi} dv\, \underset{u^2}{\cancel{\sqrt{u^4}}}\, 2u\sqrt{1+u^4} = \frac{1}{2}\int_0^{2\pi} dv \int_1^2 du\, \sqrt{\underbrace{1+u^4}_{1+u^4=t}}\ \overbrace{\boxed{2u^3}}^{=\frac{1}{2}\frac{d(1+u^4)}{du}} \cdot 2$$

$$(1+u^4)' = 4u^3$$