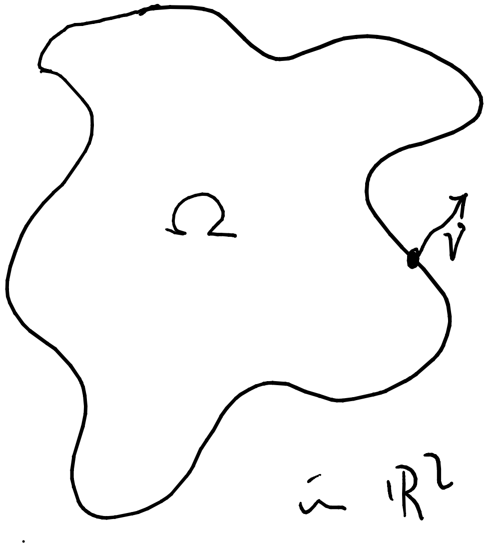
(GAUSS - GREEN - OSTROGRADSKIJ)

$$\underbrace{\int_{\Omega} \underbrace{f_x(x,y)\,dx\,dy}}_{\text{integrale doppio}} = \underbrace{\int_{\partial\Omega^+} \overbrace{f(x,y)\,dy}^{\text{forme diff.}}}_{\text{integrale di una forma su una curva}} \longleftarrow$$

$\partial\Omega^+$ (sostegno di) una curva parametrica PERCORSA IN SENSO ANTIORARIO

$$\boxed{\int_{\partial\Omega} f\,\nu_1\,dl} \qquad dl \to |\gamma'(t)|\,dt$$

$\nu$ versore normale esterno a $\partial\Omega$




$|\nu| = 1$

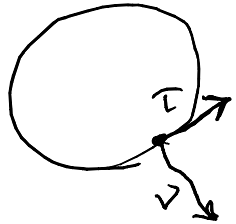
$\nu = (\nu_1, \nu_2, \dots, \nu_n)$ in $\mathbb{R}^n$

in $\mathbb{R}^2$ $\nu = (\nu_1, \nu_2)$

$$\boxed{\int_{\partial\Omega} f(\gamma(t)) \cdot \frac{\dot\gamma_2(t)}{\cancel{\sqrt{\dot\gamma_1^2(t)+\dot\gamma_2^2(t)}}} \cdot \cancel{|\dot\gamma(t)|}\, dt}$$

$$= \int_{\partial\Omega^+} f\, dy$$

$\gamma(t)$




vettore tangente $\tau(t) = \begin{pmatrix} \dot\gamma_1(t) \\ \dot\gamma_2(t) \end{pmatrix}$

vettore normale esterno $\boxed{n = \begin{pmatrix} \dot\gamma_2(t) \\ -\dot\gamma_1(t) \end{pmatrix}}$ $\quad \nu = \frac{n}{|n|}$ VERSORE normale esterno

$$\int_\Omega f_y\, dx\, dy = \underbrace{-\int_{\partial\Omega^+} f(x,y)\, dx}_{\text{forma diff.}} = \underbrace{\int_{\partial\Omega} f \nu_2\, d\ell}_{\text{integrale curvilineo della funzione } f\nu_2}$$

$\operatorname{div} A =$ A campo

$= (A_1)_{x_1} + (A_2)_{x_2} + \dots + (A_n)_{x_n}$

$\operatorname{div} A = \nabla \cdot A$

$$\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}$$

$$\int_\Omega \operatorname{div} A \, dx_1 \dots dx_n = \int_{\partial\Omega} A \cdot \nu \, d\sigma$$

$\boxed{\text{Th. GAUSS}}$

Th. della

$\boxed{\text{FEYNMAN}}$

LEZIONI DI FISICA

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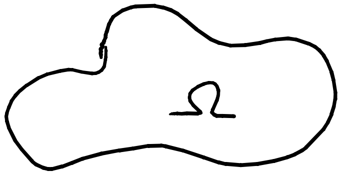
Area racchiusa da una curva piana

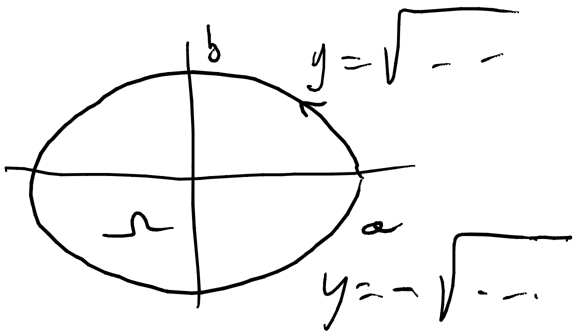



$$\text{Area di } \Omega = \int_\Omega 1 \, dx dy = \int_{\partial\Omega^+} x \, dy = -\int_{\partial\Omega^+} y \, dx$$

NON E' CHIUSA

$$\text{area di } \Omega = \int_{\partial\Omega^+} \frac{1}{2} (x \, dy - y \, dx)$$

NON E' CHIUSA




$$\gamma(t) = \begin{cases} a\cos t = x \\ b\sin t = y \end{cases} \quad t \in [0, 2\pi]$$

$$\frac{x^2}{a^2} = \cos^2 t \qquad \frac{y^2}{b^2} = \sin^2 t$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$\iint_\Omega 1 \, dx\,dy = \int_{\partial\Omega^+} x \, dy =$$

$$y = b\sin t$$

$$dy = b\cos t \, dt$$

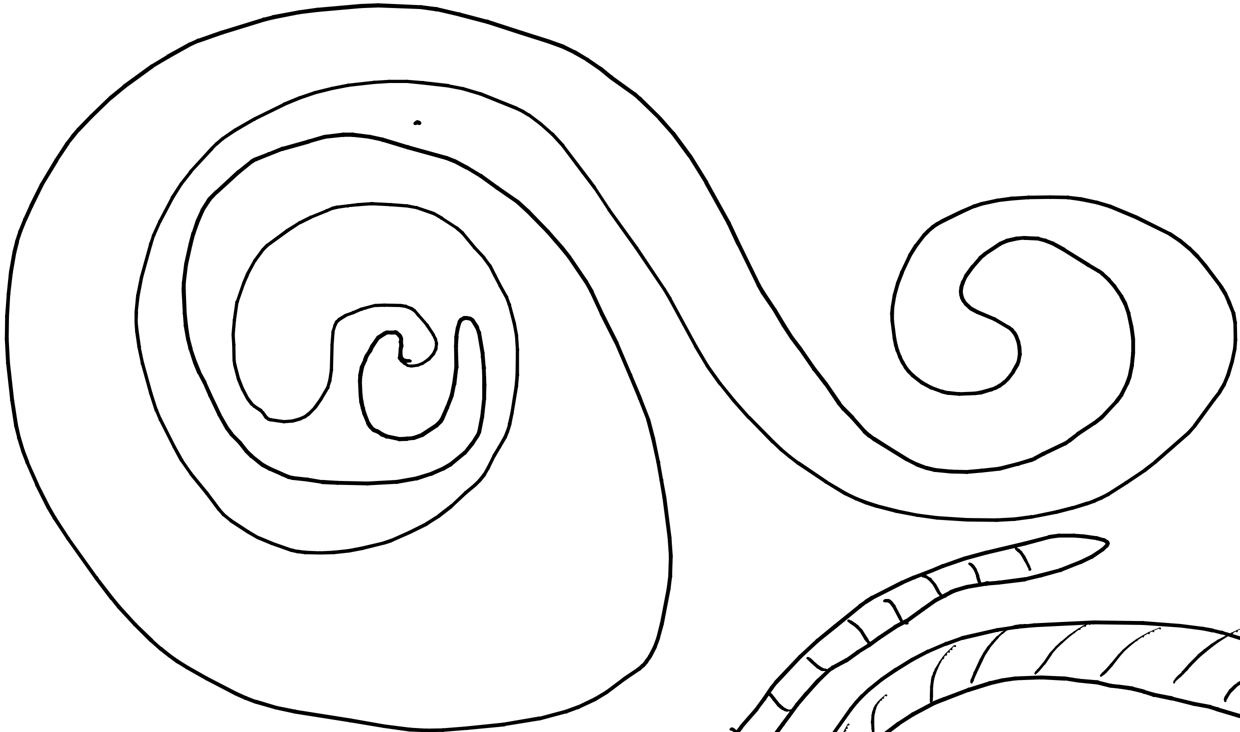
$$= \int_0^{2\pi} \underbrace{a\cos t}_{x} \, b\cos t \, dt =$$

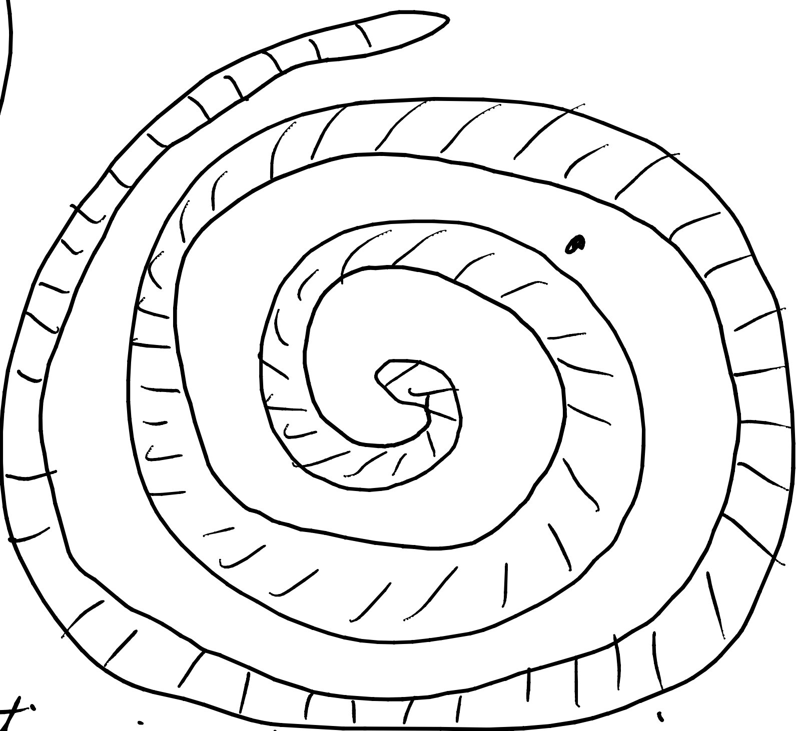
$$\cos 2t = 2\cos^2 t - 1$$

$$= ab \int_0^{2\pi} \cos^2 t \, dt =$$

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$$

$$= ab \int_0^{2\pi} \frac{1}{2} + ab \int_0^{2\pi} \frac{1}{2}\cos 2t = \pi ab + \frac{ab}{2} \underbrace{\left[\frac{1}{2}\sin 2t\right]_0^{2\pi}}_{=0} = \pi ab$$





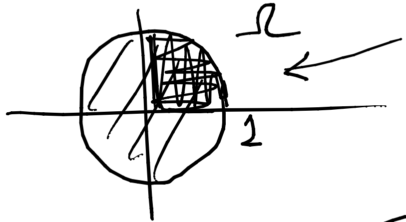
$$\int_{\gamma} \left( \frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} \right)$$

è nullo se $\gamma$ non racchiude $(0,0)$

è multiplo di $2\pi$ se gira in senso antiorario attorno all'origine

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy \qquad \Omega = \{x^2+y^2 \le 1\}$$

$[0,\pi/2]$

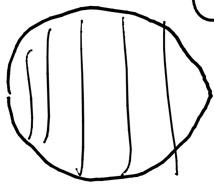



coord. polari (piane)

$$\frac{1}{\sqrt{x^2+y^2}} = k \qquad \boxed{k>0}$$

$$\sqrt{x^2+y^2} = \frac{1}{k}$$

$$x^2+y^2 = \frac{1}{k^2}$$



$\psi(x)$

$\varphi(x)$

$$x \in [-1,1] \qquad y \in \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right]$$

$$\int_0^{2\pi} d\theta \int_0^1 d\rho \left( \underbrace{\cancel{\frac{1}{\rho}}}_{f \text{ in coord. polari}} \; \underbrace{\cancel{\rho}}_{\text{jacobiano delle coord. polari}} \right) = 2\pi \cdot 1$$

$$x^2+y^2 \le 1$$

$$\rho^2\cos^2\theta + \rho^2\sin^2\theta \le 1$$

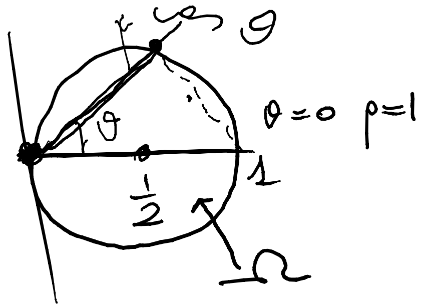
$$\rho^2 \le 1 \Rightarrow \boxed{\rho \le 1}$$

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} d\rho\, \frac{1}{\rho}\cdot\rho =$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \cos\theta = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) =$$

$$= 2$$




cerchio di centro $\left(\frac{1}{2}, 0\right)$ e raggio $\frac{1}{2}$

$$\Omega = \left\{ \left(x-\frac{1}{2}\right)^2 + y^2 \leq \frac{1}{4} \right\}$$

$$= x^2 - x + \cancel{\frac{1}{4}} + y^2 \leq \cancel{\frac{1}{4}}$$

$$\Omega = \left\{ \underbrace{x^2+y^2} - x \leq 0 \right\}$$

in coordinate polari

$$\underbrace{\rho^2\cos^2\theta + \rho^2\sin^2\theta} - \rho\cos\theta \leq 0$$

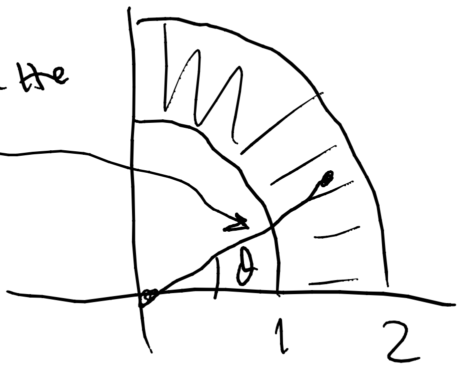
$$\boxed{\rho > 0}$$

$$\rho^2 \leq \rho\cos\theta \rightarrow 0 < \rho \leq \cos\theta$$

1) $\cos\theta > 0$ $\boxed{\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}$

2) $\boxed{\rho \in \left]0, \cos\theta\right[}$

la lunghezza è $\rho$



$\theta \in \left[0, \frac{\pi}{2}\right]$

$\rho \in [1, 2]$

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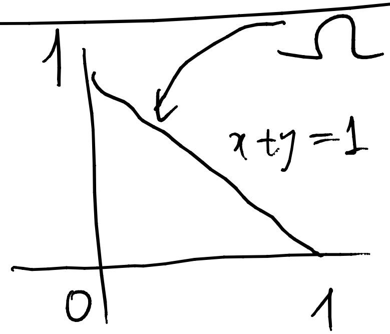
$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy =$$




$\boxed{\rho > 0}$ sempre

$$\begin{cases} x > 0 \\ y > 0 \\ x+y < 1 \end{cases}$$

1) $\rho \cos\theta > 0$
2) $\rho \sin\theta > 0$
3) $\rho(\cos\theta + \sin\theta) < 1$

$\Downarrow$

da 1) $\cos\theta > 0$
2) $\sin\theta > 0$

$\longrightarrow \theta \in \left[0, \frac{\pi}{2}\right]$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\cos\theta + \sin\theta}} d\rho\, \frac{1}{\rho}\, \rho =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + \sin\theta}\, d\theta =$$

formule parametriche

$t = \tan\frac{\theta}{2}$ $\quad \theta = 2\arctan t$

$\sin\theta = \frac{2t}{1+t^2}$ $\quad \cos t = \frac{1-t^2}{1+t^2}$ $\quad d\theta = \frac{2}{1+t^2}\, dt$

$\longrightarrow$ 3) $\rho < \frac{1}{\cos\theta + \sin\theta}$

$$= \int_0^{\operatorname{tg}\frac{\pi}{4}} \frac{1}{\frac{1-t^2+2t}{\cancel{1+t^2}}} \cdot \frac{2}{\cancel{1+t^2}}\, dt = \boxed{2\int_0^1 \frac{dt}{1-t^2+2t}} \longrightarrow$$

funzione razionale: integrazione standard

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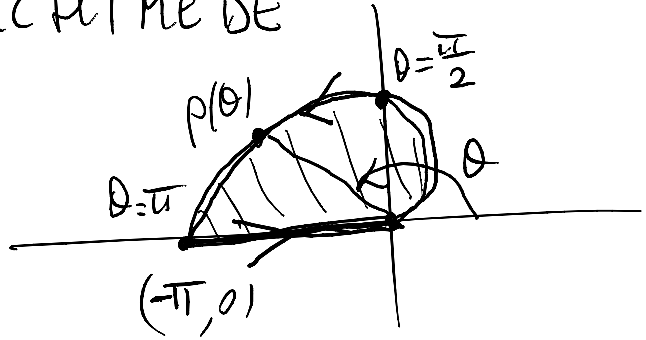
SPIRALE D'ARCHIMEDE




$$\boxed{\rho = \theta} \qquad \rho \in [0, \theta]$$

$$\theta \in [0, \pi]$$

$$\boxed{\int_\Omega 1\, dx} = \text{volume di } \Omega = \int_{\varphi^{-1}(\Omega)} \rho\, d\rho\, d\theta$$

$$\int_0^\pi d\theta \int_0^\theta \rho\, d\rho = \int_0^\pi d\theta \left[\frac{1}{2}\rho^2\right]_0^\theta = \int_0^\pi \frac{1}{2}\theta^2\, d\theta = \frac{1}{6}\theta^3\Big|_0^\pi = \frac{1}{6}\pi^3$$

SPIRALE D'ARCHIMEDE

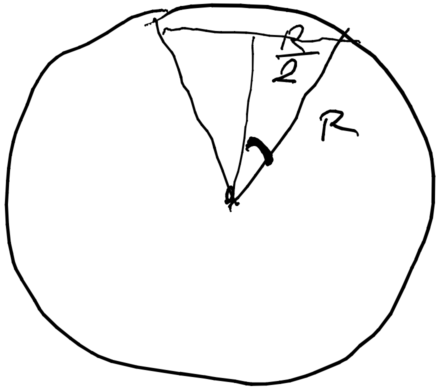
$$\int_{\gamma_{\Omega^+}} x\,dy = \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \text{ in generale}$$

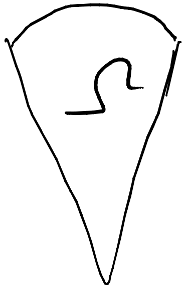
$$\begin{cases} x(\theta) = \theta\cos\theta \\ y(\theta) = \theta\sin\theta \end{cases} \quad \theta \in [0, \pi]$$

$$= \int_0^{\pi} \theta\cos\theta\,(\sin\theta + \theta\cos\theta)\,d\theta \; + \qquad \dot{y}(\theta) = \sin\theta + \theta\cos\theta$$

$$+ \cancel{\int_{\overrightarrow{\text{segmento } (-\pi,0)(0,0)}} x\,dy}$$

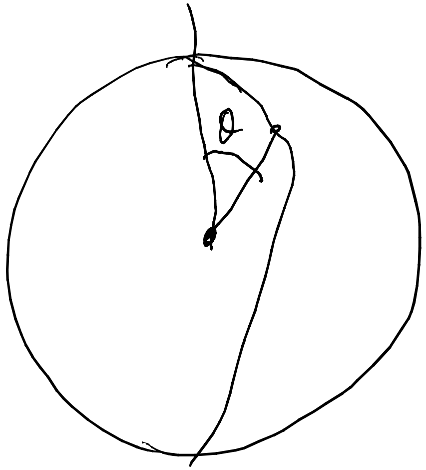
$$\boxed{\begin{array}{l} \sigma(t) = \begin{cases} -\pi + t \\ 0 \end{cases} \quad t \in [0, \pi] \\ \Downarrow \\ \dot{\sigma}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad dy = 0 \cdot dt \\ \text{parametrizzazione del segmento sull'asse } x \end{array}}$$




coord. sferiche

jacobiano $\rho^2 \sin\theta$




$$\mathrm{vol}\ \Omega = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{6}} d\theta \int_0^1 d\rho\ \underbrace{\rho^2 \sin\theta}_{\text{jacobiano}} =$$

$$= \underbrace{\int_0^{2\pi} d\varphi}_{\parallel} \underbrace{\int_0^{\frac{\pi}{6}} \sin\theta\, d\theta}_{\parallel} \underbrace{\int_0^1 \rho^2 d\rho}_{\parallel} = \frac{\cancel{2}}{3}\pi \left(\frac{2-\sqrt{3}}{\cancel{2}}\right) = \frac{\pi}{3}\left(2-\sqrt{3}\right)$$

$$-\cos\theta \Big|_0^{\frac{\pi}{6}} \qquad \frac{1}{3}$$

$$2\pi \qquad \left(-\frac{\sqrt{3}}{2}+1\right)$$

$$\phi : \Delta \to \mathbb{R}^3 \qquad \phi(u,v) = \begin{pmatrix} \phi_1(u,v) \\ \phi_2(u,v) \\ \phi_3(u,v) \end{pmatrix}$$

$$\nu = \phi_u \times \phi_v \qquad \phi_u = \begin{pmatrix} (\phi_1)_u \\ (\phi_2)_u \\ (\phi_3)_u \end{pmatrix}$$

$$\nu \neq 0 \quad \forall (u,v) \in \Delta$$

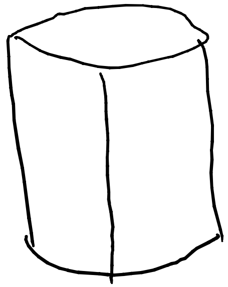
UNA SUPERFICIE PARAMETRICA

$\phi$ si dice REGOLARE se

1) $\phi \in C^1(\Delta)$

2) $\nu \neq 0$ dappertutto su $\Delta$

3) $\phi$ sia iniettiva su $\mathring{\Delta}$



Area di $\Phi = \int_{\Delta} |\nu| \, du \, dv = \int_{\Delta} |\phi_u \times \phi_v| \, du \, dv$

"simile" a $\int |\dot{\gamma}(t)| \, dt$

$f : \underline{\Omega} \to \mathbb{R}$ $\qquad$ $\Phi : \Delta \to \underline{\Omega}$

regolare

Integr. curvil.

$\int_a^b f(\gamma) |\dot{\gamma}(t)| \, dt$

$$\int_{\Phi} f \, ds = \int_{\Delta} \underbrace{f(\Phi(u,v))} \, \underbrace{|\phi_u(u,v) \times \phi_v(u,v)| \, du \, dv}_{ds}$$

$\underline{|\nu(u,v)| \, du \, dv}$

$$\phi(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$$

rappr. param. del grafico di $f$

$(u,v) \in \Omega = \mathrm{dom}\,\phi$

$$\phi_u = \begin{pmatrix} 1 \\ 0 \\ f_u(u,v) \end{pmatrix}$$

$$\phi_v = \begin{pmatrix} 0 \\ 1 \\ f_v(u,v) \end{pmatrix}$$

$$\gamma(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

$$\nu(u,v) = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix} = \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

$$\int_a^b \sqrt{1+f'(x)^2}\,dx$$

$$\text{area}\,\phi = \int_\Omega \sqrt{1+f_u^2+f_v^2}\,du\,dv$$