$$A(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

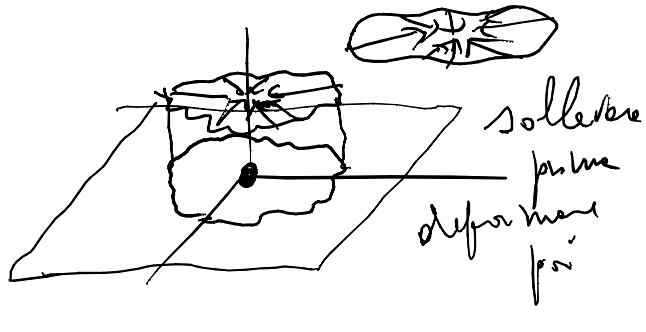
<u>A irrotazionale</u>

$$\frac{\partial}{\partial y}\left(-\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) \overset{?}{=} \frac{\partial}{\partial x}\left(-\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$-x\,\frac{\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\,2y}{(x^2+y^2+z^2)^3} = -y\,\frac{\frac{3}{2}(x^2+y^2+z)^{\frac{1}{2}}\,2x}{(x^2+y^2+z^2)^3}$$

A irrotaz. su un dominio sempl. connesso

l'omotopia prima sposta la curva e poi la deforma in un punto.



$f: \mathbb{R}^3 - \{(0,0,0)\} \to \mathbb{R} \mid \nabla f = A$ su tutto $A = \mathbb{R}^3 \setminus \{(0,0,0)\}$

integrando

$$
\begin{cases}
(1)\quad f_x = -\dfrac{x}{(x^2+y^2+z^2)^{3/2}} \\[2ex]
(2)\quad f_y = -\dfrac{y}{(x^2+y^2+z^2)^{3/2}} \\[2ex]
(3)\quad f_z = -\dfrac{z}{(x^2+y^2+z^2)^{3/2}}
\end{cases}
$$

$\Longrightarrow$ in dx

$$f(x,y,z) = -\frac{1}{2}\int \frac{2x}{(x^2+y^2+z^2)^{3/2}}\,dx + c(y,z) =$$

($2x$ è la derivata di $(x^2+y^2+z^2)$)

$$\underset{(x^2+y^2+z^2)=t}{=} c(y,z) - \frac{1}{2}\cdot\int t^{-\frac{3}{2}}\,dt =$$

$$= c(y,z) - \frac{1}{2}\left[\frac{1}{-\frac{1}{2}}\, t^{-\frac{1}{2}}\right] =$$

$$= c(y,z) + \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} + c(y,z) \Rightarrow f_y = c_y(y,z) - \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

confrontando con 2) si ha $c_y(y,z) = 0$

c è costante in y (non necess. costante in z)

Dunque $c(y,z) = d(z)$

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} + d(z)$$

derivando rispetto a z e confrontando con 3) si ottiene

$$d' \equiv 0 \Rightarrow d = \text{costante}$$

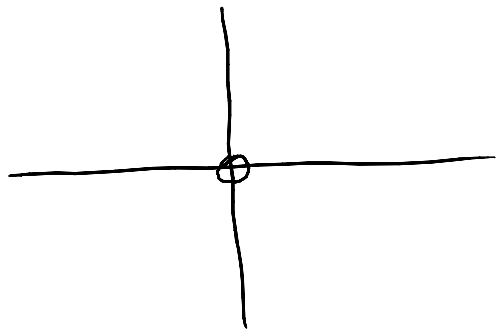
$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$ 1 primitiva

perché dom A è connesso, tutte le altre sono

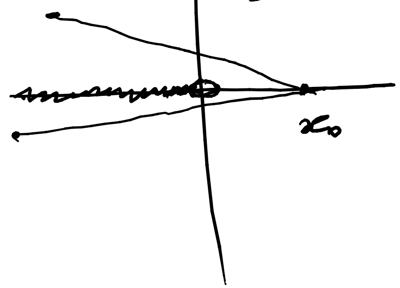
$$\boxed{\frac{1}{\sqrt{x^2+y^2+z^2}} + C}$$

$$A = \left(-\frac{y}{x^2+y^2} \, , \, \frac{x}{x^2+y^2} \right)$$

Campo irrotaz. NON INTEGRABILE



STELLA



$$\begin{cases} 1)\quad f_x = \dfrac{-y}{x^2+y^2} \\ 2)\quad f_y = \dfrac{x}{x^2+y^2} \end{cases}$$

INTEG. DI A nel suo dominio

$\nabla f \equiv A$ su dom A

1) $\Rightarrow$ $f(x,y) = -y \int \frac{1}{x^2+y^2}\,dx + c(y)$

$= \boxed{-\arctan \frac{x}{y}} + c(y)$

dom $f \subset$ dom A

strett.

è una primitiva definita per $y \neq 0$

$y>0$ convesso

dom f ↓ $\frac{\pi}{2}$ ↓ ↓ $\frac{\pi}{2}$ ↓ $x>0$, $y \to 0^+$

↑ $-\frac{\pi}{2}$ ↑

dom f $x>0$, $y \to 0^-$

$y<0$ convesso

$= \arctan \frac{y}{x}$ è anch'essa una primitiva

$\arctan t + \arctan \frac{1}{t} = \frac{\pi}{2}$ ($t>0$)

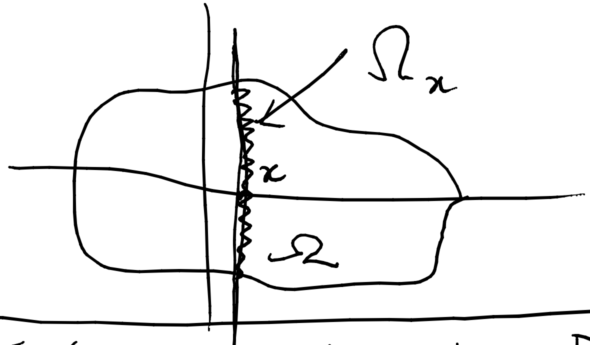
$-\arctan \frac{x}{y}$ $x>0$, $y>0$ $\qquad$ $-\arctan \frac{x}{y} + \pi$ $x>0$, $y<0$

$\frac{1}{y^2} \int \frac{-y}{\left(\frac{x}{y}\right)^2+1}\,dx$

$\frac{x}{y} = t \qquad dx = y\,dt$

$\frac{1}{y^2} \int \frac{-y}{t^2+1}\, y\,dt = \boxed{-\arctan \frac{x}{y}}$

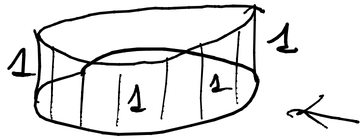
CAMPI E FORME VII



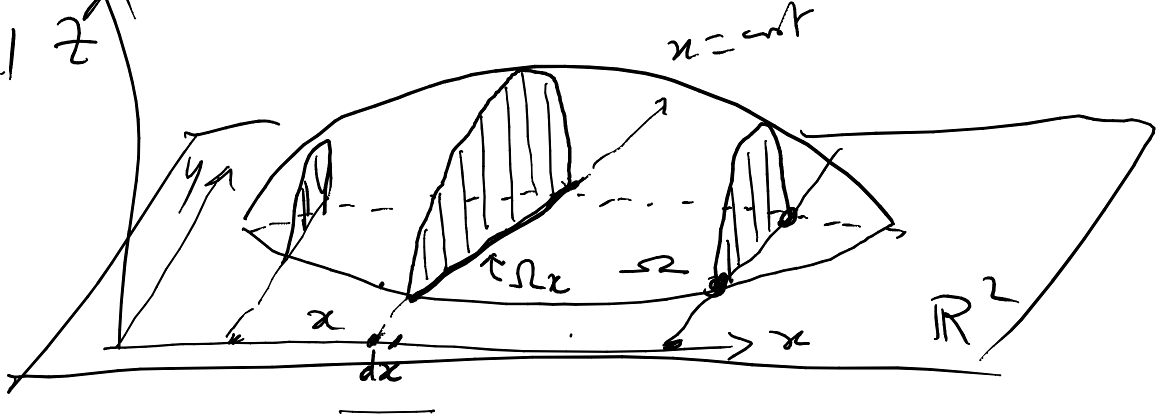
$\Omega \subseteq \mathbb{R}^2 \qquad \int_\Omega 1 \, dx\,dy$

INTEGRALI MULTIPLI

FUBINI - TONELLI 2



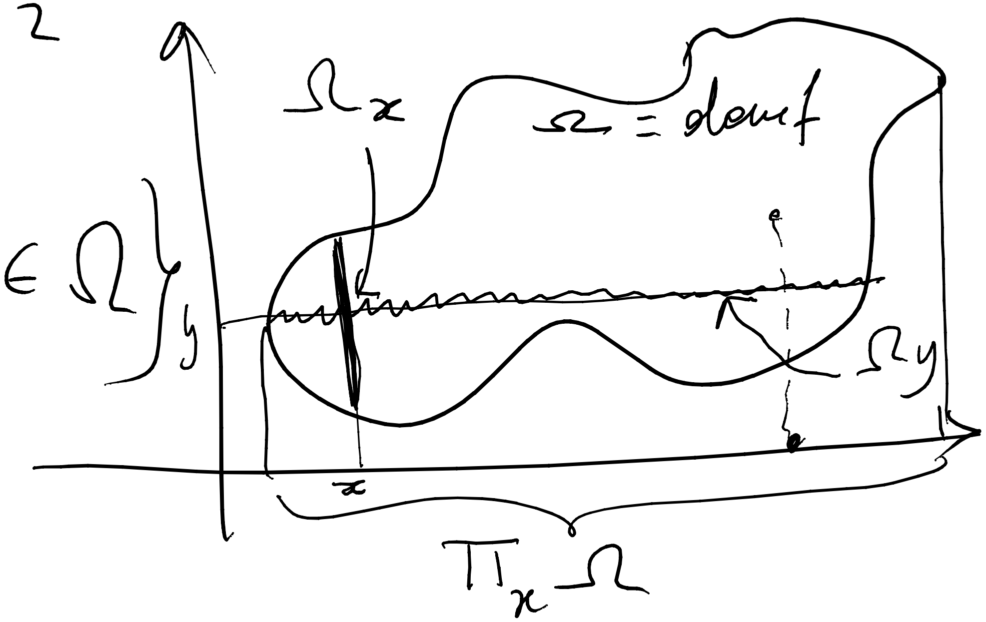
vol cilindro = area base



Area sezione di Ω con il piano xz $= \left[\int_{\Omega_x} f(x,y)\, dy \right] dx$

$$\int_\Omega f \, dx\,dy = \int \left[\int_{\Omega_x} f(x,y)\, dy \right] dx$$

$$f : \Omega \to \mathbb{R} \qquad \Omega \subseteq \mathbb{R}^2$$



$$\Pi_x \Omega = \{ x \in \mathbb{R} \; ; \; \exists y \in \mathbb{R} \;\; (x,y) \in \Omega \}$$

$$\Omega_x = \{ y \in \mathbb{R} : (x,y) \in \Omega \}$$

$$\iint_{\Omega} f(x,y)\, dx\, dy = \int_{\Pi_x \Omega} \underbrace{\left[\int_{\Omega_x} f(x,y)\, dy \right]}_{\text{funzione di } x} dx =$$

$$= \int_{\Pi_y \Omega} \left[\int_{\Omega_y} f(x,y)\, dx \right] dy$$

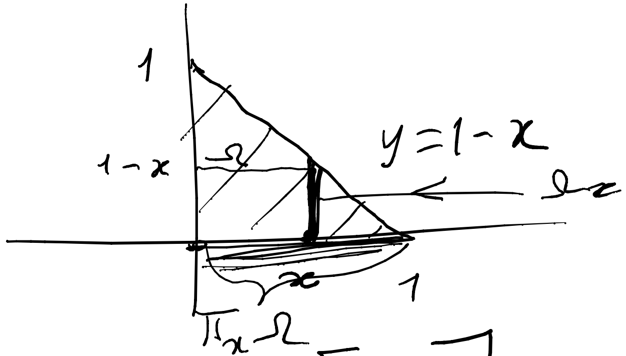
Th. integrali ITERATI   FUBINI f integrabile

TONELLI $f \geq 0$

ESEMPIO

$$\int_\Omega 1+x+y \, dx\,dy \qquad f(x,y) = 1+x+y$$

$\Omega =$ 



$$\Pi_x \Omega = [0,1]$$

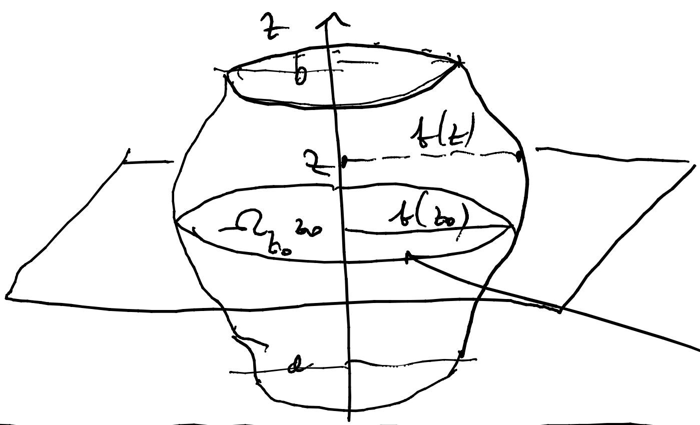
$$\Omega_x = [0, 1-x]$$

|| Fubini-Tonelli

$$\int_0^1 dx \underbrace{\int_0^{1-x} dy\,(1+x+y)}_{\text{funzione della sola } x} =$$

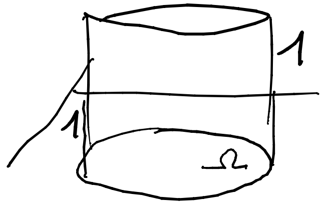
$$= \int_0^1 dx \left[(1+x)y + \frac{1}{2}y^2\right]_0^{1-x} = \int_0^1 dx \left(\underbrace{(1+x)\underbrace{(1-x)}_{y} + \frac{1}{2}(1-x)^2}_{\text{primitiva in } y=(1-x)} - \underbrace{0}_{\substack{\text{primitiva}\\ \text{in } y=0}}\right) =$$

$$= \int_0^1 dx \left[1-x^2 + \frac{1}{2}(1-x)^2\right] = \left[x - \frac{1}{3}x^3 + \frac{1}{2}\left(-\frac{1}{3}(1-x)^3\right)\right]_0^1 = 1 - \frac{1}{3} - \left(-\frac{1}{6}\right)$$



$f \geq 0$

SOLIDI DI ROTAZIONE



area Ω = volume del cilindro di base Ω e altezza 1

cerchio di centro $(0,0,z_0)$ e raggio $f(z_0)$

per generare il solido si fa ruotare il grafico di f attorno all'asse z

$\Omega_{z_0} =$ cerchio di centro $(0,0,z_0)$ e raggio $f(z_0)$ area $\Omega_{z_0} = \pi f^2(z_0)$

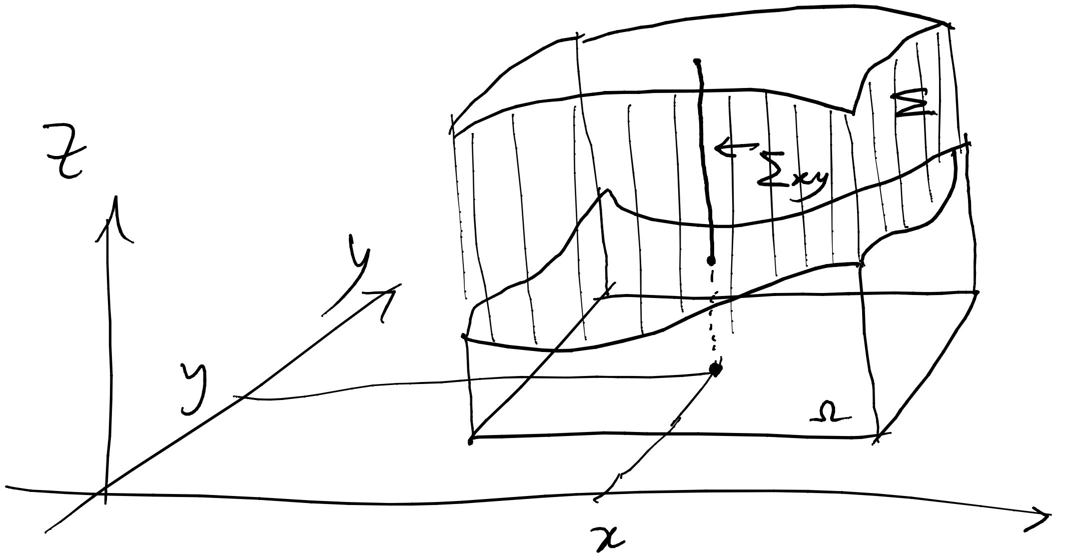
elemento di volume è $\pi f^2(z)\,dz \quad \forall z \in [a,b]$

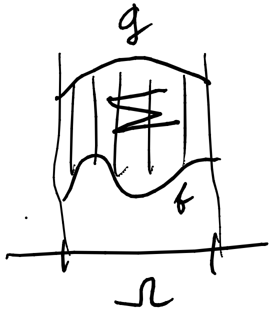
Il volume totale $= \boxed{\pi \int_a^b f^2(z)\,dz}$

$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 : z \in [a,b] \quad \underline{\sqrt{x^2+y^2} < f(z)} \right\}$$

$$\text{vol}\,\Omega = \int_\Omega 1 = \int_a^b dz \boxed{\int_{\Omega_z} 1 \, dx\,dy} = \int_a^b dz \, \pi \underbrace{f^2(z)}_{\text{quadrato del raggio}}$$

($\pi_z \Omega$)





$f(x,y) \quad f: \Omega$

$g(x,y) \quad g: \Omega$

$f \leq g$

$$\Sigma = \{ (x,y,z) \in \mathbb{R} : (x,y) \in \Omega \quad f(x,y) \leq z \leq g(x,y) \quad \forall x,y \in \Omega \}$$

$$\boxed{\mathrm{Vol}\,\Sigma}$$

$$\Pi_{x,y} \Sigma = \Omega$$

$$f(x,y) \leq z \leq g(x,y) \quad : \quad z \in [f(x,y), g(x,y)]$$

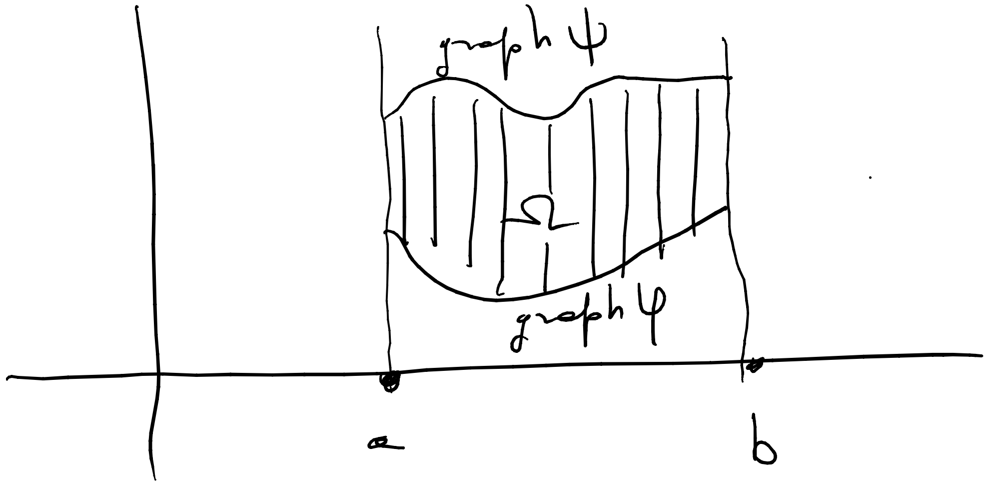
$$\mathrm{Vol}\,\Sigma = \int_{\Sigma} 1 \, dx\,dy\,dz = \int_{\Pi_{xy}\Sigma = \Omega} dx\,dy \underbrace{\int_{\Sigma_{xy}} \underbrace{f(x,y,z)}_{1} \, dz}_{g(x,y) - f(x,y)} = \boxed{\int_{\Omega} dx\,dy \, [g(x,y) - f(x,y)]}$$

ampiezza dell'intervallo sezione di Ω con le rette verticali in (x,y)

Ω è un dominio NORMALE rispetto all'asse x

se $\exists\ a < b$ e $\varphi < \psi : [a,b] \to \mathbb{R}$ tali che

$$\Omega = \{(x,y) : x \in [a,b] \quad y \in [\varphi(x), \psi(x)]\}$$



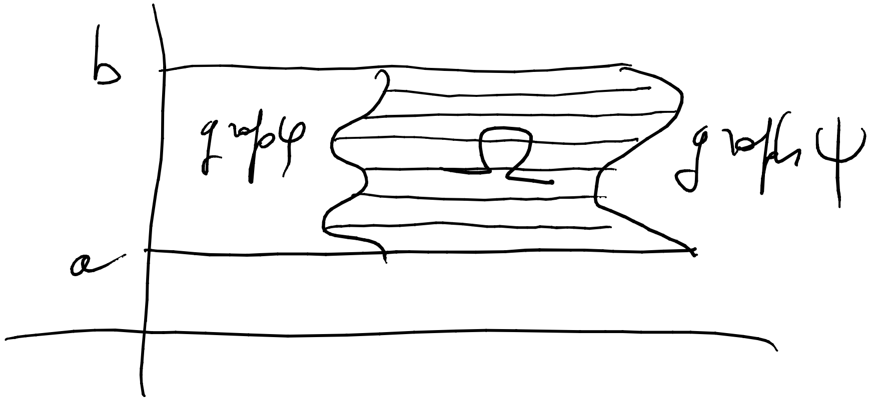
$$\int_\Omega f = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\, dy$$

$$[a,b] = \Pi_x \Omega$$

Ω è normale rispetto a y se

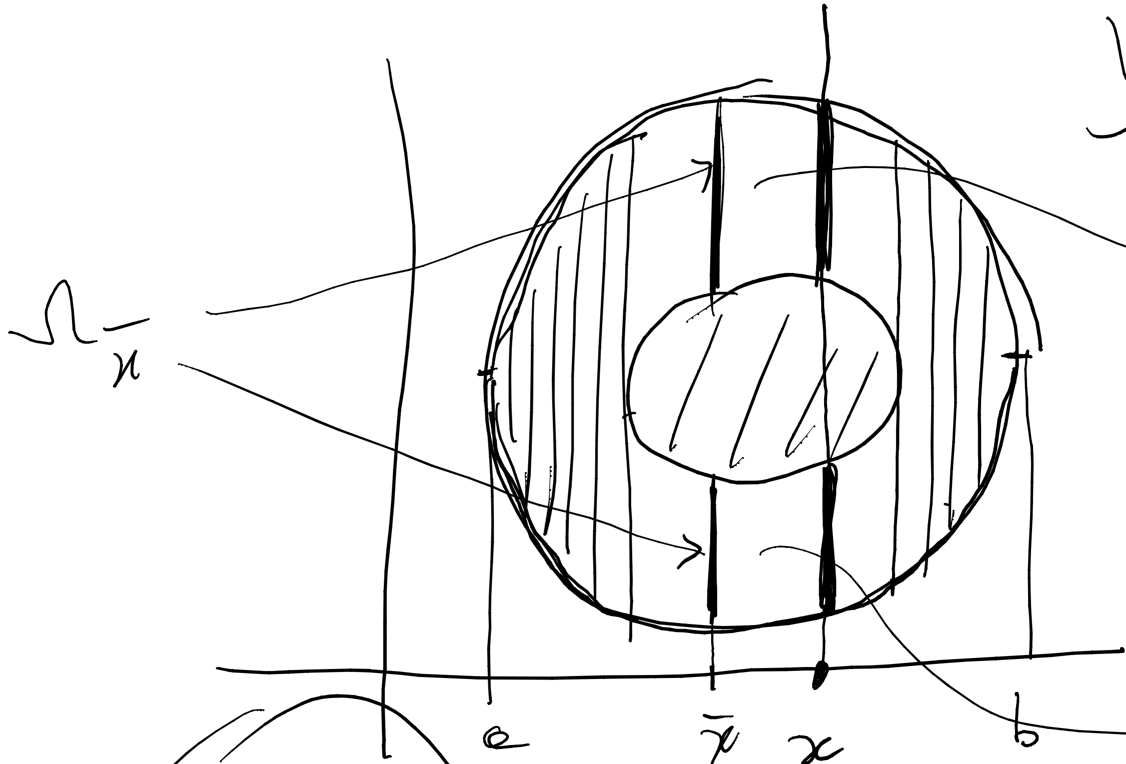
$$\exists\, a<b \quad \exists\, \varphi,\psi : [a,b]\to\mathbb{R},\ \varphi<\psi :$$

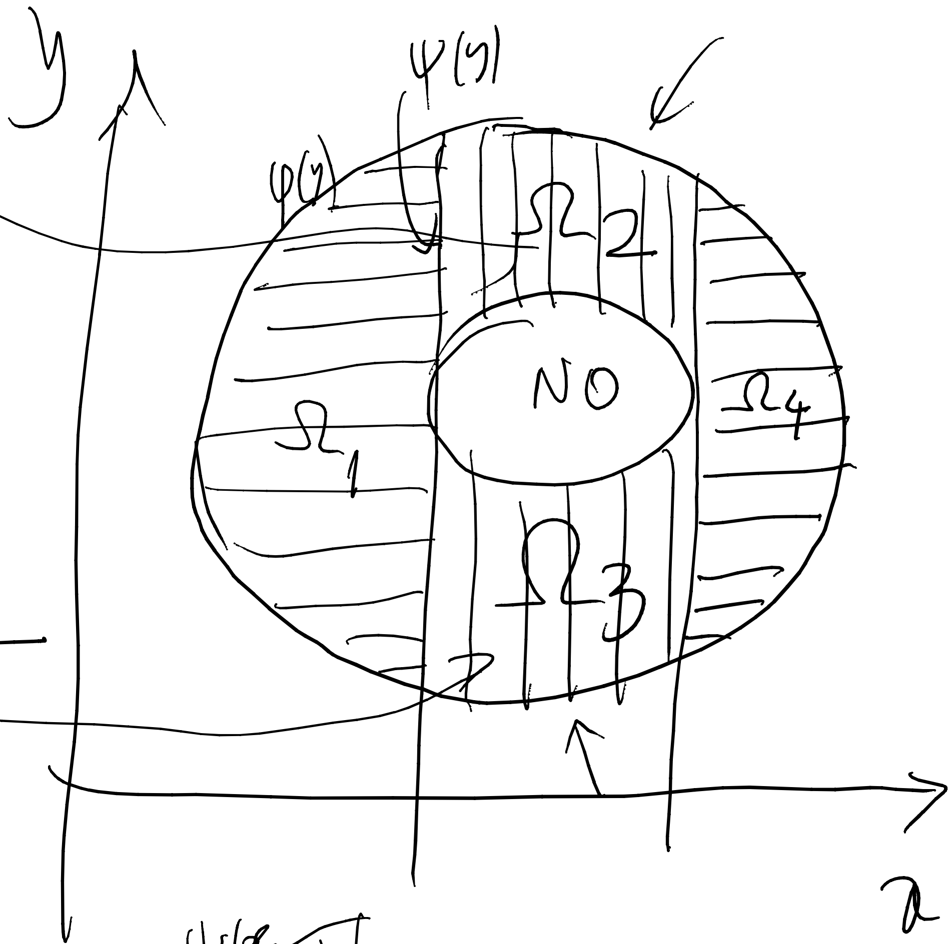
$$\Omega = \{(x,y) : y\in[a,b] \quad x\in[\varphi(y),\psi(y)]\}$$

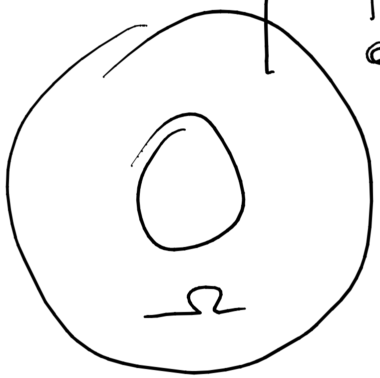


$$\int_\Omega f\,dx\,dy = \int_a^b dy \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx$$

$$\int_{\Omega_1\cup\Omega_2\cdots\cup\Omega_n} f = \sum_1^n \int_{\Omega_i}$$ se Ω_i non hanno punti interni a comune.





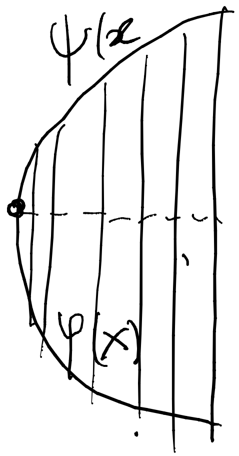


è unione

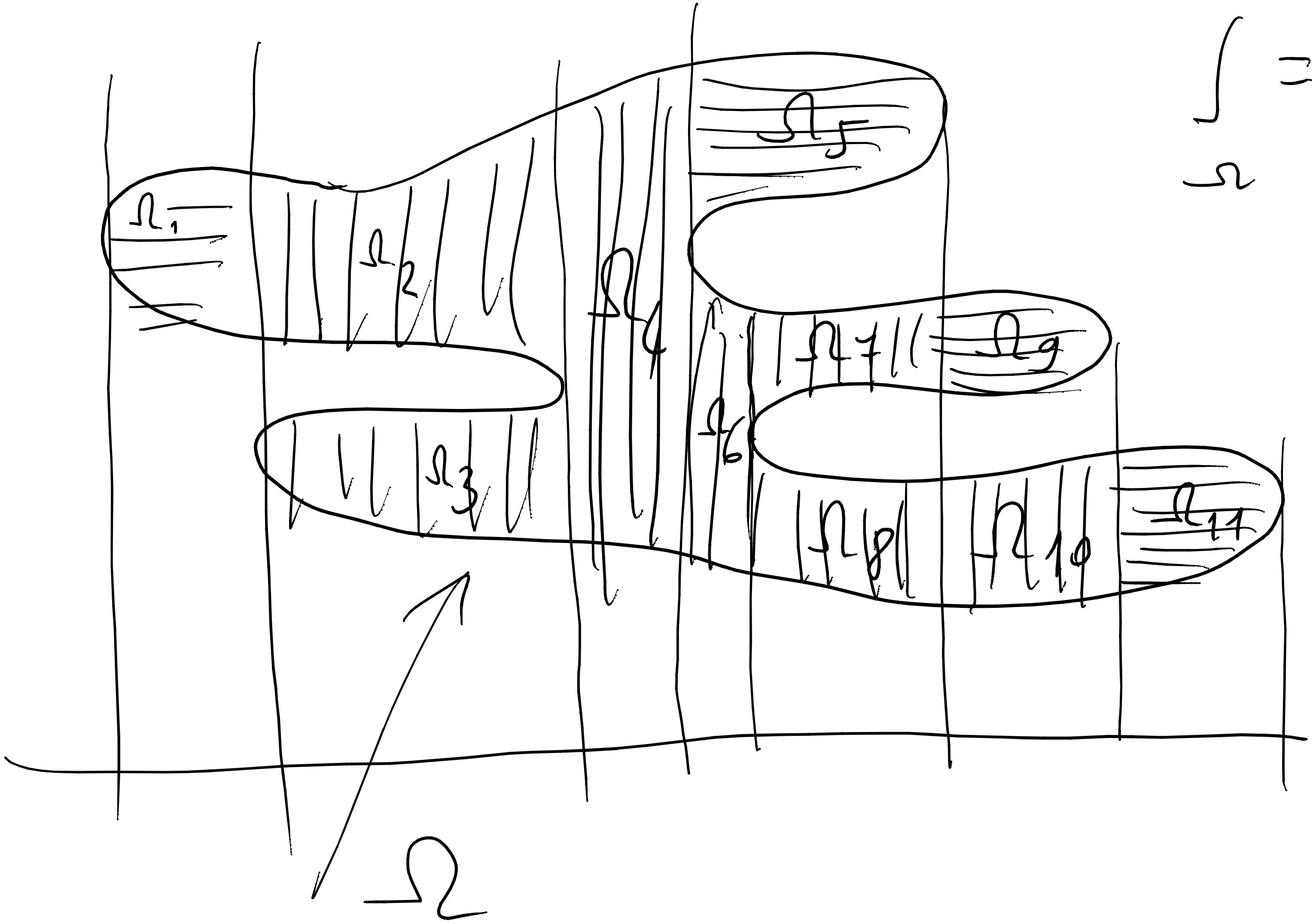
di $\Omega_1, \Omega_2, \Omega_3, \Omega_4$

privi di punti interni

comuni



$$\int_{\Omega} = \sum_{1}^{11} \int_{\Omega_i}$$



CAMBIO DI VARIABILI

$\Omega \subseteq \mathbb{R}^n$ Ω aperto

$x \in \mathbb{R}^n$

$dx \equiv dx_1 dx_2 \cdots dx_n$

$x = g(y)$ $y \in \mathbb{R}^n$

$g \in C^1(\Sigma)$

$g : \Sigma \to \mathbb{R}^n$

g sia invertibile con inversa C^1

$\Sigma = g^{-1}(\Omega)$

$$\int_\Omega f(x)\,dx = \qquad x = g(y) \quad dx = |\det g'(y)|\,dy$$

$$= \int_{g^{-1}(\Omega)} f(g(y))\,|\det g'(y)|\,dy$$

$g^{-1}(\Omega) = \Sigma$

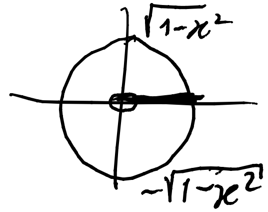
$\det g'(y) = \det \frac{\partial(x_1 \ldots x_n)}{\partial(y_1 \ldots y_n)}$

$$g(g^{-1}(\Omega)) = \Omega$$

$$g'(y) = \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1 \ldots y_n)}$$

$$\int x^2 + y^2 \, dx \, dy$$

$$\Omega = \{x^2 + y^2 < 1\}$$



$$\begin{cases} 0 < \rho < 1 \\ 0 < \theta < 2\pi \end{cases} \quad g^{-1}(\Omega)$$

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad \rho = \sqrt{x^2 + y^2}$$

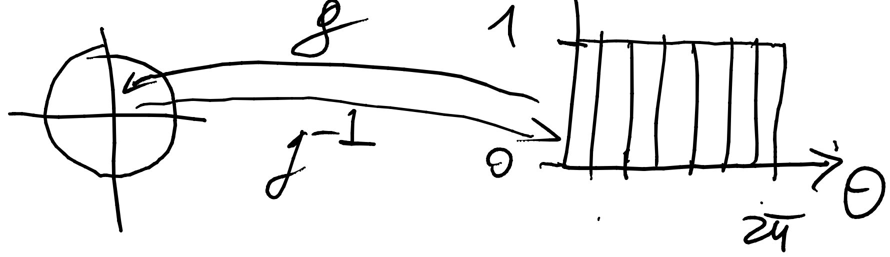
$$\det \frac{\partial(x,y)}{\partial(\rho,\theta)} = \rho$$

$$\int \underbrace{\rho^2}_{f(x,y)} \cdot \underbrace{\rho}_{|\det g'|} \, d\rho =$$

$$= \int_0^{2\pi} d\theta \boxed{\int_0^1 \rho^3 \, d\rho} =$$

$$= \int_0^{2\pi} d\theta \; \frac{1}{4}\rho^4 \Big|_0^1 = \int_0^{2\pi} d\theta \; \frac{1}{4} = \frac{1}{4} 2\pi = \frac{\pi}{2}$$

$$g^{-1}(\{x^2 + y^2 \leq 1\}) \quad \rho$$



$$\int_{\{y>0;\ x^2+y^2\leq 1\}} \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = ?$$