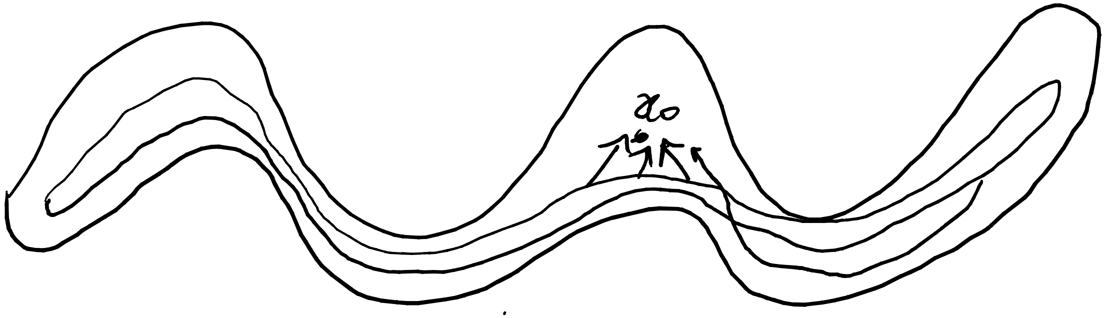
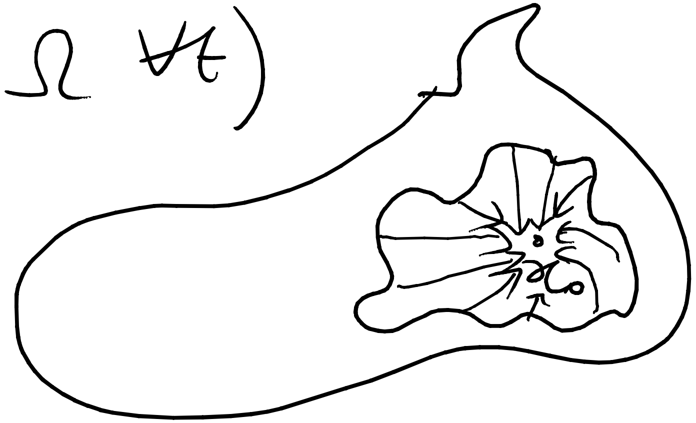
INSIEMI SEMPLICEMENTE CONNESSI

Ω si dirà SEMPLICEMENTE CONNESSO se ogni curva chiusa $\gamma : [0,1] \to \Omega$ è omotopa, in Ω ad una curva costante ($\sigma(t) = x_0 \in \Omega \ \forall t$)

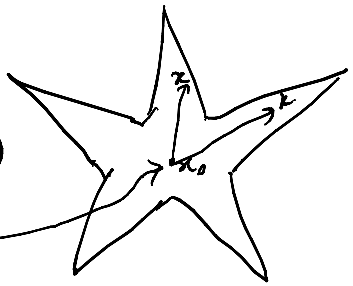


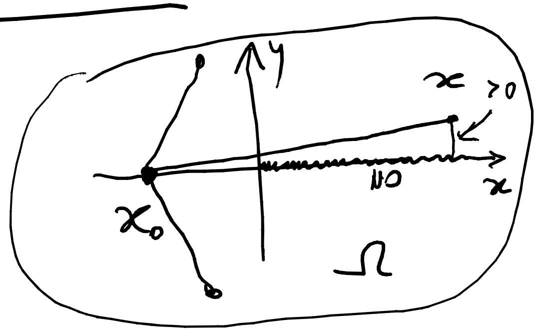


Ω si dirà STELLA se $\exists\, x_0 \in \Omega : \overline{x_0 x} \subseteq \Omega \ \forall x \in \Omega$

segmento

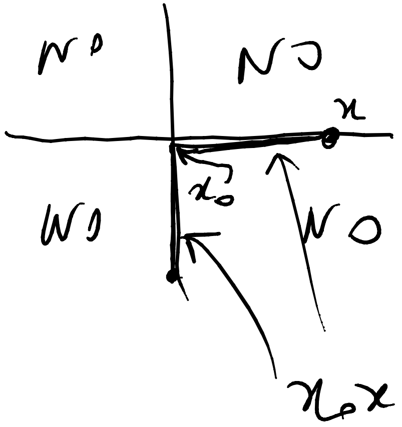
x_0 POLO



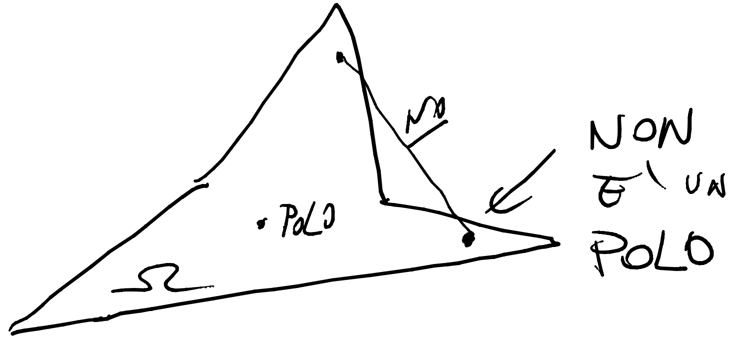


$\Omega = \mathbb{R}^2 \setminus \{(x,0) : x > 0\}$

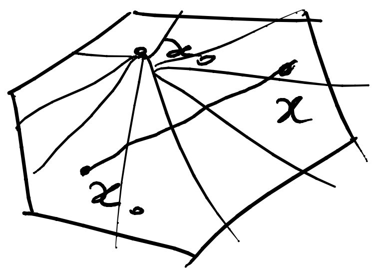
STELLA



ASSE X ∪ ASSE Y

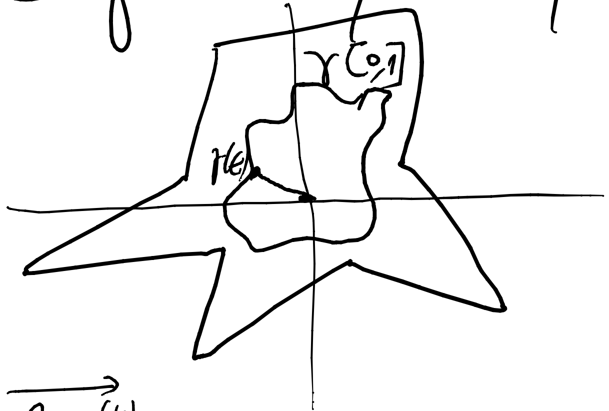


Ω è CONVESSO $\Rightarrow$ Ω è stellato



Th. Ω STELLA $\Rightarrow$ Ω sempl. connesso.

DIM. supponiamo $\boxed{x_0 = 0}$ e sia γ una qualunque curva chiusa a valori in Ω



$$h(\lambda, t) = \underbrace{(1-\lambda)\gamma(t)}_{\text{punto del segmento } \overrightarrow{O\gamma(t)}} \quad \forall \lambda \in [0,1] \quad \forall t \in [0,1]$$

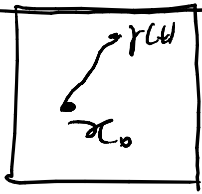
$$h : [0,1] \times [0,1] \to \Omega$$

h è continua (prodotto di funzioni continue $(1-\lambda)$ e $\gamma(t)$)

$h(0,t) = \gamma(t)$ $(\lambda = 0)$ $\quad h(1,t) \equiv 0$

$\boxed{\text{h è a valori in } \Omega\text{?}}$ $h(\lambda,t)$ è un punto del segmento di estremi 0 e $\gamma(t)$ perché $(1-\lambda) \in [0,1]$. Poiché Ω è stella $(1-\lambda)\gamma(t) \in \Omega$ se $\gamma(t) \in \Omega$

Se x_0 non fosse 0?
Si usano i segmenti fra x_0 e $\gamma(t)$



$$h(\lambda,t) = x_0 + (1-\lambda)\left[\gamma(t) - x_0\right]$$

OMOTOPIA DI $\gamma(t)$ sulla curva costante $\sigma(t) = x_0$

OSSERV. Se $\sigma(t) \equiv x_0 \ \forall t$ allora $\int_\sigma A = 0$ $\forall$ campo $A \in C^0$ $\quad \sigma : [0,1] \to \text{dom} A$

$$\boxed{\int_0^1 A(\sigma(t))\,\dot\sigma(t)\,dt} = 0$$ perché $\dot\sigma \equiv 0$

Sia $A \in C^1(\Omega)$, Ω semplicemente connesso. Allora A è integrabile se e solo se A è irrotazionale

$\Updownarrow$

CNS $A \in C^1(\Omega)$, Ω sempl. connesso sia integrabile è che

$$(A_i)_{x_j} \equiv (A_j)_{x_i} \quad \forall i, j = 1 .. N$$

ESEMPIO

$\alpha = x\,dx + y\,dy$ $\quad \boxed{A = \begin{pmatrix} x \\ y \end{pmatrix}}$ è integrabile sul suo dominio.

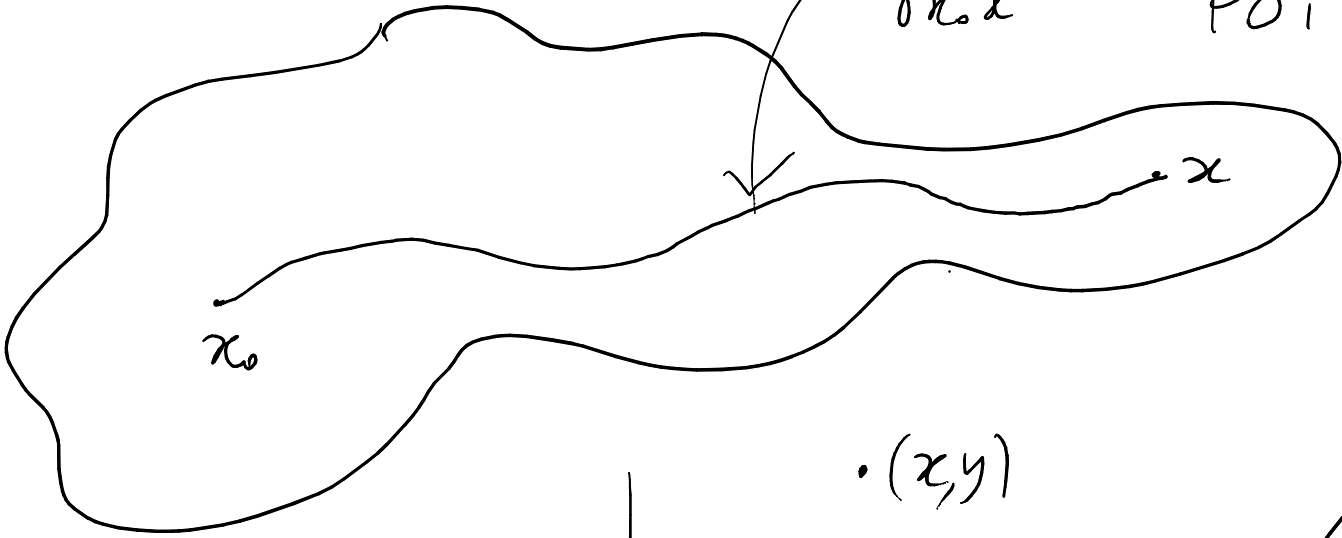
$\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\quad A : \mathbb{R}^2 \to \mathbb{R}^2$

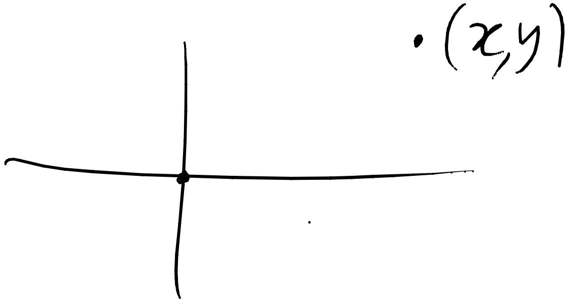
Il dominio del campo è CONVESSO $\Longrightarrow$ è semplicemente connesso.

$$\boxed{\frac{\partial}{\partial y} x = 0 \qquad \frac{\partial}{\partial x} y = 0 \qquad \underline{\underline{\text{chiuse}}}}$$

A integrabile $\qquad f(x) = \int_{\gamma_{x_0 x}} A$

COSTRUZIONE DEL POTENZIALE





$x_0 = (0,0)$

$$A = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A : \mathbb{R}^2 \to \mathbb{R}^2$$

$\mathbb{R}^2$ convesso

$$f(x,y) = \int_{\overrightarrow{(0,0)\,(x,y)}} A$$

$$\gamma(t) = t\begin{pmatrix} x \\ y \end{pmatrix} \quad t \in [0,1]$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \gamma(t) = \begin{pmatrix} tx \\ ty \end{pmatrix} \quad t \in [0,1]$$

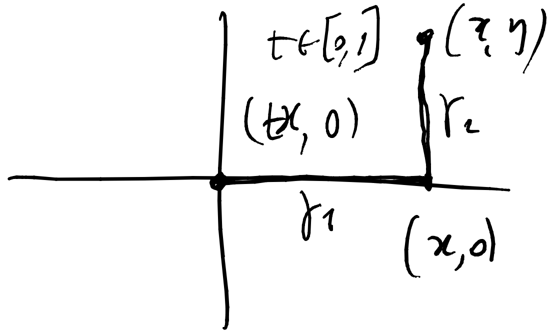
$$\dot\gamma(t) = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A(\gamma(t)) = \begin{pmatrix} tx \\ ty \end{pmatrix}$$

$$f(x,y) = \int_0^1 (tx)x + (ty)y \, dt =$$

$$= \int_0^1 (x^2+y^2)\, t \, dt = (x^2+y^2)\,\frac{1}{2}t^2 \Big|_0^1 =$$

$$= \frac{1}{2}(x^2+y^2)$$



$$\int_{\gamma_1} A + \int_{\gamma_2} A = \int_0^1 \overset{A(\gamma_1(t))}{\begin{pmatrix} tx \\ 0 \end{pmatrix}} \cdot \overset{\dot\gamma_1}{\begin{pmatrix} x \\ 0 \end{pmatrix}} +$$

$$+ \int_0^1 \overset{A(\gamma_2(t))}{\begin{pmatrix} 0 \\ ty \end{pmatrix}} \overset{\dot\gamma_2}{\begin{pmatrix} 0 \\ y \end{pmatrix}}$$

DETERMINARE TUTTE LE PRIMITIVE

$f, g : \Omega \to \mathbb{R}$ $\qquad \boxed{\nabla f \equiv \nabla g \equiv A}$

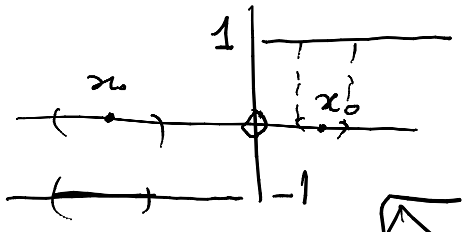
primitive d'$A : \Omega \to \mathbb{R}^N$

$$\boxed{\nabla(f-g) \equiv 0}$$

$h = f-g$ è una funzione con gradiente identicamente nullo

$\varphi'(x) \equiv 0 \Rightarrow \varphi$ costante

$\varphi(x) = \dfrac{x}{|x|}$ def. in $\mathbb{R} \setminus \{0\}$



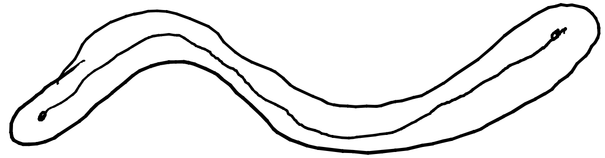
localmente costante MA NON COSTANTE

$$\varphi' \equiv 0$$

$$\varphi(x) - \varphi(y) = (x-y)\underbrace{\varphi'(\xi)}_{=0} \Rightarrow \varphi(x) = \varphi(y)$$

Ω aperto connesso $\quad f : \Omega \to \mathbb{R} \quad \nabla f \equiv 0$ su Ω

Allora f è costante.



DIM. $\quad \forall x, y \ \exists \gamma : [0,1] \to \Omega$: γ continua $\gamma(0) = x \quad \gamma(1) = y$

Se Ω è aperto si può scegliere γ regolare a tratti

$\to h(t) = f(\gamma(t))$

$\nabla f \equiv 0 \Rightarrow \nabla f$ è continua

$f \in C^1 \Rightarrow$ diff.le

$\to h'(t) = \underbrace{\nabla f(\gamma(t))}_{\equiv 0} \dot{\gamma}(t) \equiv 0$

$h' \equiv 0$ su $[0,1]$ intervallo

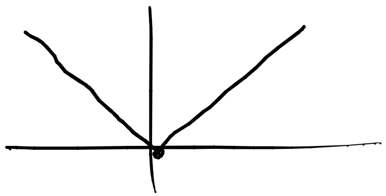
$\Rightarrow h$ è costante in $[0,1]$

$h : [0,1] \to \mathbb{R} \qquad \gamma : [0,1] \to \Omega \quad f : \Omega \to \mathbb{R}$

$h(0) = h(1)$

$f(x) = f(\gamma(0)) \quad f(\gamma(1)) = f(y)$

$$\underbrace{\gamma(t)}_{C^0 \text{ non } C^1} = \begin{pmatrix} t \\ |t| \end{pmatrix}$$



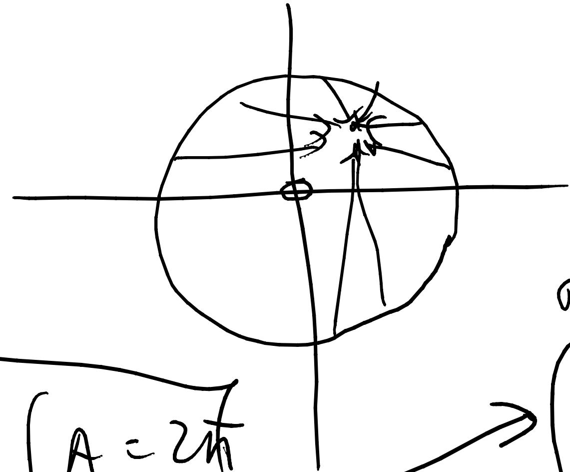
$$\underbrace{\sigma(t)}_{C^1} = \begin{pmatrix} t^3 \\ |t|^3 \end{pmatrix}$$

$$\dot\sigma(0) = \lim_{t\to 0} \frac{\sigma(t)-\sigma(0)}{t} = \lim_{t\to 0}$$

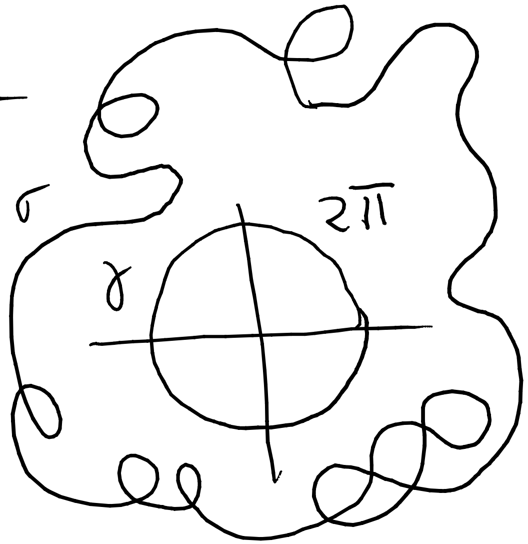
$$\frac{|t|^3 - 0}{t} \longrightarrow 0 \qquad \boxed{\dot\sigma(0) = 0}$$

$$A = \left(\frac{y}{x^2+y^2} , \frac{-x}{x^2+y^2} \right)$$

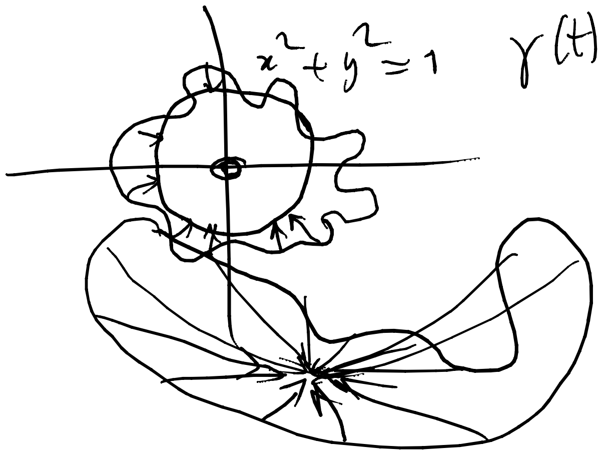
$$\operatorname{dom} A = \mathbb{R}^2 \setminus \{(0,0)\}$$



$$\boxed{\int_\gamma A = \int_\sigma A = 2\pi}$$



$$\nabla \frac{1}{x^2+y^2} = \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} = A$$



$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad t \in (0, 2\pi)$$

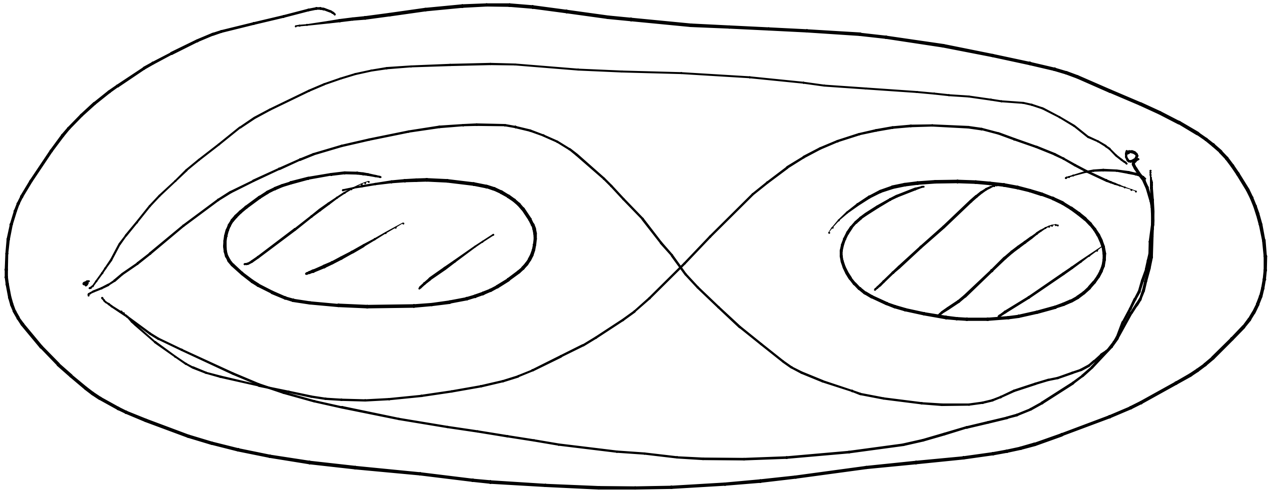
$$f(x,y) = \frac{1}{x^2+y^2}$$

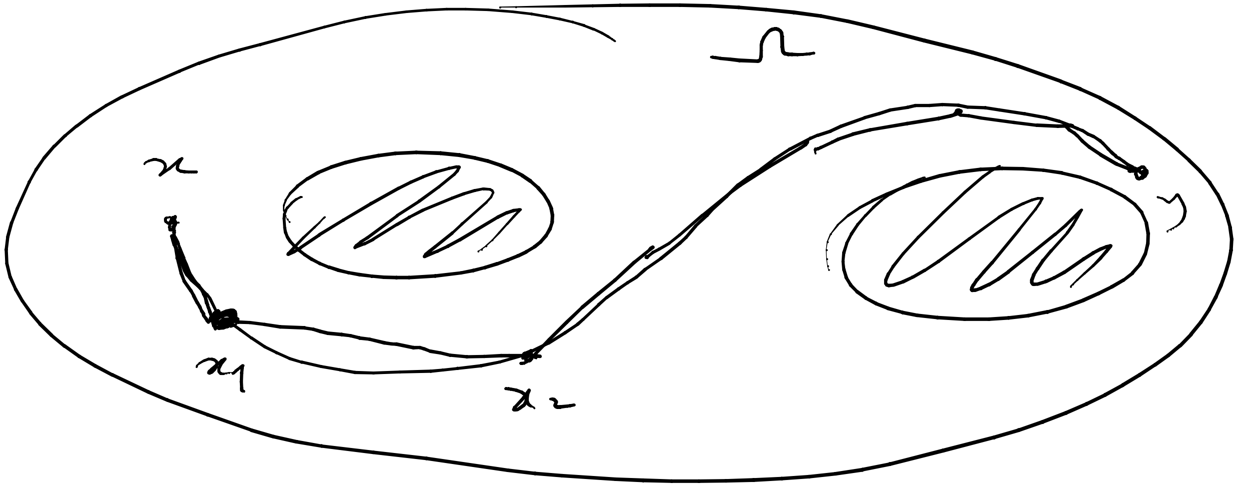
$\nabla f(x,y)$ è integrabile e f è una primitiva

$$\boxed{a(x,y) = \frac{-2x}{(x^2+y^2)^2} \qquad b(x,y) = \frac{-2y}{(x^2+y^2)^2}}$$

$$\int_{\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}} A = \int_0^{2\pi} \frac{-2\cos t}{1}(-\sin t) + \frac{-2\sin t}{1}(\cos t)\, dt =$$

$$= \int_0^{2\pi} 2\sin t\cos t - 2\sin t\cos t\, dt = 0$$

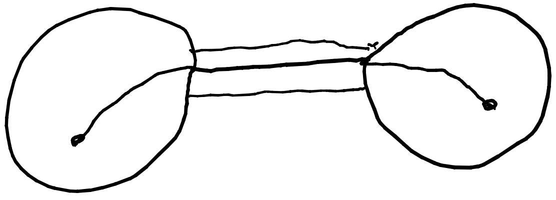




$\nabla f = 0$ in Ω

$f(x_1) = f(x)$

$f(x_2) = f(x_1) = f(x)$



Se f è una primitiva di A

Tutte le altre sono del tipo $f + \varphi$

dove $\boxed{\nabla \varphi \equiv 0}$

φ è costante su OGNI PARTE CONNESSA di dom A