FORMULA DI TAYLOR

Note Title 5/13/2020

DISPENSE AN 2.8 e AN. 2.9

DERIVATE di ordine superiore $\quad \underline{f : \Omega \to \mathbb{R} \quad \Omega \subseteq \mathbb{R}^n}$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} \qquad i,j = 1..n$$

$$\left[\begin{array}{l} \partial_{x_i}(\partial_{x_j} f) = \partial_{x_i x_j} f \\ \qquad i,j,k = 1..n \\ \partial_{x_i x_j x_k} f = \partial_{x_i}(\partial_{x_j}(\partial_{x_k} f)) \end{array}\right.$$

$$f_{x_1} \quad f_{x_2} \quad \cdots \quad f_{x_n}$$

$$f_{x_1} \xrightarrow{\partial_{x_1}} \underbrace{(f_{x_1})_{x_1}}_{= f_{x_1 x_1}} \qquad f_{x_1} \xrightarrow{\partial_{x_2}} (f_{x_1})_{x_2} = f_{x_1 x_2} \quad \cdots \quad f_{x_1} \xrightarrow{\partial_{x_n}} (f_{x_1})_{x_n} = f_{x_1 x_n}$$

Una funzione genera n derivate parziali, n^2 derivate seconde, n^3 derivate terze ...

SOLO ENUNCIATO Th. (CLAIRAUT-SCHWARZ) $f \in C^2$ allora

pronuncia: Cleró scvarz

$$\boxed{f_{x_i x_j} = f_{x_j x_i}} \quad i,j = 1..n$$

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

uguali

solo 3 derivate da calcolare: f_{xx}, f_{yy}, ed una sola fra f_{xy} e f_{yx} (che sono uguali)

$f(x,y) = x \cos y \quad f_x = \cos y \quad f_y = -x \sin y$

$f_{xx} = 0 \quad f_{yy} = -x \cos y \quad f_{xy} = -\sin y$

$f_{yx} = (-x \sin y)_x = -\sin y$

$$\begin{matrix} f_{xx} & f_{xy} & \cancel{f_{xz}} \\ f_{yx} & f_{yy} & \cancel{f_{yz}} \\ \cancel{f_{zx}} & \cancel{f_{zy}} & f_{zz} \end{matrix}$$

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$f_{xzyx} = (((f_x)_z)_y)_x$

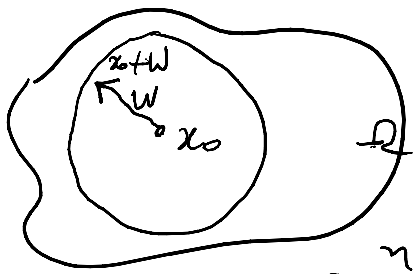
da cui $\longrightarrow f_{xzyx} =$

$= f_{xzxy} = f_{xxzy} =$

$= f_{xxyz}$

Clairaut appl. f_{xz}

formula di Taylor



$$w=(w_1,\dots,w_n)$$

$x_0 \in \mathring{\Omega}$ (interno ad Ω)

$\Omega \subseteq \mathbb{R}^n \qquad x_0, w \in \mathbb{R}^n$

dim. spazio

$$f(x_0+w)=\sum_{k=0}^{N}\frac{1}{k!}\sum_{i_1\dots i_k=1}^{n}\left[f_{x_{i_1}x_{i_2}\dots x_{i_k}}(x_0)\,w_{i_1}w_{i_2}\dots w_{i_k}\right]+R^N(w)$$

grado del polinomio di Taylor

PEANO

$$\lim_{w\to 0}\frac{R^N(w)}{|w|^N}=0$$

LAGRANGE

$$x_0 \to x_0+\xi w \qquad \xi\in[0,1]$$

$$f:\Omega\to\mathbb{R} \qquad \Omega\subseteq\mathbb{R}^3$$

x_1, x_2, x_3

$i_1=2 \quad i_2=2 \quad i_3=1$

$$f_{x_{i_1}x_{i_2}x_{i_3}}=f_{x_2x_2x_1} \qquad w_{i_1}w_{i_2}w_{i_3}=w_2w_2w_1=w_1w_2^2$$

$$f(x_0+w) = \underset{\text{term. grad. } 0}{f(x_0)} + \underbrace{\frac{1}{1!}\sum_{i=1}^{n} f_{x_i}(x_0)w_i}_{df(x_0,w)} + \underbrace{\frac{1}{2!}\sum_{i,j=1}^{n} f_{x_i x_j}(x_0)\, w_i w_j + \dots}$$

$$\dots \boxed{f_{x_i x_j}(x_0)\, w_i w_j + f_{x_j x_i}(x_0)\, w_j w_i} + \dots$$

$$\shortparallel$$

$$2 f_{x_i x_j}(x_0)\, w_i w_j$$

Nel caso $n=2$, ponendo

$$x_1 = x \qquad x_2 = y \qquad w=(h,k)$$

si ottiene

$$f(x_0+h, y_0+k) = f(x_0,y_0) + f_x(x_0,y_0)h + f_y(x_0,y_0)k +$$

$$+ \frac{1}{2}\left[f_{xx}(x_0,y_0)h^2 + f_{yy}(x_0,y_0)k^2 + 2 f_{xy}(x_0,y_0)hk\right] +$$

$$+ R^2(h,k)$$

$$\boxed{\left(\sum_1^n w_i \partial_{x_i}\right)^2 \quad \text{ove } \partial_{x_i}\partial_{x_j} = \partial_{x_i x_j}}$$

$f: \mathbb{R} \to \mathbb{R}$

$$f(x_0+w) = \sum_0^N \frac{1}{k!} f^{(k)}(x_0)\, w^k + R^N(w)$$

VERSIONE CHE RICHIEDE IL NUMERO DI DERIVATE STRETTAMENTE INDISPENSABILE

$f: \Omega \to \mathbb{R}$ $\Omega \subseteq \mathbb{R}^n$ - n variabili $x_1 \dots x_n$.

$w \in \mathbb{R}^n$ $x_0 \in \Omega$

uguali

① $k \equiv (k_1, k_2, \dots, k_n)$ $k_i \geq 0$ intero

multiindice

k_i numero di derivazioni da fare rispetto a x_i

② $k! \equiv k_1!\, k_2! \dots k_n!$ $(0! = 1)$

Es. $k_1 = 7$ $k_2 = 0$ $k_3 = 0 \dots k_6 = 0$ 6 variabili indip.

$$\frac{\partial^{(7)} f}{\partial x_1^7} = f_{x_1 x_1 x_1 x_1 x_1 x_1 x_1} \qquad k = (7,0,0,0,0,0) \qquad k! = 7!\underbrace{0!0!0!0!0!}_{1} = 7!$$

③ $|k| \equiv k_1 + k_2 + \dots + k_n$

(4) $\partial^k f = \dfrac{\partial^{|k|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}$

(5) $w^k = w_1^{k_1} w_2^{k_2} \dots w_n^{k_n}$

FORMULA DI TAYLOR in più VARIABILI

$$f(x_0+w) = \sum_{|k|\le N} \frac{1}{k!} \partial^k f(x_0)\, w^k + R^N(w) =$$

$$= \sum_{\substack{k_1+k_2+\dots+k_n \le N \\ k_i \text{ interi } \ge 0}} \frac{1}{k_1! \dots k_n!} \frac{\partial^{|k|} f(x_0)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} w_1^{k_1} \dots w_n^{k_n} + R^N(w_1 \dots w_n)$$

? Come elencarli?

Esempio

$N = 4 \quad n = 3$

k_1	k_2	k_3	
4	0	0	$\leftrightarrow f_{x_1x_1x_1x_1}$
3	1	0	
3	0	1	

k_1	k_2	k_3
2	2	0
2	1	1
2	0	2
1	3	0
1	2	1
1	1	2
1	0	3
0	4	0
0	3	1
0	2	2
0	1	3
0	0	4

tutti i multiindici di modulo 4

Ordine "decrescente" somma delle "cifre" costante

PER OGNI MULTIINDICE DELL'ELENCO VA CALCOLATO IL TERMINE CORRISPONDENTE della formula

AD ESEMPIO $k_1 = 3$ $k_2 = 1$ $k_3 = 0$ DA' $k = (3,1,0)$ e

$|k| = 3+1+0 = 4$ $k! = 3!\,1!\,0! = 6$, da cui il termine relativo

a tale scelta è $\frac{1}{6} \underset{\partial^{(k)} f(x_0)}{\underbrace{f_{x_1 x_1 x_1 x_2}(x_0)}} \underbrace{w_1^3 w_2}_{w^k}$