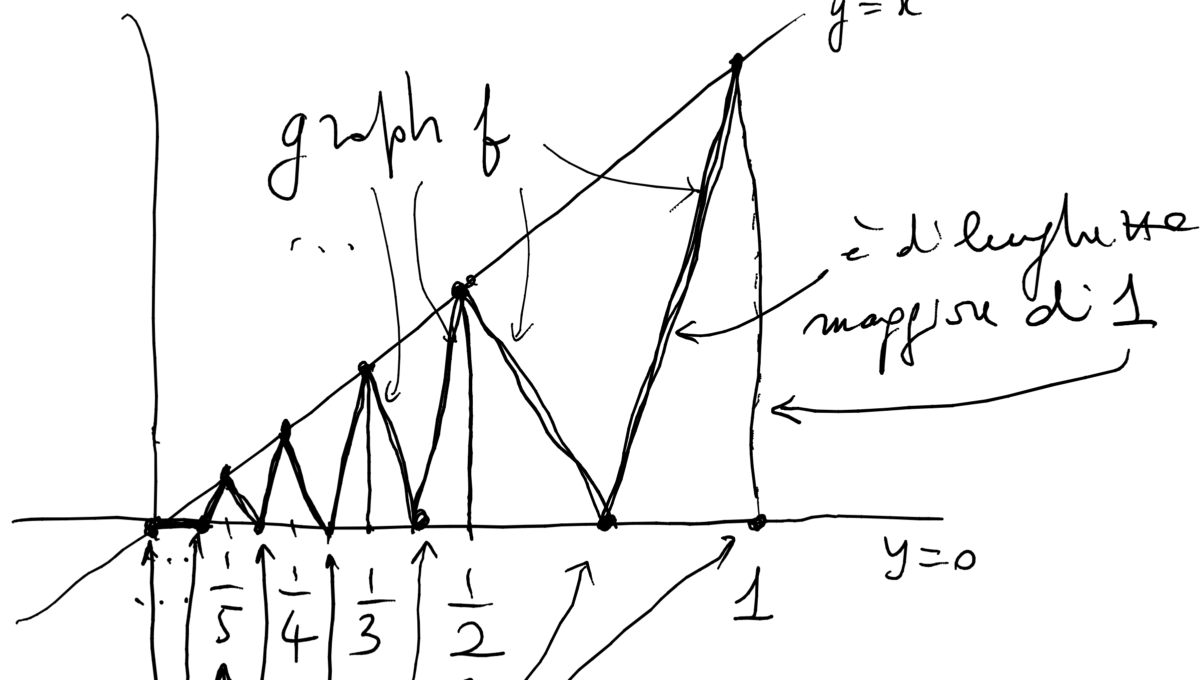
Esempio di curva di classe C^0 non rettificabile



$$\gamma:[0,1]\to\mathbb{R}^2$$

$$\gamma(t)=\begin{pmatrix}t\\ f(t)\end{pmatrix}$$

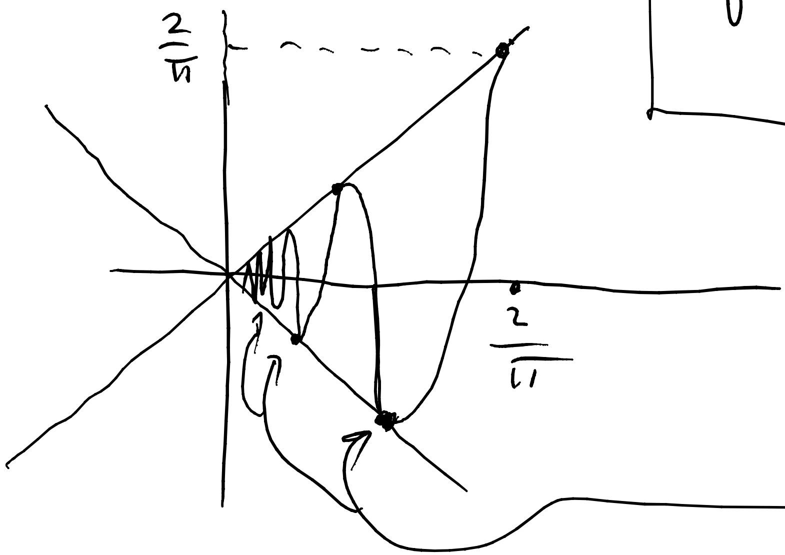
$$\gamma(0)=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\Lambda(\Pi_n)\ge \underbrace{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n}}_{n\text{-esima somma parziale della serie armonica}}$$

$$\boxed{\sup_n \Lambda(\Pi_n)=+\infty}$$

[Π_n si ottiene da $[0,\frac{1}{n},\frac{1}{n-1},\dots,\frac{1}{2},1]$ intercalando un punto arbitrario in ogni intervallo]

Alternative



$$f(t) = \begin{cases} 0 & t=0 \\ t\sin\frac{1}{t} & t\in\,]0,\frac{2}{\pi}] \end{cases}$$

$$|f(t)| = |t|\,|\sin\tfrac{1}{t}| \le |t|$$

$$\sin\frac{1}{t} = 1 \qquad \frac{1}{t} = \frac{\pi}{2} + 2k\pi$$

$$\sin\frac{1}{t} = -1 \qquad \frac{1}{-\frac{\pi}{2}+2k\pi} \qquad t = \frac{1}{\frac{\pi}{2}+2k\pi} \sim O\left(\frac{1}{k}\right)$$

stesso conto di prima

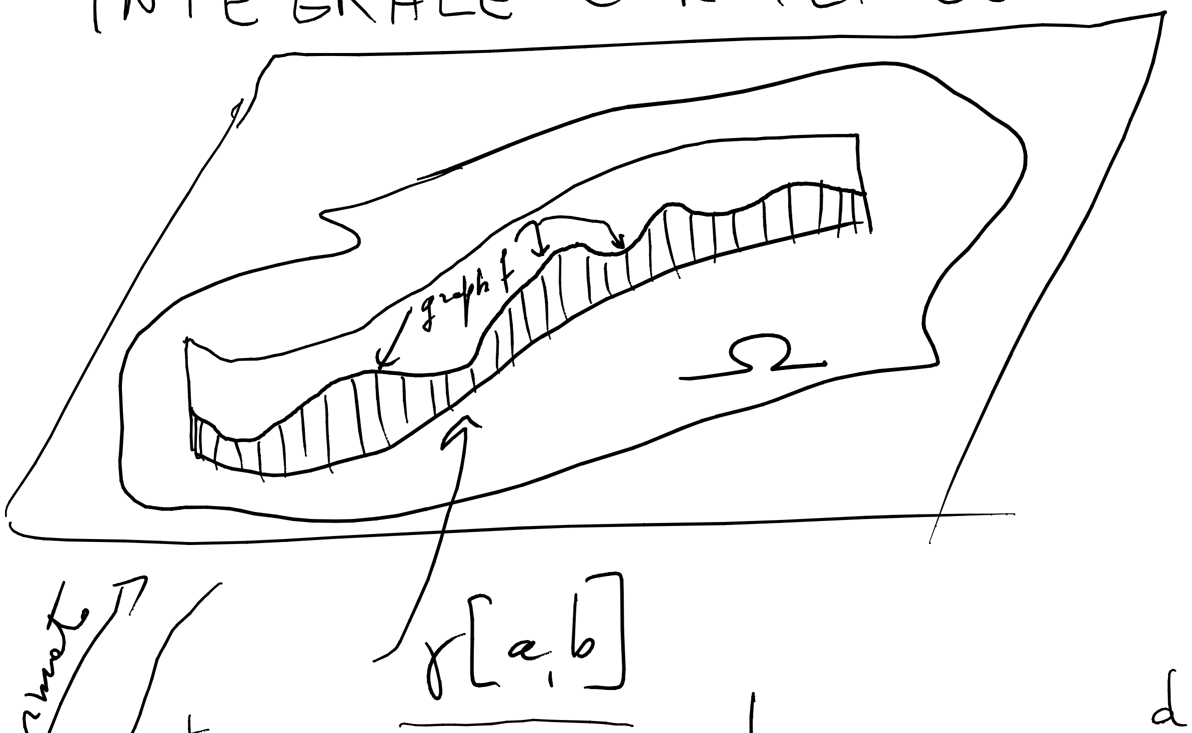
$$\gamma \in C^1 \Rightarrow \Lambda(\gamma) < +\infty$$

$$\gamma \in C^0 \not\Rightarrow \Lambda(\gamma) < +\infty$$

BV

γ è rettificabile se e solo se le sue componenti sono BV (Bounded Variation)

INTEGRALE CURVILINEO



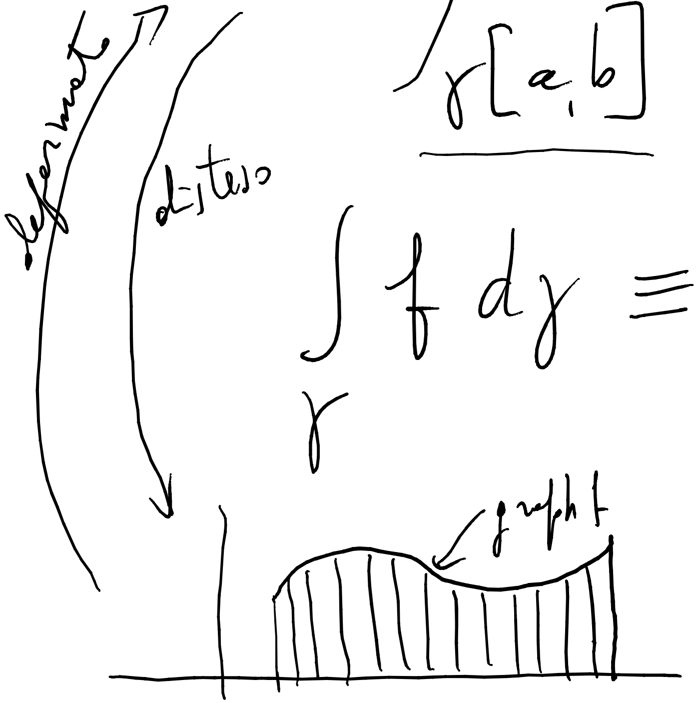
$$\boxed{f:\Omega\to\mathbb{R}} \qquad \Omega\subseteq\mathbb{R}^n$$

$$\gamma:[a,b]\longrightarrow\Omega$$ gener. regolare

$$\int_\gamma f\,d\gamma \qquad \int_\gamma f\,dl$$

$$\int_\gamma f\,d\gamma \equiv \int_a^b f(\gamma(t))\,\overbrace{\boxed{|\dot\gamma(t)|\,dt}}^{d\gamma}$$

distanza curvilinea percorsa dal punto nell'intervallino dt



$$\boxed{\begin{array}{l}\text{se } f\equiv 1\\ \int_a^b 1\,|\dot\gamma(t)|\,dt = \\ \quad = \Lambda(\gamma)\end{array}}$$

$\underbrace{f(x,y) = x}_{\text{dato}}$ $\quad \underbrace{\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}}_{\text{dato}} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \gamma : \mathbb{R} \to \mathbb{R}^2$

$f : \mathbb{R}^2 \to \mathbb{R}$

$\underbrace{t \in [0,1]}_{\text{dato}}$

$f(\gamma(t)) = t$

$$\int_\gamma f \, d\gamma = \int_0^1 \underset{\substack{\uparrow \\ f(\gamma(t))}}{t} \sqrt{1+4t^2}\, dt =$$

$$|\dot\gamma(t)| = \sqrt{1^2 + (2t)^2}$$

$$= \frac{1}{8} \int_0^1 \sqrt{1+4t^2}\, \underbrace{8t\,dt} = \ldots\ldots$$

$1+4t^2 = u$

$du = 8t\,dt$

$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

$$f(\gamma(t)) \equiv f(\underbrace{t, t^2}_{\gamma(t)})$$

ESEMPIO

$$f(x,y,z) = x^2 e^y \sin z \qquad \gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \qquad t \in [0, 2\pi]$$

f composto γ vuol dire mettere cos t al posto di x, sin t al posto di y e t al posto di z.

$$\boxed{\int_0^{2\pi} \underbrace{\cos^2 t}_{x^2}\, \underbrace{e^{\sin t}}_{e^y}\, \underbrace{\sin t}_{\sin z} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt}$$

ESERCIZIO (di analisi 1 e 2)

$\boxed{\text{Calcolare l'integrale nel caso in cui } f(x,y,z) = x e^y \sin z}$

$\gamma : [a,b] \to \mathbb{R}^n$ è detta EQUIVALENTE a $\sigma : [c,d] \to \mathbb{R}^n$

$\boxed{\gamma, \sigma \in C^1}$ se esiste $\rho : [c,d] \to [a,b]$ invertibile (C^1) tale che $\boxed{\dot\rho \neq 0 \ \forall t \in [c,d]}$

$$\sigma(s) = \gamma(\rho(s)) \qquad \dot\sigma(s) = \dot\gamma(\rho(s))\dot\rho(s)$$

da cui

$\gamma[a,b] = \sigma[c,d]$

$$\Lambda(\sigma) = \int_c^d |\dot\sigma(s)|\, ds =$$

$$= \int_c^d |\dot\rho(s)\, \dot\gamma(\rho(s))|\, ds =$$

$$\underset{\dot\rho > 0}{=} \int_c^d |\dot\gamma(\rho(s))|\, \dot\rho(s)\, ds =$$

$$\underset{\rho(s) = t}{=} \int_{\rho(c)=a}^{\rho(d)=b} |\dot\gamma(t)|\, dt = \Lambda(\gamma)$$

$[c,d] = [0, \pi]$

$[a,b] = [0, 2\pi]$

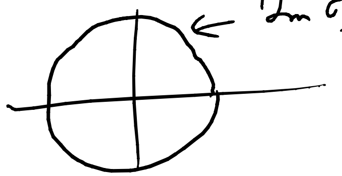
$\rho(t) = 2t$

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad t \in [0, 2\pi]$$

$$\sigma(s) = \begin{pmatrix} \cos 2s \\ \sin 2s \end{pmatrix} \quad s \in [0, \pi]$$

$\dot\rho > 0$ stesso verso

$\dot\rho < 0$ verso opposto



Im σ, γ

$\dot\rho < 0 \quad \forall s \in [c,d] \qquad \rho(c) = b \qquad \rho(d) = a$

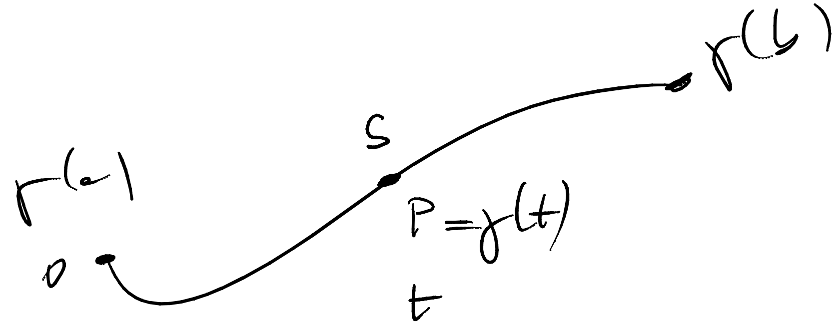
$$\int_c^d |\dot\rho(s)\dot\gamma(\rho(s))|\,ds = \int_c^d |\dot\rho(s)|\,|\dot\gamma(\rho(s))|\,ds \overset{\dot\rho<0}{=} -\int_c^d |\dot\gamma(\rho(s))|\,\dot\rho(s)\,ds =$$

$$\underset{\rho(s)=t}{=} -\int_{\rho(c)=b}^{\rho(d)=a} |\dot\gamma(t)|\,dt = -\int_b^a |\dot\gamma(t)|\,dt = \int_a^b |\dot\gamma(t)|\,dt$$

ASCISSA CURVILINEA

Pietre MILIARI

METRO FLESSIBILE (da Sarto)



$$s(t) = \int_a^t |\dot\gamma(s)|\,ds \qquad \dot\gamma \in C^0 \quad \gamma \in C^1$$

$$\underset{\overset{\|}{0}}{s(t) - s(a)} = \int \dot s(t)\,dt = \Lambda(\gamma|_{[a,t]}) \qquad \dot s(t) = |\dot\gamma(t)|$$

$s(t)$ = strada percorsa dall'origine a $\gamma(t)$.