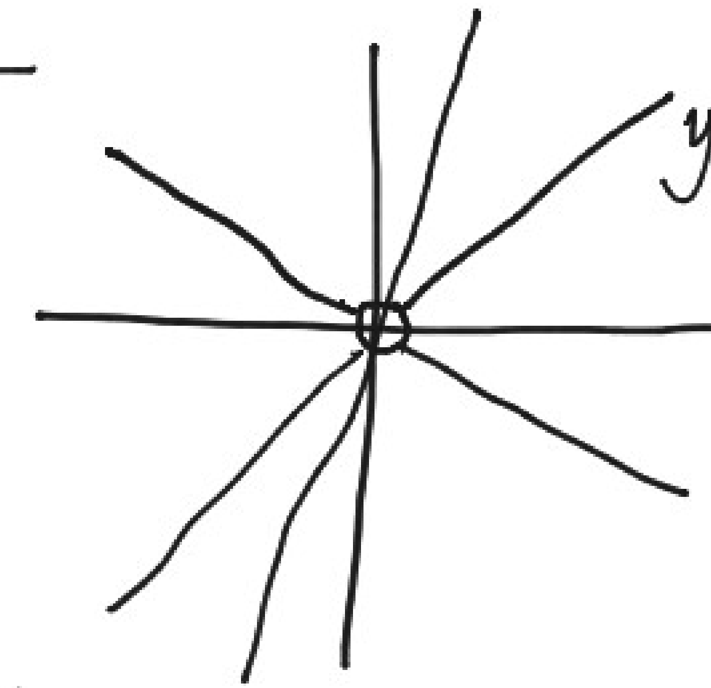
$$\lim_{(0,0)} \frac{x^3+y^3}{x^4+y^2}$$

$y = \alpha x \qquad \alpha \in \mathbb{R}$



$y = \alpha x$ ↑

Fissato α, calcoliamo

$$\lim_{(0,0)} f(x,y) = 0$$

$y = \alpha X$

$$\left.\frac{x^3+y^3}{x^4+y^2}\right|_{y=\alpha x} =$$

restrizione di f all'insieme $\{(x,y)\in\mathbb{R}^2 ; y = \alpha x\}$

$$= \frac{x^3+\alpha^3x^3}{x^4+\alpha^2x^2} = x\,\frac{1+\alpha^3}{x^2+\alpha^2}$$

$x \to 0$ se $(x,y)\to 0$

$\dfrac{x^2+\alpha^2}{} \to \dfrac{1+\alpha^3}{\alpha^2}$ se $\alpha \neq 0$

Se, invece, $\alpha = 0 \Rightarrow y = 0$ e $x \neq 0$

e $f(x,0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}$

che NON HA LIMITE se $x \to 0$

$$\lim_{(0,0)} \frac{x^2 y}{x^4 + y^2}$$ restrizioni a $\boxed{y = \alpha x}$ $\boxed{\alpha \neq 0}$

$$\frac{x^2(\alpha x)}{x^4 + \alpha^2 x^2} = \frac{x^3}{x^2} \frac{\alpha}{x^2 + \alpha^2} \to 0$$

$x \to 0$: $\frac{x^3}{x^2} \to 0$, $\frac{\alpha}{x^2+\alpha^2} \to \frac{1}{\alpha}$, $\alpha \neq 0$

se, invece,

$$\alpha = 0 \Rightarrow y = 0$$

$$f(x,0) \equiv 0 \quad \forall x \neq 0$$

$$\lim_{0} f(x,0) = 0$$

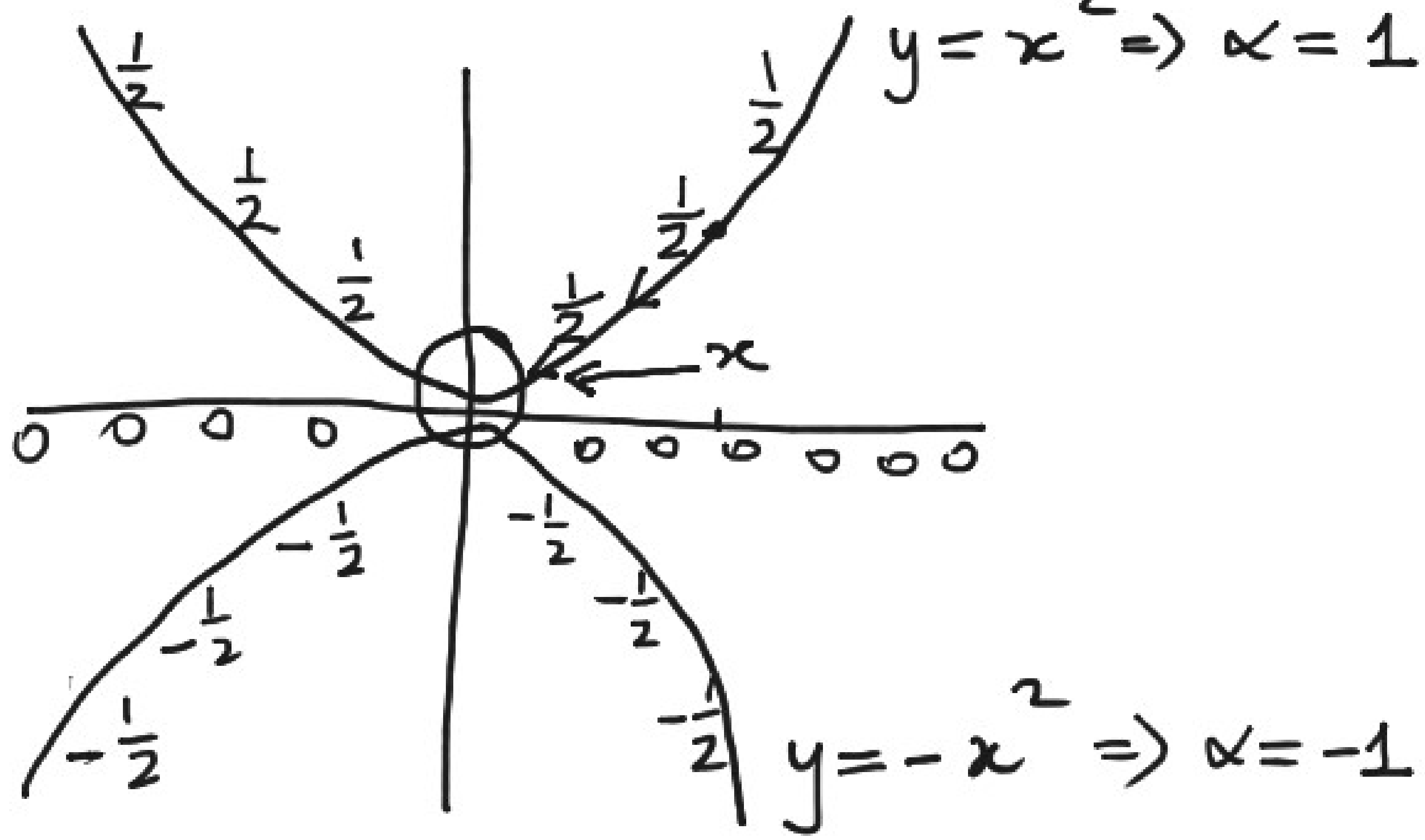
Sull'asse y, non considerato prima $f(0,y) \equiv 0$ per $y \neq 0$ e infine:

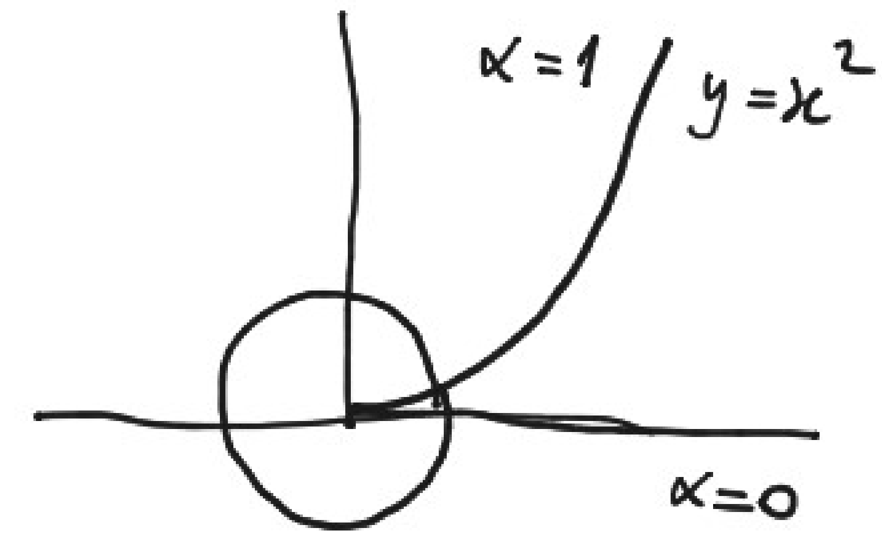
IL LIMITE SU OGNI RETTA PER (0,0) VALE 0.

<u>NON BASTA</u> per poter dire che $f \to 0$

Consideriamo la parabola $y = \alpha x^2$

$$f(x, \underbrace{\alpha x^2}_{y}) = \frac{x^2(\alpha x^2)}{x^4 + \alpha^2 x^4} \underset{x \neq 0}{=} \frac{\alpha}{1+\alpha^2}$$





$$\lim_{(x,y)\to 0} x \lg y = 0 \,?$$

NO!

$$\lim_{x\to 0^+} x \lg x = 0$$

$$x \lg x = \frac{\lg x}{\frac{1}{x}}$$

$y = \alpha x$ $\qquad$ $\alpha x > 0$

$$f(x,y) \equiv f(x,\alpha x) = x(\lg \alpha x) = \begin{cases} \boxed{\alpha > 0 \Rightarrow x > 0} \\ x(\lg x + \lg \alpha) = \\ x \lg x + x \lg \alpha \to 0 \\ \alpha < 0 \Rightarrow x < 0 \\ \alpha x = |\alpha||x| \\ x(\lg|x| + \lg|\alpha|) \\ \to 0 \qquad \text{limitato} \end{cases}$$

$$x \lg|x| = \frac{x}{|x|} \underbrace{|x| \lg |x|}_{\to 0}$$

cambio di variabile $t = |x|$

$t \lg t$ NON È DEFIN- INO

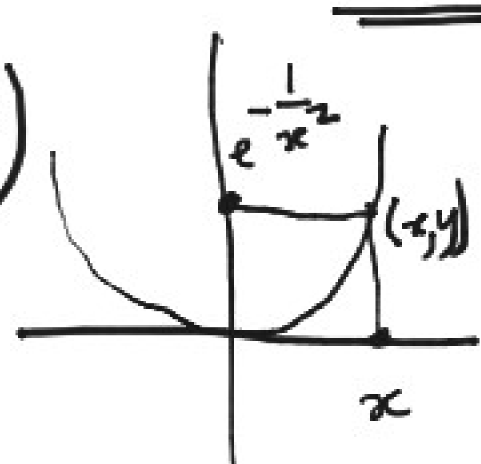
$y = e^{-1/x^2}$

$$\boxed{\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0} \quad (*)$$

$$f\Big(x, \underbrace{e^{-\frac{1}{x^2}}}_{y}\Big) = x \lg e^{-\frac{1}{x^2}} = x \cdot \left(-\frac{1}{x^2}\right)$$

NON tende a 0 per $x \to 0$ e NON ESISTE

$x \to 0 \Rightarrow y \to 0$ per $(*)$



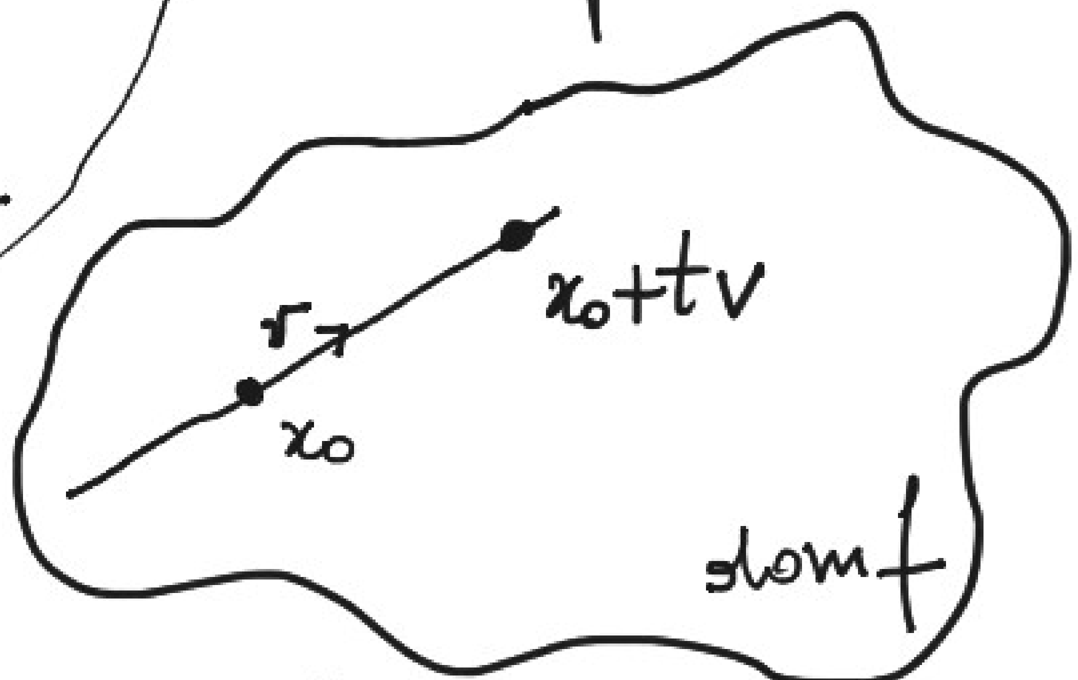
DERIVATE DIREZIONALI E PARZIALI

$f: \Omega \to \mathbb{R}$ $\quad \Omega$ aperto $\subseteq \mathbb{R}^n$

sia $x_0 \in \Omega$

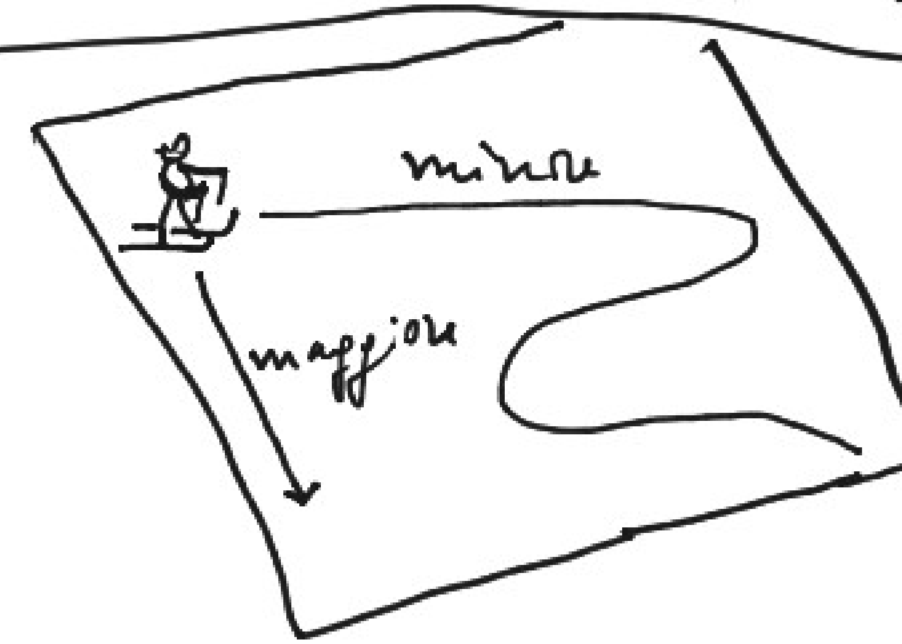
$v \in \mathbb{R}^n$

$v \neq 0$



$$h(t) = f(x_0 + tv)$$

restrizione di f alla retta $x_0 + tv$



la pendenza dipende dalla direzione di movimento!

DUNQUE...

DEF. Si definisce DERIVATA DIREZIONALE di f in x_0 nella direzione di v il limite $\lim_{t \to 0} \frac{1}{t}\left[f(x_0+tv) - f(x_0)\right] =$ se esiste finito.

$$= \lim_{0} \frac{1}{t}\left[h(t) - h(0)\right] = h'(0)$$

Si denota con

$$\underline{\frac{\partial f}{\partial v}(x_0)}, \quad \underline{f_v(x_0), \quad \partial_v f(x_0)} \qquad \underline{\text{sinonime}}$$

Le derivate direzionali $f_{e_1}(x_0), f_{e_2}(x_0), \dots, f_{e_n}(x_0)$

$e_1 \dots e_n$ base canonica di $\mathbb{R}^n$ si chiamano **DERIVATE PARZIALI**.

$x_0 = (x_0^1, \dots, x_0^n)$

$$\partial_{e_1} f(x_0) = \lim_{t \to 0} \frac{f(x_0 + te_1) - f(x_0)}{t} =$$

$$= \lim_{t \to 0} \frac{1}{t} \left[f(x_0^1 + t, x_0^2 + 0 \cdot t, x_0^3, \dots, x_0^n) - f(x_0^1, \dots, x_0^n) \right]$$

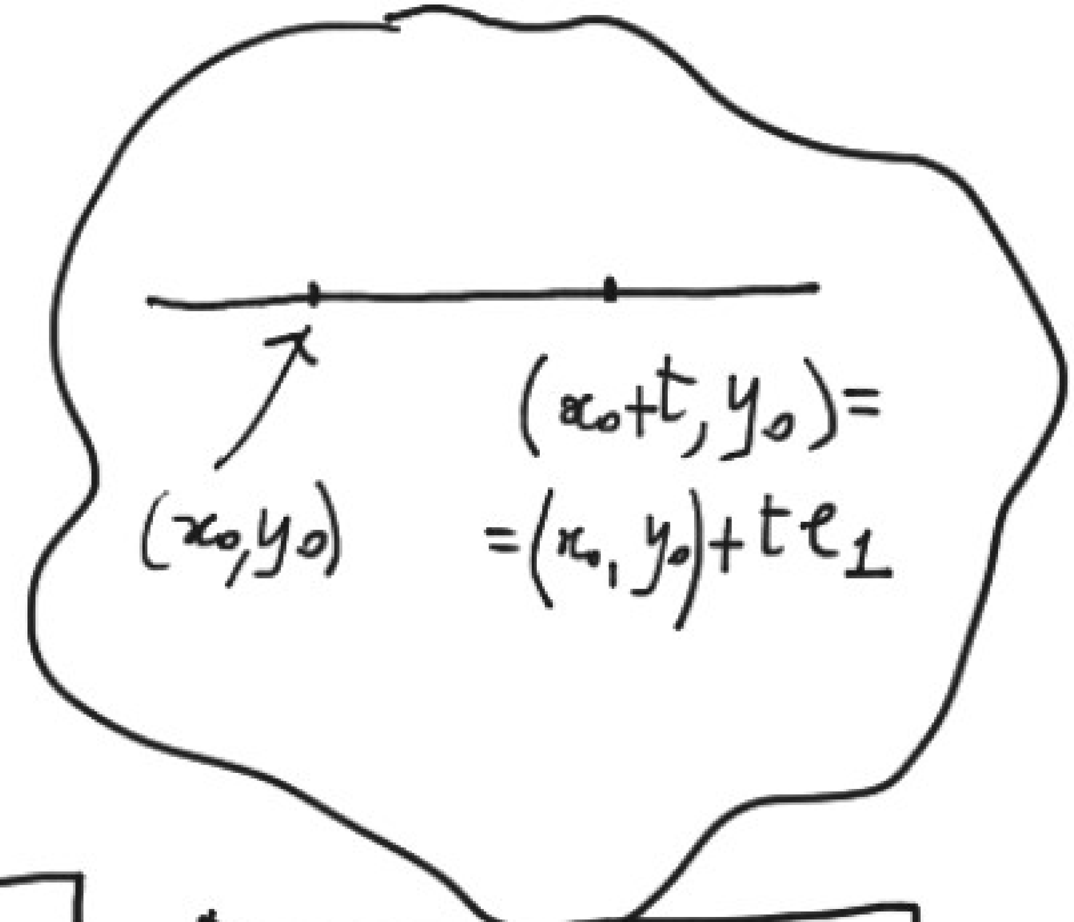
che è la derivata in x_0^1 della funzione

$$y \to f(y, x_0^2, \dots, x_0^n) \qquad 1 \text{ variabile } y$$

$$\partial_x \sin(x^2 y) = \cos(x^2 y)\, y\, 2x$$

y si considera <u>costante</u>

$$\partial_y \frac{x}{y} \underset{y \neq 0}{=} x\, \partial_y \frac{1}{y} = -\frac{x}{y^2}$$



$$\boxed{\frac{\partial f}{\partial e_i} \to \frac{\partial f}{\partial x_i}} \qquad \boxed{\partial_{e_i} f \to \partial_{x_i} f} \qquad \boxed{f_{e_i} \to f_{x_i}}$$

notazioni per derivate parziali

$\partial_x(\sin x + \cos y) = \cos x + 0 \to$ derivata della $\cos y$, costante in x.

UNA FUNZIONE PUÒ AVERE TUTTE LE DERIVATE DIREZIONALI (NULLE) ED ESSERE DISCONTINUA

$$f(x,y) = \begin{cases} \left(\dfrac{x^2 y}{x^4+y^2}\right)^2 & (x,y) \neq (0,0) \\ 0 & (0,0) \end{cases}$$

definita dappertutto in $\mathbb{R}^2$

f ha tutte le derivate direzionali in $(0,0)$, e valgono 0.

$v = (\alpha, \beta) \quad \alpha^2+\beta^2 \neq 0 \Rightarrow v \neq 0$

$$\frac{f(0+\alpha t, 0+\beta t) - f(0,0)}{t} = \frac{1}{t}\left(\frac{(\alpha^2 t^2)(\beta t)}{\alpha^4 t^4 + \beta^2 t^2}\right)^2 = \frac{1}{t}\,\frac{t^6}{t^4}\left(\frac{\alpha^2 \beta}{\alpha^4 t^2 + \beta^2}\right)^2 =$$

$$= t \left(\frac{\alpha^2 \beta}{\alpha^4 t^2 + \beta^2} \right)^2 \xrightarrow[t \to 0]{} 0$$ poiché

se $\boxed{\beta \neq 0}$ $\quad \left(\frac{\alpha^2 \beta}{\alpha^4 t^2 + \beta^2} \right)^2 \xrightarrow[t \to 0]{} \frac{\alpha^4 \beta^2}{\beta^4} \leftarrow$

Se invece

$\boxed{\beta = 0} \Rightarrow \alpha \neq 0 \quad t \neq 0 \left(\lim_{t \to 0} \right)$

$\Downarrow$

$\underline{\alpha^4 t^2 \neq 0} \Rightarrow \frac{\alpha^2 \beta}{\alpha^4 t^2 + \beta^2} = \frac{0}{\alpha^4 t^2} \underset{\forall t \neq 0}{\equiv} 0$

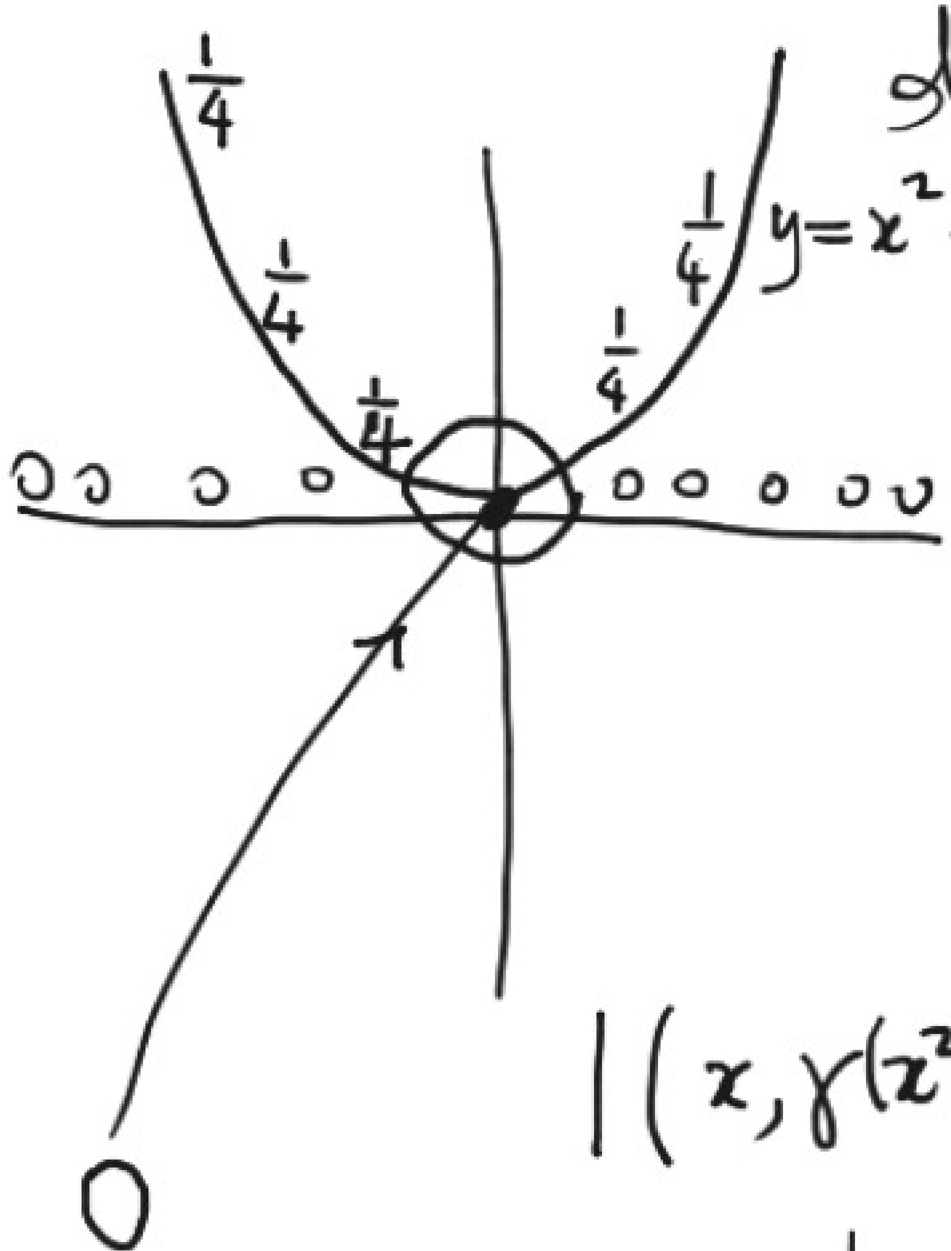
quindi tutte le derivate direzionali sono 0

$$f(x,\gamma x^2) = \left(\frac{x^2\gamma x^2}{x^4+\gamma^2 x^4}\right)^2 = \left(\frac{\gamma}{1+\gamma^2}\right)^2 \neq 0$$

$x \neq 0$ se $\gamma \neq 0$

dom f

mentre $f(0,0) = 0$



$y = x^2 \Rightarrow \gamma = 1$

$\Rightarrow$

$$f(x,\gamma x^2) = \frac{1}{4}$$

$$f(0,0) = 0$$

$$\Rightarrow \left| f(x,\gamma x^2) - f(0,0) \right| = \frac{1}{4}$$

$\forall x \neq 0$

mentre

$$\|(x,\gamma(x^2))\| = \sqrt{x^2+\gamma^2 x^4} = |x|\sqrt{1+\gamma^2 x^2} \underset{x \to 0}{\longrightarrow} 0$$

e dunque la condizione di Cauchy non si verifica per $\bar{\varepsilon} < 1/4$