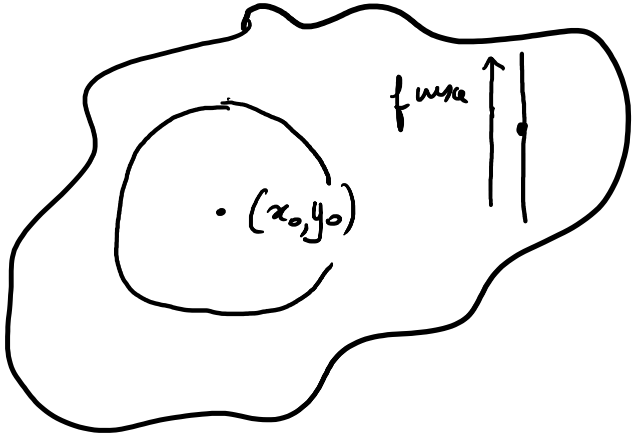
Il teorema delle funzioni implicite di Ulisse Dini

$f:\Omega\to\mathbb{R}$ $\quad \Omega\subseteq\mathbb{R}^2$



$$f(x,y)=0$$
$$\Downarrow$$
$$y=\varphi(x)$$
$$f(x,\varphi(x))\equiv 0$$

Hp. 1) f continua in Ω

2) $f(x_0,y_0)=0$

3) (x_0,y_0) interno ad Ω

$((x_0,y_0)\in\mathring{\Omega})$

4) $y\longrightarrow f(x,y)$ strett. crescente $\quad y'>y \Rightarrow f(x,y')>f(x,y)$

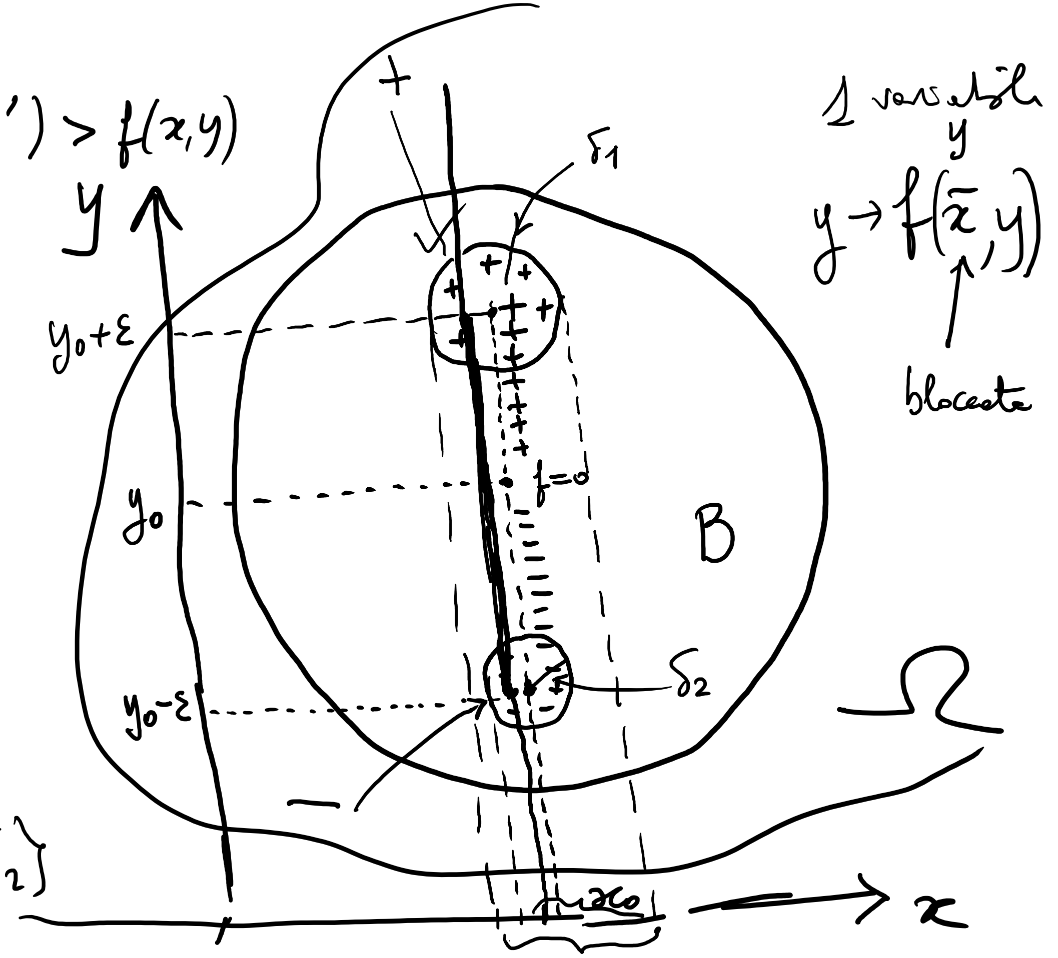
$\forall (x,y)\in\Omega$

Ts 1) $\exists\,\delta,\varphi\,,\ \delta>0\quad \varphi:[x_0-\delta,\,x_0+\delta]\to\mathbb{R}$:

2) $f(x,\varphi(x))=0\quad \forall x\in[x_0-\delta,\,x_0+\delta]$

3) $y_0=\varphi(x_0)$

4) φ è continua in $[x_0-\delta,\,x_0+\delta]$



$\bar{x}\in[x_0-\delta,\,x_0+\delta]\quad \delta=\min\{\delta_1,\delta_2\}$

DIM.

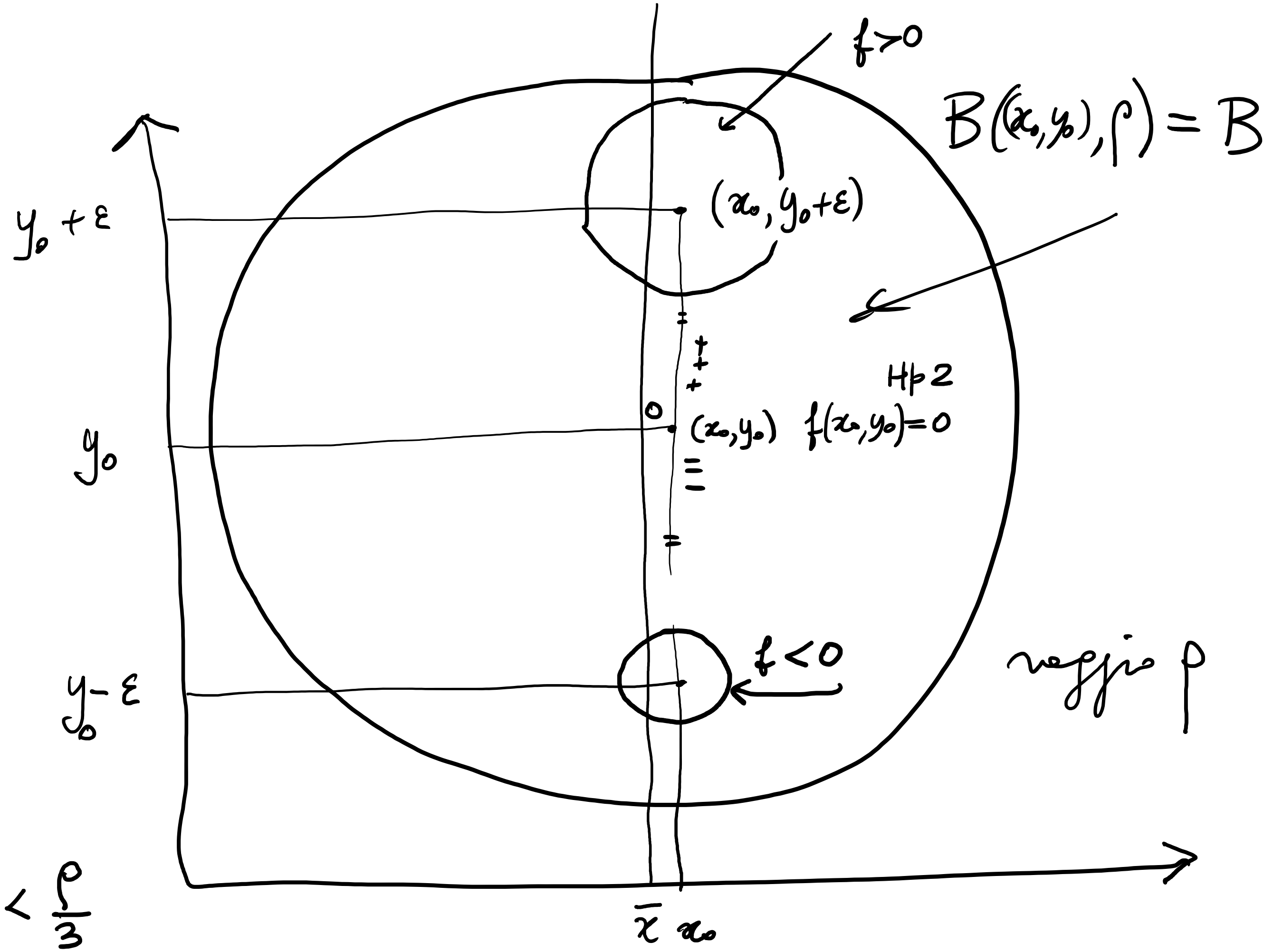
$\exists \rho > 0 : B((x_0, y_0), \rho) \subseteq \Omega$

$\varepsilon < \frac{\rho}{2}$ Hp 3

Hp 4

$f(x_0, y_0+\varepsilon) > 0$ perché $y_0+\varepsilon > y_0$

$f(x_0, y_0-\varepsilon) < 0$ " $y_0-\varepsilon < y_0$



$\delta_1, \delta_2 < \frac{\rho}{3}$

Per il th. della perm. del segno

$\exists \delta_1 > 0 \quad ; \quad f(x,y) > 0 \quad \forall (x,y) \in B((x_0, y_0+\varepsilon), \delta_1)$

$\exists \delta_2 > 0 \quad ; \quad f(x,y) < 0 \quad \forall (x,y) \in B((x_0, y_0-\varepsilon), \delta_2)$

$\delta = \min\{\delta_1, \delta_2\}$

$\bar{x} \in [x_0-\delta, x_0+\delta]$

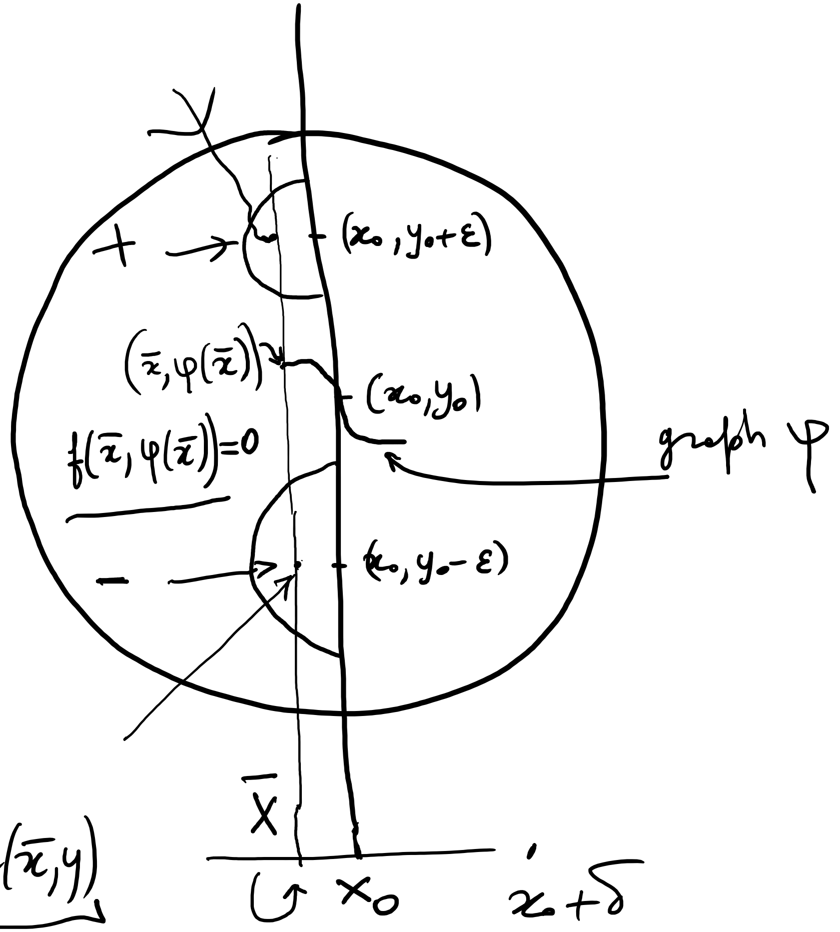
$$(*)\begin{cases} f(\bar{x}, y_0-\varepsilon) < 0 \\ f(\bar{x}, y_0+\varepsilon) > 0 \end{cases}$$

$$|(\bar{x}, y_0-\varepsilon) - (x_0, y_0-\varepsilon)| = \sqrt{(\bar{x}-x_0)^2+(0)^2} = |\bar{x}-x_0| < \delta \le \delta_2$$

$$|(\bar{x}, y_0+\varepsilon) - (x_0, y_0+\varepsilon)| < \delta < \delta_1$$

$\boxed{y \to f(\bar{x}, y)}$ su $[y_0-\varepsilon, y_0+\varepsilon]$ $(**)$

$B((x_0,y_0), \rho) \subseteq \Omega$ è convesso $\Rightarrow$ tutto il segmento che congiunge $(\bar{x}, y_0+\varepsilon)$ e $(\bar{x}, y_0-\varepsilon)$ è contenuto in Ω



Applicando il th. degli zeri alla funzione continua $\underbrace{y \to f(\bar{x}, y)}$ esiste almeno uno zero $\bar{y}$. Poiché è strett. monotona tale zero è unico. Si pone allora $\varphi(\bar{x}) = \bar{y}$.

$$\boxed{f(x, \varphi(x)) = 0 \quad \forall x \in B(x_0, \delta)}$$