Chapter 1
Heidelberg Lectures on Fundamental Groups

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As a prelude to the PIA conference, in February 2010 Amnon Besser and I gave introductory lecture series at Universität Heidelberg, following the kind request of Jakob Stix. These notes constitute a revised version of the ones I distributed during the lectures. They begin with a quick introduction to Grothendieck's concept of the algebraic fundamental group. After a reminder on basic results concerning fundamental groups of curves, we move on to discuss what is arguably the most famous open problem in the area, Grothendieck's Section Conjecture. The next section presents in detail a beautiful application of the ideas involved in the conjecture: Parshin's 'hyperbolic' proof of the geometric case of Mordell's conjecture. The final section gives an overview of the most important features of anabelian geometry.

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1.1 Grothendieck's Fundamental Group

Grothendieck's theory of the algebraic fundamental group is a common generalization of Galois theory and the theory of covers in topology. Let us briefly recall both. The proofs of all statements in this section can be found in [48].

Let k be a field. Recall that a finite dimensional k-algebra A is *étale* (over k) if it is isomorphic to a finite direct product of separable extensions of k. Fix a separable closure $k_s|k$. The $\mathrm{Gal}(k_s|k)$-action on k_s induces a left action on the set of k-algebra homomorphisms $\mathrm{Hom}_k(A,k_s)$. The rule $A \mapsto \mathrm{Hom}(A,k_s)$ is a contravariant functor. The Main Theorem of Galois theory in Grothendieck's version is the following statement.

Theorem 1.1 *The contravariant functor $F : A \mapsto \mathrm{Hom}_k(A,k_s)$ gives an anti-equivalence between the category of finite étale k-algebras and the category of finite sets with continuous left $\mathrm{Gal}(k_s|k)$-action.*

Note that the functor F depends on the choice of the separable closure k_s. The latter is not a finite étale k-algebra but a *direct limit* of such. Also, one checks that $\mathrm{Gal}(k_s|k)$ is naturally isomorphic to the *automorphism group* of the functor F (i.e. the group of isomorphisms $F \stackrel{\sim}{\to} F$).

Now to the topological situation. Let X be a connected, locally connected and locally simply connected topological space. Recall that a *cover* of X is a space Y equipped with a continuous map $p: Y \to X$ subject to the following condition: each point of X has an open neighbourhood V for which $p^{-1}(V)$ decomposes as a disjoint union of open subsets U_i of Y such that the restriction of p to each U_i induces a homeomorphism of U_i with V.

Given a point $x \in X$, the fundamental group $\pi_1(X,x)$ has a natural left action on the fibre $p^{-1}(x)$ defined as follows: given $\alpha \in \pi_1(X,x)$ represented by a closed path $f: [0,1] \to X$ with $f(0) = f(1) = x$ as well as a point $y \in p^{-1}(x)$, we define $\alpha y := \widetilde{f}(1)$, where $\widetilde{f}$ is the unique lifting of the path f to Y with $\widetilde{f}(0) = y$. One checks that this indeed gives a well-defined left action of $\pi_1(X,x)$. It is called the *monodromy action.*

Theorem 1.2 *The functor* Fib_x *sending a cover* $p: Y \to X$ *to the fibre* $p^{-1}(x)$ *equipped with the monodromy action induces an equivalence of the category of covers of X with the category of left $\pi_1(X,x)$-sets.*

Here again, the functor Fib_x depends on the choice of the point x. It is in fact *representable* by a cover $\pi: \widetilde{X}_x \to X$, i.e. we have an isomorphism of functors $\mathrm{Fib}_x \cong \mathrm{Hom}(\widetilde{X}_x, \ )$. The space $\widetilde{X}_x$ can be constructed as the space of homotopy classes of paths starting from x, the projection π mapping the class of a path to its other endpoint. As a consequence, we have isomorphisms

$$\mathrm{Aut}(\widetilde{X}_x) \cong \mathrm{Aut}(\mathrm{Fib}_x) \cong \pi_1(X,x).$$

Here is an important consequence. Call a cover $Y \to X$ *finite* if it has finite fibres; for connected X these have the same cardinality, called the *degree* of X.

Corollary 1.3 *For X and x as in Theorem 1.2, the functor* Fib_x *induces an equivalence of the category of* finite *covers of X with the category of finite continuous left* $\widehat{\pi_1(X,x)}$*-sets.*

Here $\widehat{\pi_1(X,x)}$ denotes the *profinite completion* of $\pi_1(X,x)$, i.e. the inverse limit of the natural inverse system of its finite quotients.

We can now come to Grothendieck's common generalization in algebraic geometry. Let S be a connected scheme. Recall that a *finite étale cover* of S is a finite flat surjection $X \to S$ such that each fibre at a point $s \in S$ is the spectrum of a finite étale $\kappa(s)$-algebra. Fix a geometric point $\bar{s}: \mathrm{Spec}(\Omega) \to S$. For a finite étale cover $X \to S$ we consider the geometric fibre $X \times_S \mathrm{Spec}(\Omega)$ over $\bar{s}$, and denote by $\mathrm{Fib}_{\bar{s}}(X)$ its underlying set. This gives a set-valued functor on the category of finite étale covers of X.

We *define* the *algebraic fundamental group* $\pi_1(S,\bar{s})$ as the automorphism group of this functor. By definition an automorphism of $\mathrm{Fib}_{\bar{s}}$ induces an automorphism of the set $\mathrm{Fib}_{\bar{s}}(X)$ for each finite étale cover X; in this way we obtain a natural left action of $\pi_1(S,\bar{s})$ on the set $\mathrm{Fib}_{\bar{s}}(X)$.

Theorem 1.4 (Grothendieck) *Let S be a connected scheme, and $\bar{s}\colon \mathrm{Spec}\,(\Omega)\to S$ a geometric point.*

1. *The group $\pi_1(S,\bar{s})$ is profinite, and its action on $\mathrm{Fib}_{\bar{s}}(X)$ is continuous for every X in Fet_S.*
2. *The functor $\mathrm{Fib}_{\bar{s}}$ induces an equivalence of the category of finite étale covers of S with the category of finite continuous left $\pi_1(S,\bar{s})$-sets.*

Here the functor $\mathrm{Fib}_{\bar{s}}$ is *pro-representable*, which means that there exists a (filtered) inverse system $P=(P_\alpha,\phi_{\alpha\beta})$ of finite étale covers and a functorial isomorphism $\varinjlim \mathrm{Hom}(P_\alpha,X)\cong \mathrm{Fib}_{\bar{s}}(X)$. The automorphism group of each finite étale cover $P_\alpha\to S$ is finite, and $\pi_1(S,\bar{s})$ is their inverse limit; this explains its profiniteness. In fact, Grothendieck showed that one may choose as a pro-representing system the system of all *Galois covers* $P_\alpha\to X$, i.e. those finite étale covers for which $\mathrm{Aut}\,(P_\alpha|S)$ acts transitively on geometric fibres. These are turned into an inverse system by choosing a distinguished point $p_\alpha\in \mathrm{Fib}_{\bar{s}}(P_\alpha)$ for each α; for each pair α,β there is then at most one S-morphism $P_\beta\to P_\alpha$ sending p_β to p_α. We define this map to be $\phi_{\alpha\beta}$ (if it exists).

Remark 1.5 Any two fibre functors on the category of finite étale S-schemes are (non-canonically) isomorphic. One way to prove this is to use pro-representability of the fibre functor which reduces the construction of an isomorphism between functors to the construction of a compatible system of automorphisms of the Galois objects P_α transforming one system of maps $\phi_{\alpha\beta}$ to another. This can be done by means of a compactness argument.

An isomorphism between two fibre functors $\mathrm{Fib}_{\bar{s}}$ and $\mathrm{Fib}_{\bar{s}'}$ is called a *path* from $\bar{s}$ to $\bar{s}'$. It induces an isomorphism of fundamental groups $\pi_1(S,\bar{s})\xrightarrow{\sim}\pi_1(S,\bar{s}')$. In the topological situation such an isomorphism is induced by the choice of a (usual) path between base points, whence the name in the algebraic situation. As in topology, two isomorphisms $\pi_1(S,\bar{s})\xrightarrow{\sim}\pi_1(S,\bar{s}')$ induced by different paths differ by an inner automorphism of $\pi_1(S,\bar{s})$.

Remark 1.6 Historically, the case of a normal scheme was known earlier. If S is an integral normal Noetherian scheme, denote by K_s a fixed separable closure of the function field K of S, and by K_S the composite of all finite subextensions $L|K$ of K_s such that the normalization of S in L is étale over S. Then $K_S|K$ is a Galois extension, and it can be shown that the Galois group $\mathrm{Gal}(K_S|K)$ is canonically isomorphic to the fundamental group $\pi_1(S,\bar{s})$ for the geometric point $\bar{s}\colon \mathrm{Spec}\,(\overline{K})\to S$, where $\overline{K}$ is the algebraic closure of K containing K_s.

The following examples show that the algebraic fundamental group indeed yields a common generalization of the algebraic and topological cases:

Examples 1.7

1. For $X = \mathrm{Spec}\,(k)$, $\bar{x}: \mathrm{Spec}\,(\bar{k}) \to \mathrm{Spec}\,(k)$ we have

$$\pi_1(X, \bar{x}) \cong \mathrm{Gal}(k_s|k).$$

   This holds basically because finite étale $\mathrm{Spec}\,(k)$-schemes are spectra of finite étale k-algebras.

2. For X of finite type over $\mathbf{C}$ and $\bar{x}: \mathrm{Spec}\,(\mathbf{C}) \to X$ there is a canonical isomorphism

$$\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, \bar{x})} \xrightarrow{\sim} \pi_1(X, \bar{x})$$

   where on the left hand side we have the profinite completion of the topological fundamental group of X with base point $\mathrm{Im}\,(\bar{x})$, and X^{an} denotes the complex analytic space associated with X.
   This isomorphism relies on a deep algebraization theorem for finite topological covers of schemes of finite type over $\mathbf{C}$.

A base point preserving morphism of schemes induces a continuous homomorphism of fundamental groups. To construct it, let S and S' be connected schemes, equipped with geometric points $\bar{s}: \mathrm{Spec}\,(\Omega) \to S$ and $\bar{s}': \mathrm{Spec}\,(\Omega) \to S'$, respectively. Assume given a morphism $\phi: S' \to S$ with $\phi \circ \bar{s}' = \bar{s}$. For a finite étale cover $X \to S$ consider the base change $X \times_S S' \to S'$. The condition $\phi \circ \bar{s}' = \bar{s}$ implies that $\mathrm{Fib}_{\bar{s}}(X) = \mathrm{Fib}_{\bar{s}'}(X \times_S S')$. This construction is functorial in X, and thus every automorphism of the functor $\mathrm{Fib}_{\bar{s}'}$ induces an automorphism of $\mathrm{Fib}_{\bar{s}}$, which defines the required map $\phi_*: \pi_1(S', \bar{s}') \to \pi_1(S, \bar{s})$.

The above functoriality, together with Example 1.7 (1), defines the maps in the following exact sequence which is fundamental not only because it involves fundamental groups.

Proposition 1.8 *Let X be a quasi-compact and geometrically integral scheme over a field k. Fix an algebraic closure $\bar{k}$ of k, and let $k_s|k$ be the corresponding separable closure. Write $\overline{X} := X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k_s)$, and let $\bar{x}$ be a geometric point of $\overline{X}$ with values in $\bar{k}$. The sequence of profinite groups*

$$1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}(k_s|k) \to 1 \tag{1.1}$$

induced by the maps $\overline{X} \to X$ and $X \to \mathrm{Spec}\,(k)$ is exact.

The group $\pi_1(X, \bar{x})$ acts on its normal subgroup $\pi_1(\overline{X}, \bar{x})$ via inner automorphisms, whence a map $\phi_X: \pi_1(X, \bar{x}) \to \mathrm{Aut}\,(\pi_1(\overline{X}, \bar{x}))$. Inside $\mathrm{Aut}\,(\pi_1(\overline{X}, \bar{x}))$ we have the normal subgroup $\mathrm{Inn}(\pi_1(\overline{X}, \bar{x}))$ of inner automorphisms; the quotient is the group $\mathrm{Out}(\pi_1(\overline{X}, \bar{x}))$ of *outer automorphisms*. By the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{X}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & \mathrm{Gal}(k_s|k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & \mathrm{Aut}\,(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & \mathrm{Out}(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & 1
\end{array}$$

we get an important representation

$$\rho_X : \mathrm{Gal}(k_s|k) \to \mathrm{Out}(\pi_1(\overline{X},\bar{x})).$$

called the *outer Galois representation*. It will appear several times in subsequent sections.

Example 1.9 Assume X is a smooth proper curve of genus g, and fix a prime number ℓ different from the characteristic of k. As we shall see in a moment (Remark 2.2), the maximal abelian pro-ℓ-quotient of $\pi_1(X \times_k \bar{k}, \bar{x})$ is isomorphic to the Tate module $T_\ell(\overline{J})$ of the Jacobian $\overline{J}$ of $X \times_k \bar{k}$. Taking the pushout of the sequence (1.1) by the natural map $\pi_1(X \times_k \bar{k}) \to T_\ell(\overline{J})$ we obtain an extension of $\mathrm{Gal}(\bar{k}|k)$ by $T_\ell(\overline{J})$. By the same argument as above it gives rise to a Galois representation $\mathrm{Gal}(k_s|k) \to \mathrm{Aut}(T_\ell(\overline{J})) \cong GL_{2g}(\mathbf{Z}_\ell)$. It is none but the usual Galois representation on torsion points of the Jacobian, a central object of study in number theory. The outer Galois representation can thus be viewed as a non-abelian generalization.

1.2 Fundamental Groups of Curves

In the first part of this section k denotes an algebraically closed field of characteristic $p \geq 0$ and X a proper smooth curve over k. We recall some basic structure results about the fundamental group of X and of its open subschemes; as they concern the groups up to isomorphism, we drop base points from the notation.

Theorem 2.1 (Grothendieck [13]) *Let $U \subset X$ be an open subcurve (possibly equal to X), and $n \geq 0$ the number of closed points in $X \setminus U$. Then $\pi_1(U)^{(p')}$ is isomorphic to the profinite p'-completion of the group*

$$\Pi_{g,n} := \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n \mid [a_1,b_1]\ldots[a_g,b_g]\gamma_1 \ldots \gamma_n = 1 \rangle.$$

Here $G^{(p')}$ denotes the *maximal prime-to-p quotient* of the profinite group G, i.e. the inverse limit of its finite quotients of order prime to p; for $p = 0$ we define it to be G itself.

For $k = \mathbf{C}$ the theorem follows via Example 1.7 (2) from the well-known structure of the topological fundamental group; in this particular case the underlying algebraization theorem is just the Riemann existence theorem of complex analysis. One deduces the result for k of characteristic 0 using a rigidity theorem (see e.g. [48], Proposition 5.6.7 and Remark 5.7.8) which says that the fundamental group of a smooth curve does not change under extensions of algebraically closed fields of characteristic 0 (this also holds in positive characteristic, but only for proper curves).

In positive characteristic Grothendieck proved the result by first lifting the curve to characteristic 0 and then proving a specialization theorem establishing an isomorphism between maximal prime-to-p quotients of the fundamental groups of the curve and its lifting. Thus this case also relies on the topological result over $\mathbf{C}$. The only case where a proof avoiding the topological argument is known at present is for $k = \overline{\mathbf{F}}_p$. There Wingberg [51] was able to prove that the maximal pro-ℓ quotients (for $\ell \neq p$ a prime) of $\pi_1(U)$ have the above structure using class field theory and delicate group-theoretic arguments.

Remark 2.2 For X proper and $\ell \neq p$ a prime the theorem implies that the maximal abelian pro-ℓ-quotient of $\pi_1(X)$ is isomorphic to $\mathbf{Z}_\ell^{2g}$. On the other hand, for J the Jacobian of X the Tate module $T_\ell(J)$ has the same structure. This is not a coincidence: by a theorem of Serre and Lang ([25], see also [48], Theorem 5.6.10) every finite étale cover of J of ℓ-power degree is a quotient of some cover given by

$$0 \to {}_{\ell^n}J \to J \stackrel{\ell^n}{\to} J \to 0;$$

on the other hand, given some embedding $X \to J$ obtained by sending a point x to the divisor class of $x - \xi$ for a fixed base point ξ, the induced map on fundamental groups becomes an isomorphism on the maximal prime-to-p abelian quotient ('abelian prime-to-p covers are obtained via pullback from the Jacobian').

There is also a generalization to open curves: if $U \subset X$ is an open subcurve, one still identifies the maximal abelian pro-ℓ-quotient of $\pi_1(U)$ with the ℓ-adic Tate module of the Jacobian $\widetilde{J}$ of U; the latter is a commutative group variety which is an extension of J by the $(n-1)$-st power of the multiplicative group $\mathbf{G}_m$, in accordance with the theorem. See e.g. ([20], (2.7)).

The *maximal pro-p quotient* $G^{(p)}$ of G is defined as the inverse limit of finite quotients of p-power order, and we have:

Theorem 2.3 (Shafarevich [43]) *Assume $p > 0$. Then $\pi_1(X)^{(p)}$ is a free pro-p group of finite rank equal to the p-rank of the Jacobian variety of X.*

For an open subcurve $U \neq X$ the group $\pi_1(U)^{(p)}$ is a free pro-p group of infinite rank equal to the cardinality of k.

Here recall that the *p-rank* of an abelian variety A over an algebraically closed field k of characteristic $p > 0$ is the dimension of the $\mathbf{F}_p$-vector space given by the kernel of the multiplication-by-p map on the k-points of A. It is a nonnegative integer bounded by $\dim A$.

Using methods of étale cohomology one can give a quick proof of this theorem (see e.g. [11] for details). It is based on the group-theoretic fact that a pro-p-group G is free if and only if the Galois cohomology groups $H^i(G, \mathbf{Z}/p\mathbf{Z})$ vanish for $i > 1$. In the case $G = \pi_1(X)^{(p)}$ they can be identified with the étale cohomology groups $H^i_{\text{ét}}(X, \mathbf{Z}/p\mathbf{Z})$ of X using arguments of cohomological dimension and the latter groups are known to vanish for $i > 1$. The rank is then equal to that of the maximal abelian quotient, i.e. the dual of $H^1_{\text{ét}}(X, \mathbf{Z}/p\mathbf{Z})$ and thus can be determined using Artin–Schreier theory.

Remark 2.4 Observe that Theorems 2.1 and 2.3 do not elucidate completely the structure of the fundamental group of an integral normal curve over an algebraically closed field of positive characteristic; this is still unknown at the present day. The theorems give, however, a good description of its maximal *abelian* quotient: this group is the direct sum of its maximal prime-to-p and pro-p quotients, and hence the previous two theorems together suffice to describe it.

Concerning curves over non-closed fields, a much-studied object is the outer Galois representation

$$\rho_X : \mathrm{Gal}(k_s|k) \to \mathrm{Out}(\pi_1(\overline{X},\bar{x}))$$

over fields of arithmetic interest. One of the most basic results is:

Theorem 2.5 (Matsumoto [28]) *If k is a number field and X is affine such that $\overline{X}$ has non-commutative fundamental group, then ρ_X is injective.*

Recently, Hoshi and Mochizuki [17] proved that the result holds for proper curves of genus > 1 as well. One can easily decide using Theorem 2.1 which curves have noncommutative geometric fundamental group: those for which $(g,n) \neq (0,0),(0,1),(0,2),(1,0)$. These are the so-called *hyperbolic* curves: their fundamental groups are center-free and, for $g > 0$, even free.

The case $(g,n) = (0,3)$ is due to Belyi [1] and is a consequence of his famous theorem stating that every smooth proper curve definable over a number field can be realized as a finite cover of $\mathbf{P}^1$ branched above at most 3 points. The proof of the general case uses different methods.

1.3 Grothendieck's Section Conjecture

Arguably the most famous open question concerning fundamental groups of curves is Grothendieck's Section Conjecture, stated in [14]. It concerns the exact sequence

$$1 \to \pi_1(\overline{X},\bar{x}) \to \pi_1(X,\bar{x}) \xrightarrow{p_*} \mathrm{Gal}(k_s|k) \to 1 \qquad (1.2)$$

of Proposition 1.8, where $p : X \to \mathrm{Spec}\, k$ is the structure map.

Given a k-rational point $y : \mathrm{Spec}\, k \to X$, it induces by functoriality a map $\sigma_y : \mathrm{Gal}(k_s|k) \to \pi_1(X,\bar{y})$ for a geometric point $\bar{y}$ lying above y. This is not quite a splitting of the exact sequence above because of the difference of base points. But the choice of a path (see Remark 1.5) from $\bar{y}$ to $\bar{x}$ induces an isomorphism $\lambda : \pi_1(X,\bar{y}) \xrightarrow{\sim} \pi_1(X,\bar{x})$. Changing the path is reflected by an inner automorphism of $\pi_1(X,\bar{x})$; moreover, this automorphism induces the trivial automorphism of $\mathrm{Gal}(k_s|k)$, so it is in fact conjugation by an element of $\pi_1(\overline{X},\bar{x})$. The composite $\lambda \circ \sigma_y$ is then a section of the exact sequence uniquely determined up to conjugation by elements of $\pi_1(\overline{X},\bar{x})$. We thus obtain a map

$$X(k) \to \{\pi_1(\overline{X},\bar{x})\text{-conjugacy classes of sections of } p_*\}. \qquad (1.3)$$

The Section Conjecture now states:

Conjecture 3.1 (Grothendieck [14]) *If k is finitely generated over $\mathbf{Q}$ and X is a smooth projective curve of genus $g \geq 2$, then the above map is a bijection.*

Remark 3.2 Grothendieck also formulated a variant of the conjecture for open curves. Its formulation is, however, more complicated than the above because one has to circumvent the fact that for an affine curve U with smooth compactification X there are tons of sections coming from points of $X \setminus U$ (see [8]).

Injectivity is not hard to prove and was known to Grothendieck. As the argument works over other base fields as well, we include a slightly more general statement.

Proposition 3.3 *Let X is a smooth projective curve of genus $g > 0$ over a field k. Assume any of the following:*

- *k is finite;*
- *k is p-adic;*
- *k is finitely generated over $\mathbf{Q}$.*

Then the map (1.3) is injective.

Proof. The proof will show that even the sections of the exact sequence

$$0 \to \pi_1^{\mathrm{ab}}(\overline{X}) \to \Pi \to \mathrm{Gal}(k_s|k) \to 1$$

obtained from (1.2) by pushout via the abelianization map $\pi_1(\overline{X}, \bar{x}) \to \pi_1^{\mathrm{ab}}(\overline{X})$ separate the k-points of X.

Set $\Gamma := \mathrm{Gal}(\bar{k}|k)$. Fix a k-point y_0 of X and denote by s_0 the corresponding section $\Gamma \to \pi_1(X, \bar{x})$. Given another k-point y of X with corresponding section s, the composite map $\Gamma \to \pi_1(X, \bar{x}) \to \pi_1^{\mathrm{ab}}(X)$ induced by $s_0 s^{-1}$ has image in $\pi_1^{\mathrm{ab}}(\overline{X})$ and is a continuous 1-cocycle. We thus get compatible classes in $H^1(\Gamma, \pi_1^{\mathrm{ab}}(\overline{X})/m)$ for all $m > 0$. Denoting by J the Jacobian of X we have a Galois-equivariant isomorphism $\pi_1^{\mathrm{ab}}(\overline{X})/m \cong {}_m\overline{J}$ (see Remark 2.2), so we actually get maps $\mathrm{Div}^0(X) \to H^1(\Gamma, {}_m\overline{J})$ for all m, where $\mathrm{Div}^0(X)$ is the group of degree 0 divisors on X. Moreover, it is an exercise to check the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Div}^0(X) & \longrightarrow & H^1(\Gamma, {}_m\overline{J}) \\ \downarrow & & \uparrow \\ J(k) & \xrightarrow{\cong} & J(\bar{k})^{\Gamma} \end{array}$$

where the right vertical map comes from the Kummer sequence

$$J(\bar{k})^{\Gamma} \xrightarrow{m} J(\bar{k})^{\Gamma} \to H^1(\Gamma, {}_m\overline{J}).$$

By this commutativity, if we assume $s = s_0$, the class of the divisor $y - y_0$ lies in the kernel of the Kummer map $J(\bar{k})^{\Gamma} \to H^1(\Gamma, {}_m\overline{J})$ for all m, i.e. it is divisible in $J(k)$. But if k satisfies any of the assumptions above, the group $J(k)$ has trivial divisible subgroup: over a finite k it is finite, over a p-adic k it has a finite index subgroup isomorphic to a finite direct power of $\mathbf{Z}_p$ (Mattuck [27]), and for k finitely generated over $\mathbf{Q}$ it is finitely generated by the Mordell–Weil–Lang–Néron theorem ([24], Chapter I, Corollary 4.3). Therefore the class of the divisor $y - y_0$ is trivial in $J(k)$. As X has positive genus, we conclude that $y = y_0$. □

Remark 3.4 The injectivity result of the proposition holds for open hyperbolic curves U as well, with basically the same proof. The role of J is played by the generalized Jacobian $\widetilde{J}$ encountered in the second paragraph of Remark 2.2. However, over a field finitely generated over $\mathbf{Q}$ its toric part has a nontrivial divisible subgroup of k-points in general. To remedy this, one chooses a normal separated scheme S of finite type over $\mathbf{Z}$ such that U extends to a smooth relative curve over S. The classes in $H^1(\Gamma, {}_m\widetilde{J})$ that one considers are then unramified over S, and one can invoke the finite generation of units in rings of integers together with freeness of groups of divisors.

Remark 3.5 Assume k is a subfield of a finitely generated extension of $\mathbf{Q}_p$ for some prime p (e.g. a p-adic field or a finitely generated field over $\mathbf{Q}$). Consider the exact sequence

$$1 \to \pi_1(\overline{X}, \bar{x})^{(p)} \to \Pi^p \to \mathrm{Gal}(k_s|k) \to 1 \tag{1.4}$$

obtained from (1.2) by pushout via the map $\pi_1(\overline{X}, \bar{x}) \to \pi_1(\overline{X}, \bar{x})^{(p)}$. Mochizuki has shown in ([29], Theorem 19.1) as a consequence of his anabelian characterization of hyperbolic curves (Theorem 5.5 below) that the injectivity result of the proposition (and also its generalization as in the previous remark) remains valid for splittings of (1.4) that come from k-points of X. On the other hand, Hoshi [16] recently gave examples of curves over number fields where not all splittings of (1.4) come from k-points. Therefore the 'pro-p-version' of the section conjecture is false.

So much about injectivity in the section conjecture. As for surjectivity, it is widely open: at the time of writing not a single curve is known over a number field that has a rational point and for which the map (1.3) is proven to be bijective, or at least for which the finiteness of conjugacy classes of sections is known (this would then yield another proof of Mordell's conjecture). In the next section we shall see that over function fields over $\mathbf{C}$ statements of this type can actually be proven.

Let us mention, however, a nice observation that goes back to Tamagawa [49] and was first stated in [22].

Proposition 3.6 *Conjecture 3.1 is equivalent to the following seemingly weaker statement: if k is finitely generated over $\mathbf{Q}$ and X is a smooth projective curve of genus $g \geq 2$, then X has a k-rational point if and only if the sequence (1.2) splits.*

The proposition does *not* claim that for a given curve the splitting of (1.2) implies the bijectivity of (1.3); one has to consider *all* curves.

The proof is based on the following lemma which has many other applications.

Lemma 3.7 (Tamagawa) *Let X be a smooth curve over a field k. Assume k satisfies one of the assumptions as in Proposition 3.3; in the third case assume moreover that the smooth compactification of X has genus ≥ 2.*

A section $s : \mathrm{Gal}(k_s|k) \to \pi_1(X,\bar{x})$ *comes from a k-point if and only if for each open subgroup* $H \subset \pi_1(X,\bar{x})$ *containing* $s(\mathrm{Gal}(k_s|k))$ *the corresponding cover* X_H *has a k-point.*

Proof. Let $\widetilde{X}$ be the 'universal cover' of X, i.e. the normalization of X in the extension $K_X|K$ of Remark 1.6. If s is a section coming from a k-rational point P, the image $s(\mathrm{Gal}(k_s|k))$ is the stabilizer in $\pi_1(X,\bar{x})$ of a closed point Q of $\widetilde{X}$ above P. For $H \supset s(G)$ it is also the stabilizer of Q under the action of $\pi_1(X_H,\bar{x})$; in particular, it maps onto $\mathrm{Gal}(k_s|k)$ under the projection $\pi_1(X_H,\bar{x}) \to \mathrm{Gal}(k_s|k)$. This means that the image of Q by the projection $\widetilde{X} \to X_H$ is k-rational.

For the converse we choose $\bar{x}$ to be a geometric generic point. The sets $X_H(k)$ form a natural inverse system indexed by the subgroups $H \subset s(\mathrm{Gal}(k_s|k))$. Under any of the three assumptions on k these sets are compact in their natural topology. (In the first two cases this is immediate but the third case is based on a highly nontrivial input: Faltings's theorem on the finiteness of $X_H(k)$.) Hence the inverse limit of the $X_H(k)$ is nonempty. An element of the inverse limit defines a point of $\widetilde{X}$ whose image in X induces s.

The proposition follows from the lemma because the splitting of (1.2) for X by a section $s : \mathrm{Gal}(k_s|k) \to \pi_1(X,\bar{x})$ implies its splitting for X_H when $H \supset s(\mathrm{Gal}(k_s|k))$.

Remark 3.8 In recent years a birational analogue of the Section Conjecture was also studied. If K is the function field of a smooth proper curve X over a field k of characteristic 0, a k-rational point P of X induces a conjugacy class of sections of the exact sequence of Galois groups

$$1 \to \mathrm{Gal}(\overline{K}|K\bar{k}) \to \mathrm{Gal}(\overline{K}|K) \to \mathrm{Gal}(\bar{k}|k) \to 1.$$

Indeed, the local ring of P is a discrete valuation ring with fraction field K and residue field k; a decomposition group $D_P \subset \mathrm{Gal}(\overline{K}|K)$ of this valuation is isomorphic to $\mathrm{Gal}(\overline{k((t))}|k((t)))$. The natural projection $\mathrm{Gal}(\overline{k((t))}|k((t))) \to \mathrm{Gal}(\bar{k}|k)$ has a section; its image is the subgroup of $\mathrm{Gal}(\overline{k((t))}|k((t)))$ fixing the extension of $k((t))$ obtained by adjoining the n-th roots of t for all $n > 1$. The composite map $\mathrm{Gal}(\bar{k}|k) \to \mathrm{Gal}(\overline{k((t))}|k((t)) \xrightarrow{\sim} D_P \to \mathrm{Gal}(\overline{K}|K)$ is a section as required.

Each k-point P is uniquely determined by the conjugacy classes of sections it induces because the arising D_P pairwise intersect trivially by an old theorem of F. K. Schmidt [42]. One may then ask whether the analogue of the Section Conjecture holds over arithmetically interesting fields. Koenigsmann [22] observed that the answer is yes if k is a p-adic field (see also Pop [38] for a sharpened version). Over a global field there are only partial results, but in contrast to the original conjecture of Grothendieck at least examples are known of curves having rational points where the answer is positive ([15], [45]).

1.4 Parshin's Proof of Mordell's Conjecture Over Function Fields

Let B be a smooth projective connected curve over the field $\mathbf{C}$ of complex numbers, and let C be a smooth projective connected curve defined over the function field $\mathbf{C}(B)$ of B. The following statement is usually called the geometric case of Mordell's Conjecture or the Mordell Conjecture for function fields of characteristic 0.

Theorem 4.1 *Assume that there is no finite extension $K|\mathbf{C}(B)$ for which the base changed curve $C \times_{\mathbf{C}(B)} K$ can be defined over $\mathbf{C}$. Then C has only finitely many $\mathbf{C}(B)$-rational points.*

As a consequence, one gets the same result over finitely generated base fields of characteristic 0 (assuming B geometrically integral).

This famous theorem has several proofs. The first one was given by Manin [26]; Coleman later discovered that it contained a gap which he was able to fill in [4]. The first complete published proof seems to be that of Grauert [12]. Parshin himself gave two proofs ([35], [36]); it is the second one that we are going to explain now. As we shall see, it is partly inspired by the ideas explained in the previous section.

We prove the following equivalent statement.

Theorem 4.2 *Let V be a smooth projective surface equipped with a proper flat morphism $p : V \to B$ with generic fibre C as above. If V as a family over B is non-isotrivial, then the projection p has only finitely many sections.*

Recall that the family $p : V \to B$ is *isotrivial* if there is a finite flat base change $B' \to B$ such that $V \times_B B' \to B'$ is a trivial family (i.e. isomorphic to a direct product $C' \times B'$).

To see the equivalence of the two statements, note that one may find a smooth projective surface $\widetilde{V}$ over $\mathbf{C}$ whose function field is that of the curve C of Theorem 4.1, by resolution of singularities for surfaces. The inclusion $\mathbf{C}(B) \to \mathbf{C}(\widetilde{V})$ induces a rational map $\widetilde{V} \to B$ with generic fibre C; by elimination of indeterminacy we find a blowup V of $\widetilde{V}$ in finitely many points equipped with a morphism $p : V \to B$ as required. A section of p induces a section on the generic fibre. On the other hand, by properness of V any section of the projection $C \to \mathrm{Spec}\, \mathbf{C}(B)$ extends uniquely to a section of p.

Strategy of the proof of Theorem 4.2. Choose a Zariski open subset $B_0 \subset B$ such that p is smooth over B_0. Fix a point $b_0 \in B_0$, and denote by F the fibre $p^{-1}(b_0)$. Fixing a base point $v_0 \in F$, we have a homotopy exact sequence of topological fundamental groups

$$1 \to \pi_1^{\mathrm{top}}(F, v_0) \to \pi_1^{\mathrm{top}}(V_0, v_0) \xrightarrow{p_*} \pi_1^{\mathrm{top}}(B_0, b_0) \to 1$$

where $V_0 = p^{-1}(B_0)$. A section $s_0 : B_0 \to V_0$ of p over B_0 meets F in a point v_1, whence a map $s_{0*} : \pi_1^{\mathrm{top}}(B_0, b_0) \to \pi_1^{\mathrm{top}}(V_0, v_1)$. Fixing a path from v_0 to v_1 induces an isomorphism $\pi_1^{\mathrm{top}}(V_0, v_1) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(V_0, v_0)$; it is unique up to inner automorphism.

By composition s_{0*} induces a section of the map p_* above. Therefore we obtain a map

$$S : \{\text{sections of } p|_{V_0} : V_0 \to B_0\} \to \{\text{conjugacy classes of sections of } p_*\}.$$

As any section of p is determined by its restriction to B_0, the theorem follows from the two claims below. □

Claim 4.3 *The map S has finite fibres.*

Claim 4.4 *The map S has finite image.*

We begin with the proof of Claim 4.3. First we recall the notion of $K|k$-trace for abelian varieties. Given a field extension $K|k$ and an abelian variety A over K, the $K|k$-trace $\mathrm{tr}_{K|k}(A)$ is the k-abelian variety characterized by the property that

$$\mathrm{Hom}(B_K, A) \xrightarrow{\sim} \mathrm{Hom}(B, \mathrm{tr}_{K|k}(A))$$

for all k-abelian varieties B. Its existence is a theorem of Chow; see ([18], Appendix A) or [5] for modern proofs. Applying the defining property with $B = \mathrm{Spec}\,(k)$ we obtain a bijection $A(K) \xrightarrow{\sim} \mathrm{tr}_{K|k}(A)(k)$; its inverse is induced by the map $\tau : \mathrm{tr}_{K|k}(A)_K \to A$ obtained by setting $B = \mathrm{tr}_{K|k}(A)$ and taking the map corresponding to the identity. The image of τ is the maximal abelian subvariety of A defined over k.

Proof of Claim 4.3. The diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Spec}\,\mathbf{C}(B) \\ \downarrow & & \downarrow \\ V_0 & \longrightarrow & B_0 \end{array}$$

is Cartesian, so a section $s_0 : B_0 \to V_0$ induces a section $s : \mathrm{Spec}\,\mathbf{C}(B) \to C$; moreover, s_0 is uniquely determined by s. On the other hand, a section $\pi_1^{\mathrm{top}}(B_0, b_0) \to \pi_1^{\mathrm{top}}(V_0, v_0)$ induces a map on profinite completions, i.e. a map $\pi_1(B_0, b_0) \to \pi_1(V_0, v_0)$ of *algebraic* fundamental groups. For some geometric point c_0 of C above v_0 the diagram of groups

$$\begin{array}{ccc} \pi_1(C, c_0) & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{C}(B)}|\mathbf{C}(B)) \\ \downarrow & & \downarrow \\ \pi_1(V_0, v_0) & \longrightarrow & \pi_1(B_0, b_0) \end{array}$$

coming from the previous diagram of schemes commutes, the sections s and s_0 inducing compatible sections of the horizontal maps. Hence it is enough to show that the map

$$C(\mathbf{C}(B)) \to \{\text{conjugacy classes of sections of } \pi_1(C, c_0) \to \mathrm{Gal}(\overline{\mathbf{C}(B)}|\mathbf{C}(B))\}$$

has finite fibres. This is done as in the injectivity part of the section conjecture. If y_0 is a a $\mathbf{C}(B)$-point of C and y another $\mathbf{C}(B)$-point inducing the same section $\mathrm{Gal}(\overline{\mathbf{C}(B)}|\mathbf{C}(B)) \to \pi_1(C, c_0)$, then the argument given there shows that the class of the divisor $y - y_0$ is divisible in $J(\mathbf{C}(B))$. But by the Lang–Néron theorem (see [19] for a beautiful short proof) the group $J(\mathbf{C}(B))/\tau(\mathrm{tr}_{\mathbf{C}(B)|\mathbf{C}}(J)(\mathbf{C}))$ is finitely generated and as such has no nontrivial divisible element. Therefore the image of y by the embedding $C \to J$ with base point y_0 lies in the image of the trace $\mathrm{tr}_{\mathbf{C}(B)|\mathbf{C}}(J)$. But if C is non-isotrivial, the whole of C cannot lie in the trace (this can be checked using the explicit construction of the trace in [18]). Their intersection is thus a proper closed, hence finite subset of C, which shows that there can be only finitely many points y inducing the same section as y_0. □

The proof of Claim 4.4 is entirely topological. The idea is to bound the 'size' of sections of p_* in a suitable way. This is accomplished using ideas of complex hyperbolic geometry, of which we summarize here some basic facts (see [21] and [23] for proofs and much more).

Equip the complex unit disc D with the Poincaré metric given by $z \mapsto (1-|z|^2)^{-1}$. It defines a distance function d_{hyp} on D which we may use to define the *Kobayashi pseudo-distance* on any complex manifold X:

$$d_X(x,y) = \inf\left(\sum_{i=1}^{r} d_{\mathrm{hyp}}(p_i, q_i)\right)$$

where the infimum is taken over systems of points $p_i, q_i \in D$ $(1 \le i \le r)$ for which there exist holomorphic maps $f_1, \dots, f_r : D \to X$ with $f_1(p_1) = x$, $f_r(q_r) = y$ and $f_i(q_i) = f_{i+1}(p_{i+1})$. Holomorphic maps are distance-decreasing: if $\phi : X \to Y$ is a holomorphic map, then $d_Y(\phi(x), \phi(y)) \le d_X(x,y)$ (this follows from the case $X = Y = D$, where it is a consequence of Schwarz's lemma).

The pseudo-distance d_D is identically 0, so d_X does not satisfy $d_X(x,y) \neq 0$ for $x \neq y$ in general. The manifold X is said to be *hyperbolic* if $d_X(x,y) \neq 0$ for $x \neq y$, and in this case we get a distance function that can be used to define the length of a path in X. Given a holomorphic map $X \to Y$ which is topologically a cover, it is known that the hyperbolicity of Y implies that of X.

By a classical theorem of Brody, a compact manifold X is hyperbolic if and only if there is no non-constant holomorphic map $\mathbf{C} \to X$. In particular, a compact Riemann surface of genus $g > 1$ is hyperbolic and we may obtain a hyperbolic manifold from any compact Riemann surface after removing finitely may open discs. Also, a fibred complex manifold with base and fibre of this type is again hyperbolic.

So in our case we can make a hyperbolic manifold V' out of V_0 by removing the preimage of finitely many open discs in B (which we may assume to contain the finitely many deleted points of B). Write $B' = p(V')$ and assume that the fixed fibre F (and in particular the base point v_0) lies in V'. The inclusion $V' \hookrightarrow V_0$ then induces an isomorphism $\pi_1^{\mathrm{top}}(V', v_0) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(V_0, v_0)$ since V' is a deformation retract of V_0. Similarly, we have a canonical isomorphism $\pi_1^{\mathrm{top}}(B', b_0) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(B_0, b_0)$.

Lemma 4.5 *For each $C > 0$ there are only finitely many elements of $\pi_1^{\rm top}(V_0, v_0) \cong \pi_1^{\rm top}(V', v_0)$ that can be represented by paths lying in V' that have length at most C in the hyperbolic metric of V'.*

Proof. Consider the universal cover $\widetilde{V}' \to V'$; it carries a canonical holomorphic structure. Any holomorphic map $D \to V'$ lifts to $\widetilde{V}'$, therefore the definition of the pseudo-distance implies that liftings of paths of length $\leq C$ starting at v_0 stay inside a closed ball of radius C. As V' is hyperbolic, so is the cover $\widetilde{V}'$, and therefore the ball is compact. Closed paths around v_0 lift to paths with endpoints contained in a fixed orbit of $\pi_1^{\rm top}(V', v_0)$. As these orbits are discrete, they intersect the compact ball in finitely many points. □

Now fix generators $x_1, \dots, x_r$ of the finitely generated group $\pi_1^{\rm top}(B_0, b_0)$. In view of the lemma, Claim 4.4 is a consequence of:

Proposition 4.6 *There exists a constant $C > 0$ such that for every section $s : B_0 \to V_0$ the images of $x_1, \dots, x_n$ under the induced map $\pi_1^{\rm top}(B_0, b_0) \to \pi_1^{\rm top}(V_0, v_0)$ can be represented by paths lying in V' that have length at most C.*

Proof. Let $s : B_0 \to V_0$ be a section, and let $s' : B' \to V'$ be its restriction to B'. We may identify the map $\pi_1^{\rm top}(B_0, b_0) \to \pi_1^{\rm top}(V_0, v_0)$ induced by s with the map $\pi_1^{\rm top}(B', b_0) \to \pi_1^{\rm top}(V', v_0)$ induced by s' and hence we may assume that the x_i are represented by closed paths γ_i lying inside B'. As holomorphic maps are distance-decreasing, we have for $x, y \in s'(B')$ a sequence of inequalities

$$d_{s'(B')}(x,y) \geq d_{V'}(x,y) \geq d_{B'}(p(x), p(y)) \geq d_{s'(B')}(x,y)$$

induced by the maps $s'(B') \hookrightarrow V' \xrightarrow{p} B' \xrightarrow{s'} s'(B')$. Thus we have equality throughout, which shows that for each i the length of $s(\gamma_i)$ calculated with respect to $d_{V'}$ is the same as that of γ_i with respect to $d_{B'}$. This gives a uniform bound on the V'-length of the $s(\gamma_i)$. A representative of $s_*(x_i)$ in $\pi_1^{\rm top}(V_0, v_0)$ is given by $\gamma s(\gamma_i)\gamma^{-1}$, where γ is a path lying in $F \subset V'$ joining v_0 to $s(b_0)$. But F is a compact hyperbolic Riemann surface, so we may join v_0 to any point by a path of length bounded by an absolute constant (e.g. a geodesic). This proves the proposition, and thereby Claim 4.4. □

1.5 Anabelian Geometry

By 'anabelian geometry' one refers to a sheaf of conjectures formulated by Grothendieck in a famous letter to Faltings [14]. The rough idea is that a certain category of schemes defined over finitely generated fields should be determined by their geometric fundamental groups together with its outer Galois action.

There are two kinds of motivation for the conjectures. The first one comes from topology.

Fact 5.1 Recall that for a smooth proper curve X of genus ≥ 2 over $\mathbf{C}$ the topological fundamental group has a presentation

$$\Pi = \langle a_1, b_1, \ldots, a_g, b_g \,|\, [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$$

This group is non-commutative; moreover, it has trivial center. The universal cover of X is the unit disc D which is contractible. Therefore the higher homotopy groups $\pi_q(X)$ are trivial for $q \geq 2$, and so X is the *Eilenberg-MacLane space* $K(\Pi, 1)$, and as such it is determined up to homotopy.

As an algebraic curve, X may be defined over a finitely generated extension k of $\mathbf{Q}$. The hope therefore arises that the extra structure given by Galois action on Π may determine X up to algebraic isomorphism, not just up to homotopy.

The second motivation comes from the Tate conjecture.

Fact 5.2 Let k now be a number field, X_1, X_2 smooth proper curves over k, of genus ≥ 2. Assume for simplicity that both have a k-point. These k-points can be used to embed X_i in its Jacobian J_i. Write $\overline{X}_i := X_i \times_k \bar{k}$ and similarly for J_i. We know that for each prime ℓ and $i = 1, 2$

$$T_\ell(\overline{J}_i) \cong \pi_1^{\mathrm{ab}}(\overline{J}_i)^{(\ell)} \cong \pi_1^{\mathrm{ab}}(\overline{X}_i)^{(\ell)}$$

where T_ℓ stands for the ℓ-adic Tate module (Remark 2.2).

By a fundamental theorem of Faltings [9] (ex Tate conjecture) the natural map

$$\mathrm{Hom}(J_1, J_2) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(\overline{J}_1), T_\ell(\overline{J}_2))^{\mathrm{Gal}(\bar{k}|k)}$$

is bijective. In other words, Galois-equivariant homomorphisms $T_\ell(\overline{J}_1) \to T_\ell(\overline{J}_2)$ can be 'approximated ℓ-adically' by morphisms $J_1 \to J_2$.

One can ask here whether working with the whole geometric fundamental group instead of its abelian quotient can give a stronger result: does a Galois-invariant outer homomorphism $\pi_1(\overline{X}_1) \to \pi_1(\overline{X}_2)$ come from a k-morphism $X_1 \to X_2$? Or, even more economically, does a Galois-invariant outer homomorphism $\pi_1(\overline{X}_1)^{(\ell)} \to \pi_1(\overline{X}_2)^{(\ell)}$ between maximal pro-ℓ-quotients come from a map of curves?

Before formulating precise statements, let us elucidate the role of center-freeness. Recall that the representation

$$\rho_X : \mathrm{Gal}(\bar{k}|k) \to \mathrm{Out}(\pi_1(\overline{X}, \bar{x}))$$

is defined using the exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{X}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & \mathrm{Gal}(\bar{k}|k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & \mathrm{Aut}(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & \mathrm{Out}(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & 1.
\end{array}$$

Observe: when the center of $\pi_1(\overline{X},\bar{x})$ is trivial, this becomes a pushout diagram. Therefore

$$\pi_1(X,\bar{x}) \cong \mathrm{Aut}(\pi_1(\overline{X},\bar{x})) \times_{\mathrm{Out}(\overline{X},\bar{x})} \mathrm{Gal}(\bar{k}|k),$$

i.e. $\pi_1(X,\bar{x})$ is determined by $\pi_1(\overline{X},\bar{x})$ and ρ_X. When $k \subset \mathbf{C}$, it thus appears as a "transcendental object" endowed with a Galois action.

We now define a category of profinite groups as follows. Given two profinite groups G_1, G_2 together with morphisms $p_i : G_i \to G$, define $Hom_G^*(G_1,G_2)$ as the set of morphisms $G_1 \to G_2$ compatible with the p_i up to conjugation by an element of G. This set carries an action of G_1 from the left and of G_2 from the right. The latter defines a finer equivalence, so put $Hom_G^{ext}(G_1,G_2) = Hom_G^*(G_1,G_2)$ modulo action of G_2. Fixing G we thus get a category $\mathbf{Prof}_G^{\mathrm{ext}}$ with objects profinite groups with projections onto G and Hom-sets the $Hom_G^{ext}(G_1,G_2)$. Denote by $\mathbf{Prof}_G^{\mathrm{ext,open}}$ the full subcategory with the same objects but with morphisms having open image.

Sending a variety over a field k to its algebraic fundamental group gives a functor

$$\pi_1 : \{k\text{-varieties}\} \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{\mathrm{ext}}$$

where base points do not play a role any more, so we drop them from now on.

Similarly, sending a field to its absolute Galois group yields a contravariant functor

$$\mathrm{Gal} : \{\text{field extensions of } k\} \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{ext}.$$

In his letter to Faltings Grothendieck formulated the following conjecture.

Conjecture 5.3 *Let k be a finitely generated extension of $\mathbf{Q}$. Denote by $\mathbf{Hyp}_k$ the category of hyperbolic k-curves equipped with* dominating *k-morphisms. The functor*

$$\pi_1 : \mathbf{Hyp}_k \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{\mathrm{ext,open}}$$

is fully faithful.

Recall that hyperbolic k-curves are the smooth k-curves of genus g with at least $2-2g+1$ geometric points at infinity. These are precisely the smooth curves with non-trivial center-free geometric π_1. Grothendieck also speculated about extending $\mathbf{Hyp}_k$ by including some higher-dimensional varieties called 'anabelian varieties'. At present there is no precise conjectural characterization of anabelian varieties in dimensions > 1. However, there is a precisely formulated birational analogue:

Conjecture 5.4 *Let k be finitely generated over $\mathbf{Q}$. Denote by $\mathbf{Bir}_k^{\mathrm{dom}}$ the category of fields finitely generated over k together with k-morphisms. Then the contravariant functor*

$$\mathrm{Gal} : \mathbf{Bir}_k^{\mathrm{dom}} \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{ext,open}$$

is fully faithful.

Here are the most important known results about these conjectures.

Theorem 5.5 (Mochizuki [29]) *Conjecture 5.3 is true more generally for k sub-p-adic, i.e. a subfield of some finitely generated extension of a* $\mathbf{Q}_p$. *In fact, over such fields the following holds: for a hyperbolic k-curve X and an arbitrary smooth k-variety V the map*

$$\mathrm{Hom}_k^{dom}(V,X) \to \mathrm{Hom}_{\mathrm{Gal}(k)}^{\mathrm{ext,open}}(\pi_1(V),\pi_1(X))$$

is bijective. Here π_1 *may be replaced by its quotient* π_1^p *classifying covers whose base change to* $\bar{k}$ *is of p-power degree.*

This is all the more remarkable as the Tate conjecture does not hold over $\mathbf{Q}_p$! Concerning the birational version, we have:

Theorem 5.6

1. (Pop, [37], [47]) *The isomorphism version of conjecture 5.4 is true, even in positive characteristic. More precisely, if K, L are finitely generated fields (over the prime field), the natural map*

$$\mathrm{Isom}^i(K,L) \to \mathrm{Isom}^{ext}(\mathrm{Gal}(L),\mathrm{Gal}(K))$$

is bijective, where on the left Isom^i *means isomorphisms between purely inseparable extensions* $K'|K$ *and* $L'|L$.
2. (Mochizuki [29]) *Conjecture 5.4 is true more generally for k sub-p-adic.*

Here part (2) has been recently improved by Corry and Pop [6]: one can replace $\mathrm{Gal}(K)$ (and similarly $\mathrm{Gal}(L)$ by its natural quotient obtained as an extension of $\mathrm{Gal}(k)$ by the maximal pro-p quotient of the subgroup $\mathrm{Gal}(K\bar{k})$. Thus one has a birational result that is completely analogous to Theorem 5.5. However, the positive characteristic analogue is not known at present.

Remark 5.7 The statements of Theorem 5.6 are of arithmetic nature. In remarkable contrast to this, Bogomolov [2] initiated a program according to which finitely generated fields of transcendence degree at least 2 over *algebraically closed* fields should be characterized up to isomorphism by their absolute Galois group, and even by its maximal pro-ℓ nilpotent quotient of class 2, for a prime ℓ different from the characteristic. There has been important recent progress in this direction, by Bogomolov–Tschinkel [3], and in a series of preprints by Pop ([39] being the last one to date).

Observe that Pop's result does not use the augmentation $\mathrm{Gal}(K) \to \mathrm{Gal}(k)$. This hints at the possibility that 'absolute' forms of Grothendieck's conjecture hold true. And indeed, Mochizuki proved by combining Theorem 5.3 and Theorem 5.4 (1):

Theorem 5.8 (Mochizuki [31]) *Let X and Y be hyperbolic curves defined over some finitely generated extension of* $\mathbf{Q}$ *(not necessarily the same). Then the natural map*

$$\mathrm{Isom}(X,Y) \to \mathrm{Isom}^{ext}(\pi_1(X),\pi_1(Y))$$

is bijective.

Even more surprisingly, 'absolute' results hold over a finite base field:

Theorem 5.9

1. (Tamagawa [49]) *Let X and Y be smooth affine curves defined over some finite field (not necessarily the same) with profinite universal covers $\widetilde{X}$, $\widetilde{Y}$, respectively. Then the natural map*

$$\mathrm{Isom}(\widetilde{X}|X, \widetilde{Y}|Y) \to \mathrm{Isom}(\pi_1(X), \pi_1(Y))$$

is bijective. Here on the left hand side we have the set of commutative diagrams of isomorphisms

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\cong} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\cong} & Y. \end{array}$$

2. (Mochizuki [31]) *The same statement holds for proper smooth curves of genus ≥ 2 over a finite field.*

Here the profinite universal cover of an normal integral scheme S means its normalization in the field K_S of Proposition 1.6.

Remarks 5.10

1. Recently Saïdi and Tamagawa [41] proved that in the theorem above one may replace fundamental groups by their maximal prime-to-p quotients (where p is the characteristic of the base field). They also proved results with even smaller quotients but it is not known whether the statement holds for the maximal pro-ℓ quotients of the fundamental groups (where $\ell \neq p$ is a prime).
2. Before the full statement of Theorem 5.5 was proven, Tamagawa and Mochizuki used specialization arguments to derive the statement of Theorem 5.5 for isomorphisms of hyperbolic curves over number fields from Theorem 5.9 (1). Stix [44] used a similar method to prove an isomorphism statement for hyperbolic curves over global fields of positive characteristic.

Although in the oral lectures I gave a sketch of some of the ideas involved in the proofs of the above results, I feel that one cannot do them justice in just a couple of pages. On the other hand, besides the mostly well-written original papers there are quite a few detailed surveys that the interested reader may consult with profit. So let me conclude with some bibliographic indications.

The first results that can be associated with anabelian geometry, though they actually predate the formulation of the conjectures, are the theorems of Neukirch [33] and Uchida [50] concerning Galois characterization of global fields. They can now be viewed as special cases of Theorem 5.6 (1) of Pop, but in fact their methods have been highly inspirational for the proof of the general result. A nice exposition can be found in the last chapter of the book [34] by Neukirch, Schmidt and Wingberg. As for Pop's theorem, the reader may consult my Bourbaki exposé [47].

The impact of the Neukirch-Uchida techniques can also be seen in Tamagawa's proof of Theorem 5.9 (1); the recent results of Saïdi–Tamagawa mentioned above refine this method further. An introduction to these ideas can be found in [46]. But the best introduction to the contributions of Tamagawa and Mochizuki is the survey paper [32] which also includes an overview of the work of Nakamura that contains germs of many of the ideas that were developed later. Mochizuki's methods are completely different from the approach initiated by Neukirch and Uchida and are based on constructions in p-adic Hodge theory. A succinct survey can be found in the Bourbaki exposé [10] by Faltings.

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