PRIME-TO-p ÉTALE COVERS OF ALGEBRAIC GROUPS AND HOMOGENEOUS SPACES

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1. Introduction

By a classical result of Schreier [24], the fundamental group of a connected and locally connected topological group G is commutative. If moreover G is (semi-)locally simply connected, then every Galois cover $\phi : Y \to G$ carries a group structure for which ϕ is a homomorphism. Thus G is the quotient of Y by an abelian normal subgroup.

The first part of the following proposition states an analogue of this result in algebraic geometry. The second part gives a bound on the number of topological generators of the prime-to-p fundamental group. To state it, we need to introduce some notation. Recall that by Chevalley's theorem G is an extension of an abelian variety A by a linear algebraic group G_{aff} (see [4], [5], [6], [22]). Denote by g the dimension of A and by r the rank of G_{aff} (which is by definition the dimension of a maximal torus). Furthermore, denote by $\mathbf{Z}_{(p')}$ the direct product of the rings $\mathbf{Z}_\ell$ for $\ell \neq p$.

Proposition 1.1. *Let G be a connected algebraic group over an algebraically closed field of characteristic $p \geq 0$.*

a) *Every étale Galois cover $Y \to G$ of degree prime to p carries the structure of a central isogeny. In particular, the maximal prime-to-p quotient $\pi_1^{(p')}(G)$ of the étale fundamental group of G is commutative.*
b) *The group $\pi_1^{(p')}(G)$ is a quotient of $\mathbf{Z}_{(p')}^{2g+r}$.*

By convention, all Galois covers are assumed to be connected. Statement *a*) was known in the special cases where G is an abelian variety (Lang–Serre [12]) or G is affine (Miyanishi [16], Magid [14]). Note that restricting to prime-to-p covers is crucial because if one allows p-covers in characteristic $p > 0$, the statement fails disastrously already for the additive group $\mathbf{G}_a$ (Raynaud [21]).

In part *b*) the bound is sharp (take the direct product of an abelian variety and a torus). In the case when G is commutative it follows from Theorem 1.1 and ([26], §6.4, Cor. 4) that $\pi_1^{(p')}(G)$ is moreover a free $\mathbf{Z}_{(p')}$-module. However, in the non-commutative case it may contain torsion. For instance, $\pi_1(\mathrm{SO}(n))$ in characteristic 0 is of order 2. (Of

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course, in this case the bound on the number of generators given by the proposition is very far from the truth.)

We also prove a generalization to homogeneous spaces. Bounding the topological rank is more difficult than in the group case, so we state this result as a theorem.

Theorem 1.2. *Keep the assumptions of the proposition above, and let X be a variety on which G acts transitively with connected stabilizers.*

a) *Every étale Galois cover of X of degree prime to p is of the form $\tilde{G}/\tilde{H}$ for a central isogeny $\tilde{G} \to G$ and a lift $\tilde{H}$ of the stabilizer H of a point of X in $\tilde{G}$.*
b) *The group $\pi_1^{(p')}(X)$ is a quotient of $\mathbf{Z}_{(p')}^{2(g-g_H)+r-r_H}$, where g_H and r_H are the dimension of the abelian variety quotient and the rank of the subgroup H, respectively.*

In work in progress, M. Borovoi and C. Demarche [2] have obtained an extension of statement *a*) to non-connected H. In this case $\tilde{H}$ is a finite index subgroup of the preimage of H in $\tilde{G}$. However, statement *b*) fails when the isotropy group H is not connected. For example, if Γ is a finite group of order prime to p and $\Gamma \hookrightarrow \mathrm{GL}(n)$ a faithful representation, then $\pi_1^{(p')}(\mathrm{GL}(n)/\Gamma)$ is just Γ.

We also present an application of Proposition 1.1 which was our original motivation for studying the problem. It concerns commutative algebraic groups.

Theorem 1.3. *Let G be a connected commutative algebraic group over an algebraically closed field k. For n prime to the characteristic there is an exact sequence*

$$0 \to \mathrm{Pic}\, G/n\mathrm{Pic}\, G \to \mathrm{Hom}(\Lambda^2({}_nG), \mu_n) \to {}_n\mathrm{Br}\, G \to 0$$

where $\mathrm{Pic}\, G$ and $\mathrm{Br}\, G$ denote the Picard and Brauer groups of G, respectively, and ${}_nG$ stands for the n-torsion part of G.

The case where G is an abelian variety was established by Berkovich [1]. The proof in the general case uses Proposition 1.1 as an input, in the identification of the group $\mathrm{Hom}({}_nG, \mu_n)$ with the étale cohomology group $H^1(G, \mu_n)$. However, this identification can also be derived using a more elementary method; see the conclusion of Section 4.

The following almost immediate corollary of Theorem 1.3 makes a classical formula of Grothendieck ([9], corollaire 3.4) explicit in the case of a connected commutative algebraic group G.

Corollary 1.4. *Let ℓ be a prime number different from the characteristic of k. The ℓ-primary torsion part of $\mathrm{Br}\, G$ is isomorphic to $(\mathbf{Q}_\ell/\mathbf{Z}_\ell)^{(2g+r)(2g+r-1)/2-\rho}$, where g and r are defined as above, and ρ is the rank of the Néron–Severi group of A, the abelian variety quotient of G.*

Theorem 1.3 and Corollary 1.4 will be proven in Section 4.

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2. Algebraic groups

In this section we prove Proposition 1.1. Our first proof of statement a) was quite involved, using, among other ingredients, a fairly recent result of Orgogozo [20] on compatibility of the prime-to-p fundamental groups with products. As Hélène Esnault pointed out, this result together with a modification of the classical argument of Lang and Serre for abelian varieties suffices for the proof of the statement. We briefly recall the argument (see the original paper [12] or [27], Theorem 5.6.10 and Remark 5.6.11 for more details).

Denote by $m_G : G \times G \to G$ the group operation of G. Given an étale Galois cover $Y \to G$ of degree prime to p, consider the fibre product square

$$\begin{array}{ccc} Y' = (G\times G)\times_G Y & \longrightarrow & Y \\ \downarrow & & \downarrow \phi \\ G\times G & \xrightarrow{m_G} & G. \end{array}$$

One checks that Y' is also connected, hence $Y' \to G\times G$ is a Galois étale cover with group $\Gamma := \mathrm{Aut}(Y|G)$. Since the prime-to-$p$ fundamental group is compatible with direct products (Orgogozo [20], Cor. 4.9), there exist étale Galois covers $Z_1 \to G$ and $Z_2 \to G$, with groups Γ_1 and Γ_2, respectively, such that Y' is a quotient of the direct product $Z_1 \times Z_2 \to G \times G$. In particular, there is a normal subgroup $\Delta \subset \Gamma_1 \times \Gamma_2$ with $\Gamma \cong (\Gamma_1 \times \Gamma_2)/\Delta$. Replacing the Z_i by their quotients by the actions of the $\Gamma_i \cap \Delta$ we may assume $\Delta \cap \Gamma_i = \{1\}$ for $i = 1, 2$. Thus the Γ_i may be identified with normal subgroups of Γ that generate Γ and centralize each other. But by construction the base change $Y'_2 \to \{1\} \times G$ of Y' by the map $\{1\} \times G \to G \times G$ must be an étale Galois cover with group Γ_2, and similarly for the base change $Y'_1 \to G \times \{1\}$. It follows that $\Gamma = \Gamma_1 = \Gamma_2$, and Γ is commutative. Moreover, we have isomorphisms $Z_i \cong Y'_i \cong Y$ for $i = 1, 2$. This yields a map $Y \times Y \to Y'$, whence also a map $m_Y : Y \times Y \to Y$ by composing with the projection $Y' \to Y$. Fix a point 1_Y of Y in the fibre above the neutral element of G. Modifying m_Y by an automorphism of Y if necessary we may assume $m_Y(1_Y, 1_Y) = 1_Y$. A standard argument ([27], pp. 180–181) then shows that m_Y equips Y with the structure of an algebraic group.

We now turn to the proof of part b) and take up the notation introduced before the statement of the proposition. We already know that $\pi_1^{(p')}(G)$ is commutative. In the case when G is an abelian variety of dimension g it is moreover free over $\mathbf{Z}_{(p')}$ of rank $2g$ (see e.g. [27], Theorem 5.6.10 and Remark 5.6.11). Hence it will be enough to prove the following: the natural map $\alpha_* : \pi_1^{(p')}(G) \to \pi_1^{(p')}(A)$ induced by the projection $\alpha : G \to A$ is surjective and its kernel is a quotient of $\mathbf{Z}^r_{(p')}$.

Denote by $\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}$ the direct sum of the torsion abelian groups $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ for $\ell \neq p$. Since $\pi_1^{(p')}(G)$ is commutative, there is a perfect pairing

$$\pi_1^{(p')}(G) \times H^1_{\text{ét}}(G, \mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}) \to \mathbf{Q}/\mathbf{Z}$$

obtained by passing to the limit from the perfect pairings

$$\pi_1^{(p')}(G)/m\pi_1^{(p')}(G) \times H^1_{\text{ét}}(G, \mathbf{Z}/m\mathbf{Z}) \to \mathbf{Z}/m\mathbf{Z}$$

for m prime to p.

The Leray spectral sequence

$$H^p_{\text{ét}}(A, \mathbf{R}^q\alpha_*\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}) \Rightarrow H^{p+q}_{\text{ét}}(G, \mathbf{Q}_{(p')}/\mathbf{Z}_{(p')})$$

associated with the projection α yields an exact sequence

$$\begin{aligned} 0 \to H^1_{\text{ét}}(A, \alpha_*\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}) \to H^1_{\text{ét}}(G, \mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}) \to& \\ \to H^0_{\text{ét}}(A, \mathbf{R}^1\alpha_*\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}).& \end{aligned}$$

As α has connected fibres, $H^1_{\text{ét}}(A, \alpha_*\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}) \cong H^1_{\text{ét}}(A, \mathbf{Q}_{(p')}/\mathbf{Z}_{(p')})$, which shows that the map between the first two terms in the above exact sequence is the $\mathbf{Q}/\mathbf{Z}$-dual of the map $\alpha_* : \pi_1^{(p')}(G) \to \pi_1^{(p')}(A)$. It follows that the latter map is surjective.

The stalk of the étale sheaf $\mathbf{R}^1\alpha_*\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}$ at a geometric point can be identified with $H^1_{\text{ét}}(G_{\text{aff}}, \mathbf{Q}_{(p')}/\mathbf{Z}_{(p')})$. Pick a maximal torus $T \subset G_{\text{aff}}$. Miyanishi proves in ([16], Lemma 3) that the natural map $\pi_1^{(p')}(T) \to \pi_1^{(p')}(G_{\text{aff}})$ is surjective (see also Lemma 3.4 below for a more general statement). Since $\pi_1^{(p')}(T) \cong \pi_1^{(p')}(\mathbf{G}_m)^r \cong \mathbf{Z}^r_{(p')}$, this bounds the number of generators of $\pi_1^{(p')}(G_{\text{aff}})$ by r. But since α is smooth, the sheaf $\mathbf{R}^1\alpha_*\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}$ is locally constant and thus $H^0_{\text{ét}}(A, \mathbf{R}^1\alpha_*\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')})$ becomes a subgroup of $H^1_{\text{ét}}(T, \mathbf{Q}_{(p')}/\mathbf{Z}_{(p')}) \cong (\mathbf{Q}_{(p')}/\mathbf{Z}_{(p')})^r$. By taking $\mathbf{Q}/\mathbf{Z}$-duals we obtain that the kernel of $\alpha_* : \pi_1^{(p')}(G) \to \pi_1^{(p')}(A)$ is indeed a quotient of $\mathbf{Z}^r_{(p')}$.

Remark 2.1. Jakob Stix has shown us the following very quick argument for the commutativity of $\pi_1^{(p')}(G)$. The point is that by a well-known reasoning $\pi_1^{(p')}(G)$ is a group object in the category of groups. In more detail: by compatibility of the functor $\pi_1^{(p')}$ with products ([20], Cor. 4.9 again) the multiplication map of G induces a map $m : \pi_1^{(p')}(G) \times \pi_1^{(p')}(G) \to \pi_1^{(p')}(G)$ that satisfies the group axioms by functoriality, and moreover $m(1, g) = m(g, 1) = g$ for all $g \in \pi_1^{(p')}(G)$. The calculation

$$hg = m(1, h)m(g, 1) = m((1, h)(g, 1)) = m(g, h)$$

$$= m((g,1)(1,h)) = m(g,1)m(1,h) = gh$$

shows that this group law is commutative and coincides with the usual group law of $\pi_1^{(p')}(G)$.

3. Homogeneous spaces

In this section we prove Theorem 1.2. We begin the proof of part a) with some standard observations and reductions. Let X be a variety on which a connected algebraic group G acts transitively. Choose a point $x \in X$ and denote by $H \subset G$ its isotropy subgroup scheme. Then $X \cong G/H$ and the natural map $G/H_{\mathrm{red}} \to G/H$ is finite, surjective and purely inseparable, where H_{red} denotes the reduced part of H. It follows that the induced map $\pi_1(G/H_{\mathrm{red}}) \to \pi_1(G/H)$ is an isomorphism. So we may assume throughout that H is is an algebraic group; moreover, as in the statement of the theorem, we assume it is connected.

Assume now $f : \tilde{X} \to X$ is an étale Galois cover with group Γ of order prime to p. Consider the orbit map $\varphi : G \to X$, $g \mapsto g \cdot x$, and form the cartesian square

$$\begin{tikzcd} \tilde{G} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "f"'] \\ G \arrow[r, "\varphi"] & X. \end{tikzcd}$$

Since φ is a flat (hence open) surjection with connected fibres, so is $\tilde{\varphi}$, so we infer from the connectedness of $\tilde{X}$ that $\tilde{G}$ is connected as well. By Proposition 1.1 a) the map π is therefore a central isogeny with kernel Γ. The algebraic subgroup $\pi^{-1}(H) \subset \tilde{G}$ is a disjoint union of fibres of $\tilde{\varphi}$; denote by $\tilde{H}$ its neutral component. Since H is connected by assumption, the map $\pi|_{\tilde{H}} : \tilde{H} \to H$ must be an isomorphism. Moreover, the map $\tilde{\varphi}$ is invariant under $\tilde{H}$ since so is $f\tilde{\varphi} = \varphi\pi$ and f is finite étale (see e.g. [27], Corollary 5.3.3). Thus $\tilde{\varphi}$ factors through a morphism $\tilde{G}/\tilde{H} \to \tilde{X}$, which is an isomorphism since so is the morphism $G/H \to X$. This completes the proof of statement a).

We now turn to the rank estimate of part b). Given $X = G/H$ as in the statement, we may assume without harm that G acts faithfully on X. Indeed, the (scheme-theoretic) kernel K of the action is contained in H. Then G/K acts faithfully on X with connected stabilizer H/K. By the lemma on p. 154 of [15] (see also [4], Corollary 2.1.9) the quotient H/K is affine. Thus the dimension g_H of the maximal abelian quotient of H equals that of the reduced neutral component of K, and hence the dimension of the maximal abelian quotient of G/K is $g - g_H$. Thus we may replace G by G/K and H by H/K and have a natural map $\alpha_X : X = G/H \to G/G_{\mathrm{aff}} = A$ (which is in fact the Albanese morphism of X, but we shall not need this).

Case I: the base field k has characteristic 0. The fundamental group of X being topologically finitely generated ([10], Exp. II), it does not change by extensions of algebraically closed fields by the argument of ([27], p. 186; see also [20], Theorem 4.11). Thus we reduce to the case $k = \mathbf{C}$ where the algebraic fundamental group is the profinite competion of the topological fundamental group $\pi_1^{\mathrm{top}}(X)$ ([8], Exp. XII, Cor. 5.2). It is then enough to see that $\pi_1^{\mathrm{top}}(X)$ is a quotient of $\mathbf{Z}^{2g+r-r_H}$. The fibration α_X with fibre G_{aff}/H yields an exact sequence

$$\pi_1^{\mathrm{top}}(G_{\mathrm{aff}}/H) \to \pi_1^{\mathrm{top}}(X) \to \pi_1^{\mathrm{top}}(A) \cong \mathbf{Z}^{2g}.$$

Thus it remains to show that $\pi_1^{\mathrm{top}}(G_{\mathrm{aff}}/H)$ is a quotient of $\mathbf{Z}^{r-r_H}$.

Choose a maximal torus $T_H \subset H$ and then a maximal torus $T \subset G_{\mathrm{aff}}$ containing T_H. Then we have a commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & G_{\mathrm{aff}} \\ \big\downarrow & & \big\downarrow \\ T/T_H & \longrightarrow & G_{\mathrm{aff}}/H \end{array}$$

and hence a similar commutative diagram of induced homomorphisms of topological fundamental groups.

Let B be a Borel subgroup of G_{aff} containing T. The flag variety G_{aff}/B is simply connected, and therefore the natural map $\pi_1^{\mathrm{top}}(B) \to \pi_1^{\mathrm{top}}(G_{\mathrm{aff}})$ is surjective. Also, $B = TU$, where $U \subset B$ denotes the unipotent part. Since U is isomorphic to an affine space, the natural map $\pi_1^{\mathrm{top}}(T) \to \pi_1^{\mathrm{top}}(B)$ is surjective as well, whence the surjectivity of the composite map $\pi_1^{\mathrm{top}}(T) \to \pi_1^{\mathrm{top}}(G_{\mathrm{aff}})$ (note that Miyanishi's argument mentioned in the previous section is an algebraic variant of this reasoning).

As $G_{\mathrm{aff}} \to G_{\mathrm{aff}}/H$ is a quotient map with connected fibres, the map $\pi_1^{\mathrm{top}}(G_{\mathrm{aff}}) \to \pi_1^{\mathrm{top}}(G_{\mathrm{aff}}/H)$ is also surjective, hence so is the map $\pi_1^{\mathrm{top}}(T/T_H) \to \pi_1^{\mathrm{top}}(G_{\mathrm{aff}}/H)$. But T/T_H is a torus of rank $r - r_H$, therefore $\pi_1^{\mathrm{top}}(T/T_H) \cong \mathbf{Z}^{r-r_H}$, which completes the proof of the characteristic 0 case.

Case II: k has characteristic $p > 0$. The argument will be a variant of the above but will use more background from the structure theory of algebraic groups.

Recall from ([7], III.3.8 and [22], Corollary 5, p. 440) that

$$G = G_{\mathrm{aff}} G_{\mathrm{ant}},$$

where $G_{\mathrm{ant}} \subset G$ denotes the largest anti-affine subgroup (that is, the largest subgroup with $\mathcal{O}(G_{\mathrm{ant}}) = k$); moreover, G_{ant} is connected and central in G. In positive characteristic G_{ant} is in fact a semi-abelian variety ([3], Proposition 2.2).

As before, choose a maximal torus $T_H \subset H$ and then a maximal torus $T \subset G_{\mathrm{aff}}$ containing T_H. Concerning TG_{ant} we shall prove below the following key proposition.

Proposition 3.1. *The natural map*

$$\pi_1(TG_{\mathrm{ant}})^{(p')} \to \pi_1(G)^{(p')}$$

is surjective.

Assuming the proposition, consider the commutative diagram

$$\begin{array}{ccc} TG_{\mathrm{ant}} & \longrightarrow & G \\ \downarrow & & \downarrow \\ (TG_{\mathrm{ant}})/T_H & \longrightarrow & G/H \end{array}$$

The right vertical map induces a surjection on fundamental groups because the pullback of any connected étale cover of G/H to G remains connected, as in the proof of statement a). Proposition 3.1 and the induced diagram on prime-to-p fundamental groups then imply the surjectivity of the map

$$\pi_1((TG_{\mathrm{ant}})/T_H)^{(p')} \to \pi_1(G/H)^{(p')}.$$

Observe now that TG_{ant} is a semi-abelian variety which is an extension of an abelian variety of dimension g by a torus of rank r. To prove this fact, remark first that T contains the maximal torus of the semi-abelian variety G_{ant}, since that torus is central in G_{aff}. Thus TG_{ant} is also a semi-abelian variety, with maximal torus T and abelian quotient

$$TG_{\mathrm{ant}}/T \cong G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap T).$$

But $G_{\mathrm{ant}} \cap T$ is an affine subgroup scheme of G_{ant} containing $(G_{\mathrm{ant}})_{\mathrm{aff}}$ (the maximal torus of G_{ant}), hence the quotient $(G_{\mathrm{ant}} \cap T)/(G_{\mathrm{ant}})_{\mathrm{aff}}$ is finite. Likewise, the quotient $(G_{\mathrm{ant}} \cap G_{\mathrm{aff}})/(G_{\mathrm{ant}})_{\mathrm{aff}}$ is finite. It follows that the abelian quotient of TG_{ant} is isogenous to $G_{\mathrm{ant}}/(G_{\mathrm{aff}} \cap G_{\mathrm{ant}}) \cong G/G_{\mathrm{aff}} = A$.

We conclude that $(TG_{\mathrm{ant}})/T_H$ is an extension of an abelian variety of dimension g by a torus of dimension $r - r_H$, and hence its prime-to-p fundamental group is a quotient of $\mathbf{Z}_{(p')}^{2g+r-r_H}$ by the results of the previous section. This finishes the proof of case II modulo Proposition 3.1.

The proof of Proposition 3.1 will require a series of lemmas.

Lemma 3.2. *If $f : X \to Y$ is a torsor under a connected unipotent group, then the natural map $\pi_1(X) \to \pi_1(Y)$ is surjective.*

Proof. By writing the unipotent group as an iterated extension of copies of $\mathbf{G}_a$ we reduce by induction to the case where f is a $\mathbf{G}_a$-torsor. Then f has a compactification by the projective line bundle $\bar{f} : X \times^{\mathbf{G}_a} \mathbf{P}^1 \to Y$, where $\mathbf{G}_a$ acts on $\mathbf{P}^1$ by translations. The complement of X in $X \times^{\mathbf{G}_a} \mathbf{P}^1$ is a divisor which yields a section of $\bar{f}$. So the assertion follows from [8], Exposé XIII, Proposition 4.1 and Example 4.4. □

Returning to our situation, we choose a Borel subgroup B of G (or equivalently, of G_{aff}) containing T. Then BG_{ant} is a connected solvable subgroup of G, since G_{ant} is connected and central in G.

Lemma 3.3. *The natural map*

$$\pi_1(BG_{\mathrm{ant}})^{(p')} \to \pi_1(G)^{(p')}$$

is surjective.

Proof. Choose another Borel subgroup B^- such that the intersection $B \cap B^-$ is of smallest dimension. Specifically, consider the unipotent radical $R_u(G) = R_u(G_{\mathrm{aff}})$. Then $G_{\mathrm{aff}}/R_u(G)$ is a connected reductive group with Borel subgroup $B/R_u(G)$, and we take for B^- the preimage in G of an opposite Borel subgroup ([11], §26.2). Denoting by U^- the unipotent part of B^-, we have $B \cap U^- = R_u(G)$ (as schemes), since U^- contains $R_u(G)$ and the intersection $(B/R_u(G)) \cap (U^-/R_u(G))$ is trivial. It follows that

$$(BG_{\mathrm{ant}}) \cap U^- = B(G_{\mathrm{ant}} \cap G_{\mathrm{aff}}) \cap U^- = B \cap U^- = R_u(G),$$

where the first equality holds since G_{aff} contains B and U^-, and the second one since $G_{\mathrm{ant}} \cap G_{\mathrm{aff}}$ is contained in the center of G_{aff} and hence in the Borel subgroup B. Now the multiplication in G yields a map

$$f : BG_{\mathrm{ant}} \times U^- \to G$$

making $BG_{\mathrm{ant}} \times U^-$ a torsor under $(BG_{\mathrm{ant}}) \cap U^- = R_u(G)$. The image of f is open in $G = G_{\mathrm{aff}}G_{\mathrm{ant}}$, since $BU^- = BB^-$ is open in G_{aff}. Thus the pullback of any connected étale cover of G to the image $BG_{\mathrm{ant}}U^-$ is connected, and hence the natural map

$$\pi_1(BG_{\mathrm{ant}}U^-) \to \pi_1(G)$$

is surjective. But by Lemma 3.2 the map

$$\pi_1(BG_{\mathrm{ant}} \times U^-) \to \pi_1(BG_{\mathrm{ant}}U^-)$$

is surjective as well. This implies the surjectivity of the natural map

$$\pi_1(BG_{\mathrm{ant}} \times U^-)^{(p')} \to \pi_1(G)^{(p')}$$

and in turn the lemma, by using ([20], Cor. 4.9) again, and recalling that the variety U^- is an affine space. □

Lemma 3.4. *The natural map*

$$\pi_1(TG_{\mathrm{ant}})^{(p')} \to \pi_1(BG_{\mathrm{ant}})^{(p')}$$

is surjective.

Proof. First note that $B = UT$, and hence $BG_{\mathrm{ant}} = UTG_{\mathrm{ant}}$. This yields a torsor $U \times TG_{\mathrm{ant}} \to BG_{\mathrm{ant}}$ under the group scheme $U \cap TG_{\mathrm{ant}}$. The latter is commutative, finite (since its reduced neutral component is unipotent and diagonalizable), and unipotent, hence of order a power of p. Write G_1 for the quotient of $U \times TG_{\mathrm{ant}}$ by the reduced part of $U \cap G_{\mathrm{ant}}$. The natural map $G_1 \to BG_{\mathrm{ant}}$ is finite, surjective and purely inseparable, therefore it induces an isomorphism on fundamental groups ([8], exposé IX, théorème 4.10). We may thus assume $G_1 = BG_{\mathrm{ant}}$, in which case $U \times TG_{\mathrm{ant}} \to BG_{\mathrm{ant}}$ is a finite étale cover with group $\Gamma := U \cap TG_{\mathrm{ant}}$. We thus obtain an exact sequence

$$1 \to \pi_1(U \times TG_{\mathrm{ant}}) \to \pi_1(BG_{\mathrm{ant}}) \to \Gamma \to 1$$

of fundamental groups. But Γ is a p-group, and hence the natural map $\pi_1(U \times TG_{\mathrm{ant}})^{(p')} \to \pi_1(BG_{\mathrm{ant}})^{(p')}$ is an isomorphism. We now conclude as in the proof of the previous lemma. □

Combining the two lemmas above we obtain the surjectivity of the natural map

$$\pi_1(TG_{\mathrm{ant}})^{(p')} \to \pi_1(G)^{(p')},$$

as stated in Proposition 3.1.

Remark 3.5. In the course of the above proof we have in fact shown that in characteristic $p > 0$ the subgroup $TG_{\mathrm{ant}} \subset G$ is a semi-abelian variety which is an extension by T of an abelian variety isogenous to A. If $p = 0$ the same holds for the image of TG_{ant} in $G/R_u(G)$. A similar argument shows that (in all characteristics) the subgroup BG_{ant} is an extension by B of an abelian variety isogenous to A.

One shows easily that BG_{ant} is a maximal connected solvable subgroup of G and moreover all such subgroups are conjugate in G. Likewise, TG_{ant} (resp. the image of TG_{ant} in $G/R_u(G)$) is a maximal semi-abelian variety if $p > 0$ (resp. $p = 0$), and all such subgroups are conjugate. Thus BG_{ant} is a subgroup of G that is analogous to a Borel subgroup in the linear case, and TG_{ant} is analogous to a maximal torus.

4. Application to Brauer groups

In this section we prove Theorem 1.3. We need two lemmas, in which all cohomology groups are taken with respect to the étale topology.

Lemma 4.1. *Let G be a connected commutative algebraic group over an algebraically closed field k. For n prime to the characteristic the* *cohomology ring $H^*(G, \mathbf{Z}/n\mathbf{Z})$ with its cup-product structure is canonically isomorphic to the exterior algebra $\Lambda^* H^1(G, \mathbf{Z}/n\mathbf{Z})$.*

Proof. The group G is an extension of a semi-abelian variety G_1 by a a connected commutative unipotent group U. Since U is an iterated extension of $\mathbf{G}_a$'s, we may view G as an affine bundle over G_1. But then the cohomology of G with finite coefficients identifies with that of G_1. So we may assume that G is a semi-abelian variety, extension of an abelian variety A by a torus T. In this case the proof is a direct generalization of the argument for abelian varieties as in [18] Theorem 15.1. Here is a brief sketch. By formal cohomological arguments as in *loc. cit.* one sees that it is enough to prove the lemma with $\mathbf{Q}_\ell$-coefficients instead of $\mathbf{Z}/n\mathbf{Z}$-coefficients, where ℓ is a prime invertible in k. Then one invokes an algebraic lemma ([18], Lemma 15.2) based on Borel's fundamental structure theorem for Hopf algebras. It shows that it is enough to check that there exists an integer $m > 0$ such that $H^i(G, \mathbf{Q}_\ell) = 0$ for $i > m$ and moreover $H^1(G, \mathbf{Q}_\ell)$ has dimension $\leq m$ over $\mathbf{Q}_\ell$. We check these properties for $m = 2g + r$, where $g = \dim A$ and $r = \dim T$. In the Leray spectral sequence for $\alpha : G \to A$

$$E_2^{pq} = H^p(A, R^q\alpha_*\mathbf{Q}_\ell) \Rightarrow H^{p+q}(G, \mathbf{Q}_\ell)$$

the terms E_2^{pq} vanish for $p > 2g$ (because A has cohomological dimension $2g$) and $q > r$ (because T has cohomological dimension r, being affine of dimension r). This shows the first property. For the second property, observe first that there is an isomorphism ${}_nG \cong (\mathbf{Z}/n\mathbf{Z})^{2g+r}$ because the sequence

$$0 \to {}_nT \to {}_nG \to {}_nA \to 0$$

is split exact by n-divisibility of T; moreover, there are isomorphisms ${}_nT \cong (\mathbf{Z}/n\mathbf{Z})^r$ and ${}_nA \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$. Thus it suffices to apply the next lemma. □

Lemma 4.2. *Under the assumptions of Lemma 4.1 the map*

$$\mathrm{Hom}({}_nG, \mu_n) \to H^1(G, \mu_n)$$

sending a character $\chi : {}_nG \to \mu_n$ to the class of the μ_n-torsor over G obtained by pushing out the extension

$$0 \to {}_nG \to G \to G \to 0 \tag{1}$$

by χ is an isomorphism.

Proof. To prove injectivity, assume that a pushout of (1) by a nontrivial character gives a split torsor over G. By construction it is then a split extension of G by a cyclic group of order dividing n which is also a quotient of G. This is impossible because G is n-divisible.

For surjectivity assume given a μ_n-torsor $\pi : Y \to G$. Assume first Y is connected. By Proposition 1.1 *a*) there is an algebraic group structure on Y making it a central extension of G; in particular it is commutative. By the same argument as for abelian varieties ([19], Remark p. 169 or [27], Corollary 5.6.9) the multiplication-by-n map $G \to G$ factors through π, which gives in particular a character $\chi : {}_nG \to \mu_n$ by looking at the fibres above the neutral element. There is thus a map of μ_n-torsors from the pushforward of (1) by χ to Y which must be an isomorphism. If Y is not connected, it is induced from a connected μ_m-torsor for some $m|n$ which in turn comes from a character ${}_mG \to \mu_m$. The character corresponding to Y is obtained by extending this character to ${}_nG$ in a way compatible with the induction of torsors. □

Remark 4.3. It follows from the above proof that the extension (1) gives the largest abelian étale Galois cover of G of exponent n. Indeed, an easy argument using the Leray spectral sequence as in the proof of Lemma 4.1 shows that $H^1(G, \mu_n)$ is finite. As before, after choosing an isomorphism $\mu_n \cong \mathbf{Z}/n\mathbf{Z}$ we may identify connected μ_n-torsors and étale $\mathbf{Z}/n\mathbf{Z}$-Galois covers. So we infer that there is a largest finite étale Galois cover whose Galois group is abelian of exponent n, and this Galois group is isomorphic to $H^1(G, \mu_n)$. By the above proof it must be a quotient of the étale cover given by (1), but then the quotient map is an isomorphism by maximality.

Proof of Theorem 1.3. The Kummer sequence in étale cohomology coming from the exact sequence

$$1 \to \mu_n \to \mathbf{G}_m \to \mathbf{G}_m \to 1$$

of étale sheaves yields an exact sequence

$$0 \to \operatorname{Pic} G/n\operatorname{Pic} G \to H^2(G, \mu_n) \to {}_n\operatorname{Br} G \to 0$$

which we may rewrite using Lemma 4.1 as

$$0 \to \operatorname{Pic} G/n\operatorname{Pic} G \to \Lambda^2 H^1(G, \mu_n) \otimes \mu_n^{\otimes -1} \to {}_n\operatorname{Br} G \to 0.$$

We may then replace $H^1(G, \mu_n)$ by $\operatorname{Hom}({}_nG, \mu_n)$ using Lemma 4.2. □

Proof of Corollary 1.4. Setting $n = \ell^m$ in the exact sequence of Theorem 1.3 and passing to the direct limit over m we obtain an exact sequence

$$0 \to \operatorname{Pic} G \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \to \operatorname{Hom}(\Lambda^2 T_\ell(G), \mathbf{Q}_\ell/\mathbf{Z}_\ell) \to \operatorname{Br} G\{\ell\} \to 0$$

where $T_\ell(G)$ is the ℓ-adic Tate module of G and $\operatorname{Br} G\{\ell\}$ denotes the ℓ-primary torsion of $\operatorname{Br} G$. As the Hom-group in the middle is ℓ-divisible, so is $\operatorname{Br} G\{\ell\}$, hence it is isomorphic to a finite direct power of $\mathbf{Q}_\ell/\mathbf{Z}_\ell$. To determine this power explicitly, we calculate the left and middle terms.

For the left term, we use first that $\operatorname{Pic} G$ is isomorphic to $\operatorname{Pic} G/U$ as in the proof of Lemma 4.1, where U denotes the largest connected unipotent subgroup of G. So we may assume that G is an extension of

A by the torus $T = \mathbf{G}_m^r$. Now recall that by virtue of a generalization of the Barsotti–Weil formula ([25], VII, §3) the extension G of A by T is classified by a homomorphism $c : X^*(T) \to A^*$, where $A^* = \mathrm{Pic}^0 A$ is the dual abelian variety and $X^*(T)$ the character group of T. It fits into an exact sequence

$$\mathrm{X}^*(T) \xrightarrow{c} \mathrm{Pic}\, A \xrightarrow{\alpha^*} \mathrm{Pic}\, G \to 0 \tag{2}$$

whose exactness follows, for instance, from ([13], Theorem 5).

Thus $\mathrm{Pic}\, G$ is the quotient of $\mathrm{Pic}\, A$ by a subgroup of A^*. Since A^* is ℓ-divisible, we get that $\mathrm{Pic}\, G$ is an extension of $\mathrm{NS}\, A$ by an ℓ-divisible group, whence an isomorphism $\mathrm{Pic}\, G \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \cong \mathrm{NS}\, A \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell$. Moreover, we have $\mathrm{NS}\, A \cong \mathbf{Z}^\rho$ by definition of ρ. Finally, the group $\mathrm{Hom}(\Lambda^2 T_\ell(G), \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ is isomorphic to $(\mathbf{Q}_\ell/\mathbf{Z}_\ell)^{(2g+r)(2g+r-1)/2}$. This follows from the fact that the $\mathbf{Z}_\ell$-rank of $T_\ell(G)$ is $2g + r$, as calculated at the end of the proof of Lemma 4.1. □

In conclusion, we remark that in the above proof (more precisely, in Lemma 4.2) we used Proposition 1.1 in a crucial way, which in turn was based on the difficult noncommutative result of Orgogozo [20]. However, for the proof of Lemma 4.2 it is sufficient to know that μ_n-torsors over commutative algebraic groups carry a commutative group scheme structure. This fact can be proven in a much more elementary way, as we now explain.

We reduce as above to the case where G is a semi-abelian variety. A μ_n-torsor $\tilde{G}$ over G comes from a line bundle L of order n on G together with a trivialization of $L^{\otimes n}$ (see e.g. [17], p. 125). It thus corresponds to a class of order n in $\mathrm{Pic}\, G$ which, by exact sequence (2), comes from a class in $\mathrm{Pic}\, A$. As the Néron–Severi group of A is torsion free, the latter must be a class in $\mathrm{Pic}^0 A$, and therefore it is represented by a line bundle whose associated $\mathbf{G}_m$-torsor Y carries a group scheme structure making it a central extension of A by $\mathbf{G}_m$. The $\mathbf{G}_m$-torsor L^* associated with L is the pullback of Y to G, so it comes from a central extension of G by $\mathbf{G}_m$. Since $\mathcal{O}(L^*) \cong \oplus_{n=-\infty}^{\infty} H^0(G, L^{\otimes n})$, the trivialization of $L^{\otimes n}$ can be identified with a regular invertible function ψ on L^*, and then $\tilde{G}$ is the zero subscheme of the regular function $\psi - 1$ on G. When choosing the group structure on L^* we may assume that ψ takes the neutral element of L^* to 1. But then by Rosenlicht's lemma ([23], Theorem 3) ψ is a character of the group L^*, hence $\tilde{G} \subset L^*$ is a subgroup scheme. This defines a commutative group scheme structure on $\widetilde{G}$ making it a central extension of G by μ_n.

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