THE THEOREM ON SMALL SIMPLICES AND APPLICATIONS

1. Reminder on homotopies

As a warmup we recall the basic ideas from the proof of the following theorem:

Theorem 1.1. *If $f, g : X \to Y$ are homotopic continuous maps of topological spaces, the induced maps $S_\bullet(X) \to S_\bullet(Y)$ are chain homotopic.*

Recall that chain homotopy of the induced maps means that there exist maps $k_i : S_i(X) \to S_{i+1}(X)$ for all i such that $f_i - g_i = d_{i+1} \circ k_i + k_{i-1} \circ d_i$ for all i.

Some ideas from the proof will be needed later, so let us give a sketch. The first step is:

Proposition 1.2. *If $X \subset \mathbf{R}^n$ is a bounded convex subset and $x_0 \in X$, the identity map of X is chain homotopic to the map $\epsilon : S_\bullet(X) \to S_\bullet(X)$ such that*

$$\epsilon_i = \begin{cases} x \mapsto x_0 & i = 0 \\ 0 & i > 0. \end{cases}$$

Consequently, $H_i(X) = 0$ for $i > 0$ and in fact $\widetilde{H}_i(X) = 0$ for all i.

Proof. The proof uses the *cone construction:* for a simplex $\sigma : \Delta_i \to X$ define $k(\sigma) : \Delta_{i+1} \to X$ by

$$k(\sigma)(t_0, \ldots, t_{i+1}) = \begin{cases} x_0 & t_0 = 1 \\ t_0 x_0 + \sigma\left(\dfrac{t_1}{1-t_0}, \ldots \dfrac{t_{i+1}}{1-t_0}\right) & t_0 \neq 1. \end{cases}$$

(This is the cone with vertex x_0 above $\sigma(\Delta_i)$.) A direct computation seen in class shows that this map induces the required chain homotopy. □

A *functorial* morphism of complexes $\phi : S_\bullet(X) \to S_\bullet(X)$ is a collection of morphisms of complexes $\phi^X : S_\bullet(X) \to S_\bullet(X)$ for every topological space X such that for every continuous map $X \to Y$ the diagram

$$\begin{array}{ccc} S_\bullet(X) & \xrightarrow{\phi^X} & S_\bullet(X) \\ \downarrow & & \downarrow \\ S_\bullet(Y) & \xrightarrow{\phi^Y} & S_\bullet(Y) \end{array}$$

commutes.

Proposition 1.3. *Assume $\phi, \psi : S_\bullet(X) \to S_\bullet(X)$ are functorial morphisms of complexes such that $\phi_0 = \psi_0 : S_0(X) \to S_0(X)$. Then ϕ and ψ are functorially chain homotopic (i.e. there exist chain homotopies for all X satisfying similar commutative diagrams).*

Date: March 14, 2020.

Proof. Up to replacing ϕ by $\phi - \psi$ we may assume $\psi = 0$. We construct a functorial chain homotopy k such that $\phi_i = d_{i+1} \circ k_i + k_{i-1} \circ d_i$ by induction on i. Set $k_0 = 0$ and assume the k_j have been constructed for $j < i$.

Assume first $X = \Delta_i$, and let $\iota \in S_i(\Delta_i)$ be the identity map of Δ_i. Then

$$d_i(\phi(\iota) - k_{i-1}(d_i(\iota))) = \phi(d_i(\iota)) - (d_i \circ k_{i-1})(d_i(\iota)) = k_{i-2}(d_{i-1}(d_i(\iota))) = 0$$

where in the first equality we used that ϕ is a morphism of complexes, then we used the inductive assumption and finally $d_{i-1} \circ d_i = 0$.

Since $\Delta_i \subset \mathbf{R}^{i+1}$ is a bounded convex subset, its higher homology groups are 0 by the previous proposition. Therefore we find $\alpha \in S_{i+1}(\Delta_i)$ such that

$$d_{i+1}(\alpha) = \phi(\iota) - k_{i-1}(d_i(\iota)).$$

Now for all spaces X define $k_i : S_i(X) \to S_{i+1}(X)$ by $k_i(\sigma) := \sigma \circ \alpha$ for each $\sigma : \Delta_i \to X$. By construction this map is functorial. To check it is a chain homotopy, notice first that

$$d_{i+1}(k_i(\sigma)) = d_{i+1}(\sigma \circ \alpha) = \sigma(d_{i+1}(\alpha))$$

as the continuous map σ induces a morphism of complexes $S_\bullet(\Delta_i) \to S_\bullet(X)$. Similarly, since by assumption both ϕ and k_{i-1} are functorial for the continuous map $\sigma : \Delta_i \to X$, we get

$$\sigma(d_{i+1}(\alpha)) = \sigma(\phi(\iota) - k_{i-1}(d_i(\iota))) = \phi(\sigma(\iota)) - k_{i-1}(\sigma(d_i(\iota))) = \phi(\sigma) - k_{i-1}(d_i(\sigma))$$

because $\sigma(\iota) = \sigma$ and σ commutes with d_i as above. Putting everything together we obtain

$$d_{i+1}(k_i(\sigma)) = \phi(\sigma) - k_{i-1}(d_i(\sigma))$$

as required. □

Remarks 1.4.
1. The same statement holds if instead of $\phi_0 = \psi_0$ we assume $\eta \circ \phi_0 = \eta \circ \psi_0$, where $\eta : S_0(X) \to \mathbf{Z}$ is given by sending all $x \in X$ to $1 \in \mathbf{Z}$. The proof if the same, except we do not set $k_0 = 0$ but use $\widetilde{H}_0(\Delta_i) = 0$ coming from the previous proposition to construct k_0.

2. The same statement holds if we consider maps $\phi, \psi : S_\bullet(X) \to S_\bullet([0,1] \times X)$ with similar properties. [The sets $[0,1] \times \Delta_i$ are also bounded convex subsets in $\mathbf{R}^{i+2}$, so the same proof goes over.]

Proof of Theorem 1.1. Consider the maps $\tilde{\phi}, \tilde{\psi} : X \to [0,1] \times X$ given by $\tilde{\phi}(x) = (0, x)$, $\tilde{\psi}(x) = (1, x)$. They induce *functorial* morphisms of complexes $\phi, \psi : S_\bullet(X) \to S_\bullet([0,1] \times X)$ such that $\eta \circ \phi = \eta \circ \psi$, where η is as in the first remark above. So by the previous proposition and remarks they are chain homotopic.

Now let $h : [0,1] \times X \to Y$ be a homotopy between f and g. Notice that $h \circ \tilde{\phi} = f$ and $h \circ \tilde{\psi} = g$. Thus the maps $S_\bullet(X) \to S_\bullet(X)$ induced by f and g equal the compositions of $h : S_\bullet([0,1] \times X) \to S_\bullet(X)$ with the two chain homotopic maps ϕ and ψ. As such they are chain homotopic. □

2. Background from homological algebra

A *short exact sequence of complexes* is a sequence of morphisms of complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

such that the sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

are exact for all i. Now we have the following basic fact.

Proposition 2.1. *Let*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a short exact sequence of complexes of abelian groups. Then there is a long exact sequence

$$\dots \to H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \xrightarrow{\partial} H_{i-1}(A_\bullet) \to H_{i-1}(B_\bullet) \to \dots$$

The map ∂ is usually called the *connecting homomorphism* or the *boundary map.*

For the proof of the proposition we need the following equally basic lemma.

Lemma 2.2. (The Snake Lemma) *Given a commutative diagram of abelian groups*

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

with exact rows, there is an exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma).$$

Proof. The construction of all maps in the sequence is immediate, except for the map $\partial : \ker(\gamma) \to \operatorname{coker}(\alpha)$. For this, lift $c \in \ker(\gamma)$ to $b \in B$. By commutativity of the right square, the element $\beta(b)$ maps to 0 in C', hence it comes from a unique $a' \in A'$. Define $\partial(c)$ as the image of a' in $\operatorname{coker}(\alpha)$. Two choices of b differ by an element $a \in A$ which maps to 0 in $\operatorname{coker}(\alpha)$, so ∂ is well-defined. Checking exactness is left as an exercise to the readers. □

Proof of Proposition 2.1: Applying the Snake Lemma to the diagram

$$\begin{array}{ccccccccc} & & A_i/B_i(A_\bullet) & \longrightarrow & B_i/B_i(B_\bullet) & \longrightarrow & C_i/B_i(C_\bullet) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} & & \\ 0 & \longrightarrow & Z_{i-1}(A_\bullet) & \longrightarrow & Z_{i-1}(B_\bullet) & \longrightarrow & Z_{i-1}(C_\bullet) & & \end{array}$$

yields a long exact sequence

$$H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \to H_{i-1}(A_\bullet) \to H_{i-1}(B_\bullet) \to H_{i-1}(C^\bullet),$$

and the proposition is obtained by splicing these sequences together. □

3. The theorem on small simplices

Consider an open covering $\mathcal{U} = \{U_j : j \in I\}$ of a topological space X. Define subgroups $S_i^{\mathcal{U}}(X) \subset S_i(X)$ by

$S_i^{\mathcal{U}}(X) :=$ free abelian group with basis $\{\sigma : \Delta_i \to X \;:\; \operatorname{Im}(\sigma) \subset U_i \text{ for some } j \in I\}$

Together with the restrictions of the differentials $d_i : S_i(X) \to S_{i-1}(X)$ to $S_i^{\mathcal{U}}(X)$ these groups form a *subcomplex* $S_\bullet^{\mathcal{U}}(X)$ of the singular complex $S_\bullet(X)$. Another way to view this complex is to consider the images of the natural maps $S_\bullet(U_j) \to S_\bullet(X)$ for all $j \in I$ and take the subcomplex generated by them (i.e. the subgroups of the $S_i(X)$ generated by the $\operatorname{Im}(S_i(U_j) \to S_i(X))$ for all j together with the restrictions of the d_i).

Theorem 3.1 (Theorem on small simplices)**.** *The inclusion map* $S_\bullet^{\mathcal{U}}(X) \hookrightarrow S_\bullet(X)$ *induces isomorphisms* $H_i(S_\bullet^{\mathcal{U}}(X)) \xrightarrow{\sim} H_i(X)$ *for all* $i \geq 0$.

In other words, the homology groups of X can be computed using only simplices that are 'so small' that their image is contained in one of the 'small' open subsets in the covering of X.

We begin with the geometric part of the proof.

Definition 3.2. Assume given $i+1$ points $v_0, v_1, \dots, v_i$ in $\mathbf{R}^{i+1}$ (identified with vectors from the origin). The *affine i-simplex* spanned by $v_0, \dots, v_i$ is their convex hull:

$$\Delta_{v_0,\dots,v_i} = \{t_0 v_0 + \cdots + t_i v_i : t_1 + \cdots + t_i = 1, 0 \le t_j \le 1 \text{ for all } j\}$$

So the standard simplex Δ_i is the affine simplex spanned by the unit vectors in $\mathbf{R}^{i+1}$.

An affine simplex is any $\Delta_{v_0,\dots,v_i}$ defined as above. Its faces are the subsets given by $t_j = 0$ ($j = 0, \dots, i$).

The *barycenter* of the affine simplex $\Delta_{v_0,\dots,v_i}$ is the point

$$b := \frac{1}{i+1} v_0 + \cdots + \frac{1}{i+1} v_i.$$

For example, the barycenter of the unit simplex $\Delta_i \subset \mathbf{R}^{i+1}$ is the point with coordinates $(1/(i+1), \dots, 1/(i+1))$ (and not the origin!).

Construction 3.3. The *barycentric subdivision* of an affine i-simplex Δ is defined inductively as follows. For $i = 0$ the simplex is a point and the subdivision is trivial. Suppose the barycentric subdivision has been constructed for all faces of Δ. Define the subdivision of Δ by joining all vertices of the barycentric subdivisions of the faces to the barycenter b of Δ.

So for $\Delta = \Delta_i$ and $i = 1$ we get an interval divided in two equal intervals by the midpoint. For $i = 2$ we get a subdivision of the triangle in 6 smaller triangles by joining the barycenter to the 3 vertices of the triangle and to the 3 midpoints of edges of the triangle. (See the pictures on p. 120 of Hatcher's book.)

We now want to measure 'how small' a simplex in the barycentric subdivision can be. Recall that the *diameter* of a subset of $\mathbf{R}^{i+1}$ is the maximal (Euclidean) distance between two of its points. For an affine simplex this is the same as the maximal distance between two of its vertices.

Lemma 3.4. *The diameter of each small simplex in the barycentric subdivision of an affine i-simplex Δ is at most $i/(i+1) \cdot \mathrm{diameter}(\Delta)$.*

Proof. Induction on i: the cases $i = 0, 1$ are clear. Let Δ' be a subsimplex in the barycentric subdivision of Δ. One of its vertices is the barycenter b: denote the others by $w_1, \dots, w_i$. Consider the longest edge of Δ'. If it does not contain b, we are done by induction as then this edge lies in one of the faces of Δ and $i/(i+1) > (i-1)/i$. So say the edge $\overline{bw_1}$ between b and w_1 is the longest edge of Δ'. Here w_1 must be a vertex of Δ because the barycenter of a face of Δ is closer to b than the vertices of the face. Let $\Delta^1 \subset \Delta$ be the face of Δ not containing w_1, and let b_1 be its barycenter. Then

$$b = \frac{1}{i+1} w_1 + \frac{i}{i+1} b_1$$

and therefore

$$\mathrm{diameter}(\Delta') = \mathrm{length}(\overline{bw_1}) = \frac{i}{i+1} \mathrm{length}(\overline{b_1 w_1}) \le \frac{i}{i+1} \mathrm{diameter}(\Delta).$$

□

Now back to topology. Let X be a topological space. We define a morphism of complexes $\beta : S_\bullet(X) \to S_\bullet(X)$ which will be *functorial in X*.

Construction 3.5. For $i = 0$ set $\beta_0 = \mathrm{id}$.

Assume $\beta_j : S_j(X) \to S_j(X)$ has been constructed for $j < i$. We now define $\beta_i : S_i(X) \to S_i(X)$. Let (as in the proof of Proposition 1.3) $\iota : \Delta_i \to \Delta_i$ be the identity simplex. Consider $d_i(\iota) \in S_{i-1}(\Delta_i)$. By assumption we have an element $\beta_{i-1}(d_i(\iota)) \in S_{i-1}(\Delta_i)$. Define

$$\beta_i(\iota) := k^b(\beta_{i-1}(d_i(\iota))) \in S_i(\Delta_i)$$

where k^b is the chain homotopy map $S_{i-1}(\Delta_i) \to S_i(\Delta_i)$ given by the *cone construction* (seen in the proof of Proposition 1.2) such that the vertex of the cone is the barycenter b of Δ_i. Finally, given an arbitrary simplex $\Delta_i \to X$, set

$$\beta_i(\sigma) := \sigma(\beta_i(\iota)).$$

By construction, β_i is *functorial* in X.

Remark 3.6. If you follow the construction carefully, you see that if we write

$$\beta_i(\iota) = \sum n_i \sigma_i$$

with some $\sigma_i : \Delta_i \to \Delta_i$, then all the n_i are equal to ± 1 and the images of the σ_i are the simplices in the barycentric subdivision of Δ_i. Hence they are of diameter $\leq i/(i+1)\sqrt{2}$ by Lemma 3.4. Furthermore, the barycenter of $\mathrm{Im}(\sigma_i)$ will be $\sigma_i(b)$.

We may iterate β_i: the images of the simplices in $\beta^2(\iota) = (\beta \circ \beta)(\iota)$ will be small simplices in the barycentric subdivision of each $\mathrm{Im}(\sigma_i)$ by our previous observation. Hence by Lemma 3.4 again they will be of diameter $\leq (i/(i+1))^2\sqrt{2}$.

Iterating r times, we obtain that the images of the simplices in $\beta^r(\iota)$ will be of diameter $\leq (i/(i+1))^r\sqrt{2}$, hence their diameter tends to 0.

Lemma 3.7. *The maps $\beta_i : S_i(X) \to S_i(X)$ assemble to a morphism of complexes $\beta : S_\bullet(X) \to S_\bullet(X)$.*

Proof. We have to show $\beta_{i-1} \circ d_i = d_i \circ \beta_i$ for $i > 0$. We do this by induction on i. Given $\sigma : \Delta_i \to X$, we have

$$d_i\beta_i(\sigma) = d_i(\sigma(\beta_i(\iota))) = \sigma(d_i(\beta_i(\iota))) = \sigma(d_i(k^b(\beta_{i-1}(d_i(\iota))))$$

using (as in the proof of Proposition 1.3) that σ induces a morphism of complexes $S_\bullet(\Delta_i) \to S_\bullet(X)$ and that β_{i-1} is functorial. We know from the proof of Proposition 1.2 that k^b defines a certain chain homotopy on $S_\bullet(\Delta_i)$. For $i = 1$ it means $d_1 \circ k^b = \mathrm{id} - \epsilon_0$. So we obtain

$$d_1(\beta_1(\sigma)) = \sigma(\beta_0(d_1(\iota))) - \sigma(\epsilon_0(\beta_0((d_1(\iota)))) = \sigma(d_1(\iota)) - \sigma(\epsilon_0(d_1(\iota))) = d_1(\sigma)$$

using $\beta_0 = \mathrm{id}$ and $\epsilon_0(d_1(\iota)) = b - b = 0$, which is the case $i = 1$ of the lemma.

On the other hand, for $i > 1$ we have

$$\mathrm{id} = d_i \circ k^b_{i-1} + k^b_{i-2} \circ d_{i-1},$$

so

$$d_i(k^b_{i-1}(\beta_{i-1}(d_i(\iota))) = \beta_{i-1}(d_i(\iota)) - k^b_{i-2}(d_{i-1}(\beta_{i-1}(d_i(\iota))))$$

where by induction

$$d_{i-1}(\beta_{i-1}(d_i(\iota))) = \beta_{i-2}(d_{i-1}(d_i(\iota))) = 0$$

using $d_{i-1} \circ d_i = 0$. So

$$d_i k^b_{i-1}(\beta_{i-1}(d_i(\iota)) = \beta_{i-1}(d_i(\iota))$$

whence we compute as in the proof of Proposition 1.3

$$\sigma(d_i k^b_{i-1}(\beta_{i-1}(d_i(\iota))) = \sigma(\beta_{i-1}(d_i(\iota))) = \beta_{i-1}\sigma(d_i(\iota))) = \beta_{i-1}(d_i(\sigma)))$$

as required. □

Lemma 3.8. *The map $\beta : S_\bullet(X) \to S_\bullet(X)$ is* functorially *chain homotopic to the identity map of $S_\bullet(X)$. Similarly $\beta^r = \beta \circ \beta \circ \cdots \beta$ is* functorially *chain homotopic to the identity map of $S_\bullet(X)$ for all $r > 0$.*

Proof. By construction, β^r is a functorial morphism of complexes such that $\beta^r_0 = \mathrm{id}$. Thus it is functorially chain homotopic to the identity map by Proposition 1.3. □

Lemma 3.9. *Fix an open covering $\mathcal{U} = \{U_j : j \in I\}$ of X.*

Given $z \in S_i(X)$, there exists $r > 0$ such that $\beta^r(z) \in S_i^{\mathcal{U}}(X)$.

Proof. Since z is a linear combination of simplices $\sigma : \Delta_i \to X$, it is enough to show that $\beta^r(\sigma) \in S^{\mathcal{U}}(X)$ for some $r > 0$. Observe that the system $\{\sigma^{-1}(U_j) : j \in I\}$ is an open covering of Δ_i, and Δ_i is a compact metric space. Hence we can apply Lebesgue's lemma from general topology (see e.g. 'Lebesgue's number lemma' on en.wikipedia.org) according to which there exists $\varepsilon > 0$ such that every subset of diameter $< \varepsilon$ contained in Δ_i is in fact contained in one of the $\sigma^{-1}(U_j)$. Hence by Remark 3.6 for r large enough all simplices involved in $\beta^r(\iota)$ will be contained in some $\sigma^{-1}(U_j)$. This implies that all simplices involved in $\beta^r(\sigma)$ will be contained in some U_j, as required. □

Proof of Small Simplices Theorem. The theorem is obvious for $i = 0$. To prove the cases $i > 0$, consider the exact sequence of complexes

$$0 \to S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X) \to S_\bullet(X)/S_\bullet^{\mathcal{U}}(X) \to 0.$$

Part of the associated long exact sequence (Proposition 2.1) reads

$$H_{i+1}(S_\bullet(X)/S_\bullet^{\mathcal{U}}(X)) \to H_i(S_\bullet^{\mathcal{U}}(X)) \to H_i(S_\bullet(X)) \to H_i(S_\bullet(X)/S_\bullet^{\mathcal{U}}(X)).$$

So it will be enough to prove $H_i(S_\bullet(X)/S_\bullet^{\mathcal{U}}(X)) = 0$ for all $i > 0$. By definition of H_i, a class in $H_i(S_\bullet(X)/S_\bullet^{\mathcal{U}}(X))$ is represented by an element $z \in S_i(X)$ such that $d_i(z) \in S_{i-1}^{\mathcal{U}}(X)$. By Lemma 3.8, there exists a functorial chain homotopy k of $S_\bullet(X)$ such that

$$z - \beta_i^r(z) = d_{i+1}(k_i(z)) + k_{i-1}(d_i(z)).$$

Now $d_i(z) \in S_{i-1}^{\mathcal{U}}(X)$ means that $d_i(z)$ is a sum of simplices $\sigma_j : \Delta_{i-1} \to X$ such that $\sigma_j(\Delta_{i-1}) \subset U_j$ for some $U_j \in \mathcal{U}$. In other words, $\sigma_j : \Delta_{i-1} \to X$ can be identified with an element of $S_{i-1}(U_j)$. But since k is a *functorial* chain homotopy defined for *all* topological spaces and in particular for U_j, we see that $k_{i-1}(\sigma_j) : \Delta_i \to X$ identifies with an element of $S_i(U_j)$. Doing this for all j we get that $k_{i-1}d_i(z) \in S_i^{\mathcal{U}}(X)$. On the other hand, by the previous lemma we have $\beta^r(z) \in S_i^{\mathcal{U}}(X)$ for r large enough. So we get

$$z = d_{i+1}(k_i(z)) + \text{ an element of } S_i^{\mathcal{U}}(X).$$

But this means exactly that the class of z is 0 in $H_i(S_\bullet(X)/S_\bullet^{\mathcal{U}}(X))$.

Remark 3.10. Inspection of the proof shows that the statement of the theorem holds under the weaker condition that the U_j are arbitrary subsets of X such that their *interiors* form an open covering of X.

4. Applications

Let $X = U \cup V$ be a covering of the topological space X by two open sets. In the notation of the previous section we have a short exact sequence of complexes

$$0 \to S_\bullet(U \cap V) \to S_\bullet(U) \oplus S_\bullet(V) \to S_\bullet^{\mathcal{U}}(X) \to 0$$

where $\mathcal{U} = \{U, V\}$, the first map is induced by the natural inclusions $S_\bullet(U \cap V) \hookrightarrow S_\bullet(U)$ and $S_\bullet(U \cap V) \hookrightarrow S_\bullet(V)$, and the second map by the *difference* of the natural inclusions $i_U : S_\bullet(U) \hookrightarrow S_\bullet(X)$ and $i_V : S_\bullet(V) \hookrightarrow S_\bullet(X)$ (i.e. $(x, y) \mapsto i_U(x) - i_V(y)$).

Applying Proposition 2.1 gives a long exact sequence

$$\cdots \to H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(S_\bullet^{\mathcal{U}}(X)) \to H_{i-1}(U \cap V) \to \cdots$$

But by the theorem on small simplices the natural maps $H_i(S_\bullet^{\mathcal{U}}(X)) \to H_i(X)$ are isomorphisms for all i. So we obtain:

Theorem 4.1 (Mayer–Vietoris sequence). *Let $X = U \cup V$ be a covering of the topological space X by two open sets. There is a long exact sequence*

$$\cdots \to H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(X) \to H_{i-1}(U \cap V) \to \cdots$$

Remark 4.2. Again the theorem also holds if we only assume that the interiors of U and V cover X, as in Remark 3.10.

Another application concerns relative homology groups. These are defined as follows.

Given a subspace $T \subset X$ of a topological space T, define

$$S_\bullet(X,T) := \operatorname{coker}(S_\bullet(T) \hookrightarrow S_\bullet(X))$$

and

$$H_i(X,T) := H_i(S_\bullet(X,T))$$

for $i \geq 0$. These are the *relative homology groups* of X with respect to T.[1]

Define the category of pairs of topological spaces as the category whose objects are pairs (X,T) with T a subspace of X, and morphisms $(X,T) \to (X',T')$ are given by pairs of continuous maps $(\phi, \phi|_T)$ where $\phi : X \to X'$ is such that $\phi(T) \subset T'$. The assignment $(X,T) \mapsto S_\bullet(X,T)$ is a functor from the category of pairs of spaces to complexes of abelian groups, and the assignments $(X,T) \mapsto H_i(X,T)$ are functors to the category of abelian groups.

The exact sequence of complexes

$$0 \to S_\bullet(T) \to S_\bullet(X) \to S_\bullet(X,T) \to 0$$

gives rise to a long exact sequence

$$\cdots \to H_i(T) \to H_i(X) \to H_i(X,T) \to H_{i-1}(T) \to H_{i-1}(X) \to \cdots$$

by Proposition 2.1, called the *relative homology sequence.*

Remark 4.3. If $f, g : (X,T) \to (X',T')$ are morphisms of pairs of spaces as above such that f is homotopic to g on X and $f|_T$ is homotopic to $g|_T$ on T, then the induced maps $H_i(X,T) \to H_i(X',T')$ are the same. This is proven in the same way as Theorem 1.1, by constructing a chain homotopy on the complex $S_\bullet(X,T)$.

An important application of the small simplices theorem is the following

Theorem 4.4 (Excision). *Let $T_2 \subset T_1 \subset X$ be subspaces such that the closure of T_2 is contained in the interior of T_1. Then the natural maps*

$$H_i(X \setminus T_2, T_1 \setminus T_2) \to H_i(X, T_1)$$

induced by the inclusion maps $X \setminus T_2 \to X$, $T_1 \setminus T_2 \to T_1$ are isomorphisms *for all i.*

Thus the theorem says that relative homology does not change if we excise (cut out) a small subset from T_1. This will be very useful for the computation of homology groups of spaces.

[1] In some nice cases one may identify the groups $H_i(X,T)$ with the homology groups of the quotient space X/T (see Hatcher's book) but this is not true in general. However, one may construct a space (the *mapping cone* of the inclusion map $T \to X$) whose homology groups equal the $H_i(X,T)$; we'll see this later. For the moment we regard the $H_i(X,T)$ as purely algebraic constructs.

Proof. We apply the theorem on small simplices to the covering $\mathcal{U}$ of X formed by T_1 and $X \setminus T_2$. This is not necessarily an open covering, but by assumption the interiors of T_1 and $X \setminus T_2$ cover X, so the theorem applies by Remark 3.10. Note that we may identify each $S_i^{\mathcal{U}}(X)$ with the subgroup $S_i(T_1) + S_i(X \setminus T_2)$ of $S_i(X)$. Similarly, we may identify $S_i(T_1 \setminus T_2)$ with the subgroup $S_i(T_1) \cap S_i(X \setminus T_2)$ of $S_i(X)$. The well-known isomorphisms of group theory

$$S_i(X \setminus T_1)/(S_i(T_1) \cap S_i(X \setminus T_2)) \xrightarrow{\sim} (S_i(T_1) + S_i(X \setminus T_2))/S_i(T_1)$$

thus identify with isomorphisms

$$S_i(X \setminus T_1)/S_i(T_1 \setminus T_2) \xrightarrow{\sim} S_i^{\mathcal{U}}(X)/S_i(T_1)$$

and assemble to an isomorphism of complexes

$$S_\bullet(X \setminus T_1)/S_\bullet(T_1 \setminus T_2) \xrightarrow{\sim} S_\bullet^{\mathcal{U}}(X)/S_\bullet(T_1)$$

Taking homology we get

$$H_i(X \setminus T_1, T_1 \setminus T_2) \xrightarrow{\sim} H_i(S_\bullet^{\mathcal{U}}(X)/S_\bullet(T_1)). \tag{1}$$

On the other hand, the exact sequence of complexes

$$0 \to S_\bullet(T_1) \to S_\bullet^{\mathcal{U}}(X) \to S_\bullet^{\mathcal{U}}(X)/S_\bullet(T_1) \to 0$$

induces a long exact sequence

$$\cdots \to H_i(T_1) \to H_i(S_\bullet^{\mathcal{U}}(X)) \to H_i(S_\bullet^{\mathcal{U}}(X)/S_\bullet(T_1)) \to H_{i-1}(T_1) \to \cdots$$

where we may replace $H_i(S_\bullet^{\mathcal{U}}(X))$ by $H_i(X)$ using the small simplices theorem. Comparing the long exact sequence

$$\cdots \to H_i(T_1) \to H_i(X) \to H_i(S_\bullet^{\mathcal{U}}(X)/S_\bullet(T_1)) \to H_{i-1}(T_1) \to \cdots$$

with the relative homology sequence

$$\cdots \to H_i(T_1) \to H_i(X) \to H_i(X, T_1) \to H_{i-1}(T_1) \to \cdots$$

finally gives

$$H_i(S_\bullet^{\mathcal{U}}(X)/S_\bullet(T_1)) \cong H_i(X, T_1),$$

whence the result in view of (1). □