What determines a variety?

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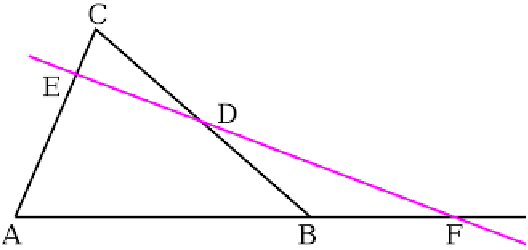
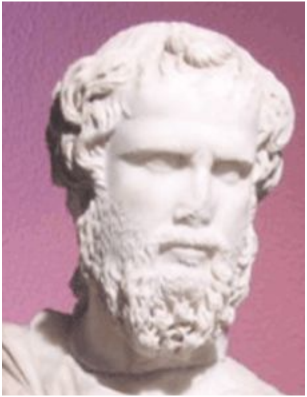
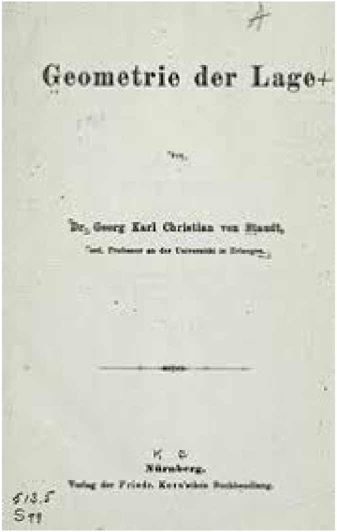
Menelaus's theorem, $\sim$ 70-140 AD





$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = -1.$$

Karl Georg Christian von Staudt, 1798–1867



Geometrie der Lage

von

Dr. Georg Karl Christian von Staudt,

Nürnberg.

Veblen-Young theorem (1908)

Given a projective geometry $\mathbf{P}$=(points, lines) of dim. $n \geq 2$ (with very few axioms),
there is a unique field K such that $\mathbf{P} \cong K\mathbb{P}^n$.

Recall: a scheme X is a

- topological space $|X|$, and a
- sheaf of rings $\mathcal{O}_X$ on the open subsets of $|X|$.

Main question

- How to read off properties of X from $|X|$?
- Does $|X|$ alone determine X?

Example: $\dim X =$ Krull dimension of $|X|$.

Negative results — easy ones

- For curves C_K, we see only the cardinality of K.
- Normalization is frequently homeomorphism.
- Purely inseparable maps are homeomorphisms.
- If K/L finite, then any K-variety X can be viewed as an L-variety with the same $|X|$. To fix this:
  - maximal choice $K = H^0(X, \mathcal{O}_X)$, equivalently
  - X is geometrically irreducible over K.

Negative results — surprising ones

– (Wiegand–Krauter, 1981) $|\mathbb{P}^2_F|$ same for all finite fields.

– (K.- Mangolte, 2009)
S_1, S_2: blow-up of $\mathbb{RP}^2$ in same number of points. Then

every Euclidean-homeo $\Phi : S_1(\mathbb{R}) \sim S_2(\mathbb{R})$

can be approximated by $\Psi : S_1(\mathbb{R}) \sim S_2(\mathbb{R})$ that are

both Euclidean and Zariski homeomorphisms.

Holds for C^0 and C^∞-approximations.

Theorem (Topology determines sheaf theory)

– K, L fields of char. 0,
– X_K, Y_L normal, projective, geom. irred. varieties,
– $|X_K| \sim |Y_L|$ homeomorphism.
Assume

1. *either* $\dim X \geq 4$,
2. *or* $\dim X \geq 3$ *and* K, L *are finitely generated* $/\mathbb{Q}$.

Then $K \cong L$ *and* $X_K \cong Y_L$.

Will outline the proof of a simpler theorem, its proof has the same basic ideas.

Theorem (Topology determines projective space)

– $\mathrm{char} L = 0$, K *arbitrary,*
– Y_L *normal, projective, geom. irreducible of dimension* $n \geq 2$
– $|\mathbb{P}^n_K| \sim |Y_L|$ *a homeomorphism. Then*

1. $Y_L \cong \mathbb{P}^n_L$ *and*
2. $K \cong L$.

Scip = set-theoretic complete intersection property

X variety, $Z \subset X$ closed subset.

- a divisor $D_Z \subset Z$ is SCI iff $D_Z = \operatorname{Supp}(D_X \cap Z)$ for some divisor D_X.
- Z irreducible: scip iff every divisor $D_Z \subset Z$ is SCI.
- $Z = \cup Z_i$ reducible: scip iff $\cup_i D_{Z_i}$ is SCI for all divisors $\emptyset \neq D_{Z_i} \subset Z_i$
- Z is generically scip iff there is a finite set $\Sigma \subset X$ such that scip holds if
  - $D_Z \cap \Sigma = \emptyset$ (makes it easier)
  - then also $D_X \cap \Sigma = \emptyset$ (makes it harder).

Algebraic geometry lemma I

Lemma 1. [Zero sets determine section] Z variety, L line bundle, $s_i \in H^0(Z, L^{n_i})$. Equivalent

1. $s_1^{m_1} = u \cdot s_2^{m_2}$ for some $u \in k[Z]^\times$,
2. $\mathrm{Supp}(s_1 = 0) = \mathrm{Supp}(s_2 = 0)$,

provided:

3. zero set is irreducible, and
4. either Z normal or zero set is disjoint from a certain finite $\Sigma(Z) \subset Z$.

Irreducibility can be guaranteed if
– $\dim Z \geq 2$ (by Bertini),
– $\dim Z \geq 1$ and k is finitely generated (by Hilbert).

Algebraic geometry lemma II

Lemma 2. (Boissière-Gabber-Serman) If X normal, there is a finite $\Sigma^{\mathrm{ncar}} \subset X$ such that every divisor disjoint from Σ^{ncar} is Cartier.

Homework I

HW 1. $\dim X \geq 2$, quasi-proj k-variety. Equivalent:
- Every irreducible curve $C \subset X$ is scip.
- k is locally finite (=algebraic over $\mathbb{F}_p$).

HW 2. $L_1, L_2 \subset \mathbb{P}^n$ linear spaces, meeting at a point.
- Then $L_1 \cup L_2$ is generically scip.
- $(\text{line}) \cup (\text{conic}) \subset \mathbb{P}^2$ is **not** generically scip.

Key observation

$D \subset \mathbb{P}^n$ divisor, $C \subset \mathbb{P}^n$ geom connected curve.
Assume $\mathrm{char}\, k = 0$. Then $D \cup C$ is generically scip iff

- $D =$ hyperplane and $C =$ line.

First special case of Theorem 2

Corollary. Assume $\mathrm{char} K = 0$. Then $|\mathbb{P}^n_L| \sim |\mathbb{P}^n_K|$ iff $K \cong L$.

Proof. Pick $\Phi : |\mathbb{P}^n_L| \sim |\mathbb{P}^n_K|$. $H_L \subset \mathbb{P}^n_L$ hyperplane, $\ell_L \subset \mathbb{P}^n_L$.

$\Rightarrow$ $H_L \cup \ell_L$ is generically scip.

$\Rightarrow$ $\Phi(H_L) \cup \Phi(\ell_L)$ is generically scip.

$\Rightarrow$ $\Phi(H_L) \subset \mathbb{P}^n_K$ is a hyperplane.

$\Rightarrow$ $\Phi(\text{linear space}) \subset \mathbb{P}^n_K$ is a linear space.

Finish by Veblen-Young.

Key observation, statement

Proposition

X normal, projective, $\rho(X) = 1$, $\mathrm{char}\, k = 0$.
$Z, W \subset X$ irreducible, $\dim(Z \cap W) = 0$.
Assume that $Z \cup W$ is generically scip. Then

1. *$Z \cap W$ is reduced and*
2. *either $k[Z \cap W] = k[Z]$ or $k[Z \cap W] = k[W]$.*

If Z, W are set-theoretic complete intersections then

3. *$Z \cap W$ is a k-point.*

Key observation, proof I

Choose L ample such that
$H^0(X, L) \to H^0(Z \cap W, L_{Z\cap W})$ is surjective.
Choose general $s_Z, s_W \in H^0(X, L)$, set
$\mathrm{Supp}(s_Z|_Z = 0) = \cup_i A_i$ and $\mathrm{Supp}(s_W|_W = 0) = \cup_j B_j$.
Generically scip $\Rightarrow \exists$ Cartier $D_{ij} \subset X$ such that
$\mathrm{Supp}(D_{ij}|_{Z\cup W}) = A_i + B_j$ (multiplicities = ?)
Linear algebra $\Rightarrow \exists$ D such that
$D|_Z = m_Z(s_Z|_Z = 0)$ and $D|_W = m_W(s_W|_W = 0)$.

Since $\rho(X) = 1$, $D = (s = 0)$ for $s \in H^0(X, L^m)$. By Lem 1

$$\begin{aligned} s^r|_Z &= u_Z \cdot s_Z^m|_Z && \text{for some} \quad u_Z \in k[Z]^\times, \\ s^r|_W &= u_W \cdot s_W^m|_W && \text{for some} \quad u_W \in k[W]^\times. \end{aligned} \qquad (*)$$

Key observation, proof II

$$s^r|_Z = u_Z \cdot s_Z^m|_Z \quad \text{for some} \quad u_Z \in k[Z]^\times, \qquad (*)$$
$$s^r|_W = u_W \cdot s_W^m|_W \quad \text{for some} \quad u_W \in k[W]^\times,$$

hence $(s_Z/s_W)^m|_{Z\cap W} = u_W|_{Z\cap W} \cdot u_Z^{-1}|_{Z\cap W}$ is in

$$\text{image of:} \quad k[W]^\times \times k[Z]^\times \to k[Z \cap W]^\times.$$

We can arrange s_Z/s_W to be an arbitrary element of $k[Z \cap W]^\times$, hence

$$k[Z \cap W]^\times / k[W]^\times \times k[Z]^\times \quad \text{is a torsion group.}$$

Key observation, proof III

$$k[Z \cap W]^\times / k[W]^\times \times k[Z]^\times \text{ is a torsion group.}$$

Apply next to $A = k[Z \cap W]$, $L_1 = k[W]$, $L_2 = k[Z]$.

HW 3, Algebra lemma. A Artin k-algebra, $\mathrm{char} k = 0$. $L_1, L_2 \subset A$ subfields. Equivalent

1. $A^\times / L_1^\times \cdot L_2^\times$ is torsion,
2. $A^\times / L_1^\times \cdot L_2^\times$ has finite rank,
3. either $A = L_1$ or $A = L_2$.

Note. Key case: A is a field.

I would like to see a simple proof.

Topology determines $\mathbb{P}^n$, proof

Recall Thm: If $\Phi : |\mathbb{P}^n_K| \sim |Y_L|$ homeomorphism, then
$Y_L \cong \mathbb{P}^n_L$ and $K \cong L$.

Assume: $\rho(Y) = 1$.

Pick $H \cup \ell \subset \mathbb{P}^n_K$ generically scip.
$\Rightarrow \Phi(H) \cup \Phi(\ell) \subset Y_L$ generically scip,
$\Rightarrow \big(\Phi(H) \cdot \Phi(\ell)\big) = 1$,
$\Rightarrow \big\{\Phi(H) : H \in |\mathcal{O}_{\mathbb{P}^n}(1)|\big\}$ is a linear system.
(Needs more argument, mainly if $\rho(Y) > 1$.)
$\Rightarrow$ It gives $Y_L \cong \mathbb{P}^n_L$.

We already proved that then $K \cong L$. $\square$

HW 4. K perfect, infinite and $\Phi : |\mathbb{P}^n_K| \sim |\mathbb{P}^n_K|$ homeo.
If Φ identity on K-points, then identity.

$\sim$ = linear equivalence
$\sim_{\mathrm{s}}$ = linear similarity: $m_1 D_1 \sim m_2 D_2$ for some $m_1, m_2 \neq 0$.
$\sim_{\mathrm{sa}}$ = linear similarity + D_1, D_2 ample and irreducible.

Main steps of the proof of Theorem 1

Step 1. $|X|$ determines $\sim_{\mathrm{sa}}$.
Step 2. $(|X|, \sim_{\mathrm{sa}})$ determines $\sim$.
Step 3. (Lieblich–Olsson) $(|X|, \sim)$ determines X.

Toward Step 1: Ampleness criterion

HW 5. X normal, projective, $\dim X \geq 3$.
Then an irreducible divisor H is $\mathbb{Q}$-Cartier and ample iff
$(*)$ For every divisor $D \subset X$ and closed points $p, q \in X \setminus D$, there is a divisor $H(p, q) \subset X$ such that

1. $H \cap D = H(p, q) \cap D$,
2. $p \notin H(p, q)$ and $q \in H(p, q)$.

Toward Step 1: Linear similarity

HW 6. X normal, projective, $\dim X \geq 3$,
H_1, H_2 irreducible, $\mathbb{Q}$-Cartier, ample.
Then $H_1 \sim_{sa} H_2$ iff

(*) Let $C_1, C_2 \subset X$ be any 2 disjoint, irred curves.
Then there is a $\mathbb{Q}$-Cartier, ample H' such that
$\operatorname{Supp}(H' \cap C_i) = \operatorname{Supp}(H_i \cap C_i)$ for $i = 1, 2$.

Toward Step 2: Linking = Liaison

Variant of scip. Fix L ample.

Defn. L-linking is free on $Z \cup W$ if

given $H_Z \sim_{\mathrm{sa}} L, H_W \sim_{\mathrm{sa}} L$, there is $H \sim_{\mathrm{sa}} L$ such that

$H \cap (Z \cup W) = (H_Z \cap Z) \cup (H_W \cap W)$.

Toward Step 2: Residue fields of points.

Proposition

$\dim X \geq 4$ *and* $\mathrm{char}\, k = 0$.
For $p, q \in X$ *equivalent:*

1. *There is a* $k(p) \hookrightarrow k(q)$.
2. *There are irreducible subvarieties* Z, W *such that*
   1. $\dim Z = 1, \dim W = 2$,
   2. $\mathrm{Supp}(Z \cap W) = \{p\}$,
   3. $q \in Z$,
   4. W *is SCI, and*
   5. L*-linking is free on* $Z \cup W$.

Toward Step 2: Isomorphism of zero-cycles

Corollary

$(|X|, \sim_{\mathrm{sa}})$ determines isomorphism of 0-dimensional reduced subschemes.

HW 7. X normal, $\mathrm{char}\, k = 0$, Z a zero-cycle of degree 0. Then Z is rationally equivalent to a zero-cycle

$$\sum [p_i - q_i] \text{ where } k(p_i) \cong k(q_i)\ \forall i.$$

Some questions I ran into

Conjecture

C_k smooth, projective curve, genus ≥ 1, k not locally finite. L very ample. For $s \in H^0(C_k, L)$ write $(s = 0) =: \{p_i(s) : i \in I\}$ and $(s = 0)_{\bar{k}} =: \{\bar{p}_i(s) : i \in \bar{I}\}$.

Then, for 'most' sections,

– $[p_i(s)] \in \mathrm{Pic}(C_k)$ are linearly independent (weak form).

– $[\bar{p}_i(s)] \in \mathrm{Pic}(C_{\bar{k}})$ are linearly independent (strong form).

Conjecture

C smooth, projective curve over $\bar{\mathbb{Q}}$.
Then for 'most' ample line bundles L,

every section of L^m has at least $g(C)$ zeros for every $m \geq 1$.

Notes.

– not sure what 'most' means.
– true for nodal rational curves.