Lecture on Mochizuki's anabeloids

Motivation: Assume $k = \mathbf{C}$, X is a hyperbolic curve over $\mathbf{C}$ (smooth of genus g with r deleted points such that $2g - 2 + r > 0$.
Consider the universal covering space $\widetilde{X} \to X$. It is known that $X \cong \mathbb{H}$, the upper half plane.
Thus there is an embedding

$$\Pi_X := \pi_1^{\mathrm{top}}(X) \hookrightarrow \mathrm{Aut}\,(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbf{R}) =: \Gamma.$$

Two subgroups $H_1, H_2 \subset \Gamma$ are commensurable if $[H_i : H_1 \cap H_2] < \infty$ for $i = 1, 2$. Given a subgroup $H \subset \Gamma$, its *commensurator* is

$$C_\Gamma(H) := \{\gamma \in \Gamma \,:\, \gamma^{-1} H \gamma \text{ is commensurable with } \ H\}$$

Fact: If $X = \mathbb{H}/\Pi_X$ is not isogenous to a Shimura curve, then $[C_\Gamma(\Pi_X) : \Pi_X] < \infty$.

Here two curves X, Y are called isogenous if there exist finite étale morphisms $X \leftarrow Z \to Y$.
In this case, consider $C := \mathbb{H}/C_\Gamma(\Pi_X)$ (it may only exist as an orbifold).
Since $[C_\Gamma(\Pi_X) : \Pi_X] < \infty$, we have a finite étale morphism $X \to C$.

Now assume that Y is a hyperbolic curve isogenous to X. That is, we have finite étale maps $X \leftarrow Z \to Y$, corresponding to subgroups $\Pi_X \supset \Pi_Z \subset \Pi_Y$ in Γ.
Lemma. We have $\Pi_Y \subset C_\Gamma(\Pi_X)$.

Indeed, we may assume $Z \to Y$ is Galois, hence $\Pi_Z \subset \Pi_Y$ is normal. Now if $\gamma \in \Pi_Y$, we have $\Pi_X \cap \gamma^{-1}\Pi_X\gamma \supset \Pi_Z \cap \gamma^{-1}\Pi_Z\gamma = \Pi_Z$, hence it is of finite index in Π_X. Applying the same argument with γ^{-1} shows the lemma.

By the lemma, we have a commutative diagram of finite étale morphisms

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & C. \end{array}$$

Thus if we denote by $\overline{\mathrm{Loc}}(X)$ the category of hyperbolic (orbi)curves isogenous to X, with morphisms finite étale morphisms (not necessarily over X), then C is a terminal object in this category. It is called a *core* of X. Its existence depends crucially on the assumption that X is not isogenous to a Shimura curve.

Mochizuki used the existence of a core (+ Riemann-Hurwitz and the finite generation of Π_C) to prove that for X as above there are only finitely many Y of given g and r isogenous to X.

Moreover, he proved that for generic X (corresponding to a dense open subset in the moduli space), in fact X is its own core. Thus every finite étale $Z \to X$, and even every Y isogenous to X corresponds to a canonically defined subgroup in Π_X, once an embedding $\Pi_X \subset \Gamma = \mathrm{PSL}_2(\mathbf{R})$ is fixed.

Now assume k is a field of characteristic 0, X a hyperbolic k-curve. Define $\overline{\mathrm{Loc}}_k(X)$ as above with morphisms k-morphisms, and call a terminal object, if exists, a k-core.

Proposition.

(1) If $k = \overline{k}$, base change to another algebraically closed field $K \supset k$ induces an equivalence of categories $\overline{\mathrm{Loc}}_k(X) \xrightarrow{\sim} \overline{\mathrm{Loc}}_K(X_K)$.

(2) If k is not algebraically closed, then there is a k-core in $\overline{\mathrm{Loc}}(X)$ if and only if there is a $\overline{k}$-core in $\overline{\mathrm{Loc}}_{\overline{k}}(X_{\overline{k}})$. [Galois descent.]

Recall now Grothendieck's theory of the fundamental group from SGA1.

Let X be a connected scheme equipped with a geometric base point $\bar{x} : \mathrm{Spec}\,(\Omega) \to X$.

For a finite étale cover $Y \to X$ we consider $\mathrm{Fib}_{\bar{x}} := Y \times_X \mathrm{Spec}\,(\Omega)$ as a set.

$Y \to \mathrm{Fib}_{\bar{x}}(Y)$ is a set-valued exact functor on the category of finite étale covers of X (preserves finite limits and colimits).

Grothendieck's definition: $\pi_1(X, \bar{x}) := \mathrm{Aut}\,\mathrm{Fib}_{\bar{x}}$.

By definition an automorphism of $\mathrm{Fib}_{\bar{x}}$ induces an automorphism of the set $\mathrm{Fib}_{\bar{x}}(Y)$ for each finite étale cover $Y \to X$; in this way we obtain a natural left action of $\pi_1(X, \bar{x})$ on the set $\mathrm{Fib}_{\bar{x}}(Y)$.

Theorem.

(1) The group $\pi_1(X, \bar{x})$ is profinite, and its action on $\mathrm{Fib}_{\bar{x}}(Y)$ is continuous for every finite étale cover $Y \to X$.

(2) The functor $\mathrm{Fib}_{\bar{x}}$ induces an equivalence of the category of finite étale covers of S with the category of finite continuous left $\pi_1(X, \bar{x})$-sets.

Here the functor $\mathrm{Fib}_{\bar{x}}$ is *pro-representable*: there exists a (filtered) inverse system $P = (P_\alpha, \phi_{\alpha\beta})$ of finite étale covers and a functorial isomorphism

$$\varinjlim \mathrm{Hom}(P_\alpha, Y) \cong \mathrm{Fib}_{\bar{x}}(Y).$$

Here $\mathrm{Aut}\,(P_\alpha|X)$ is finite for all α, and $\pi_1(X, \bar{x}) = \varprojlim \mathrm{Aut}\,(P_\alpha|X)^{\mathrm{op}}$; this explains its profiniteness.

In fact, Grothendieck showed that one may choose as a pro-representing system the system of all *Galois covers* $P_\alpha \to X$, rigidified by choosing $p_\alpha \in \mathrm{Fib}_{\bar{x}}(P_\alpha)$ for each α.

Any two fibre functors on the category of finite étale X-schemes are (non-canonically) isomorphic. One way to prove this is to use pro-representability of the fibre functor.

Now to Mochizuki's interpretation.

Definition. A connected anabeloid $\mathcal{X}$ is a category equivalent to some

$$B(G) := \{\text{finite sets with a continuous } G\text{-action}\}$$

for a profinite group G. Example: the category of finite étale covers of a connected scheme.

A morphism $\phi : \mathcal{X} \to \mathcal{Y}$ of connected anabeloids is an exact functor $\phi^* : \mathcal{Y} \to \mathcal{X}$. We say that $\mathcal{X}$ and $\mathcal{Y}$ are *isomorphic* if there is a morphism $\mathcal{X} \to \mathcal{Y}$ inducing an *equivalence* of categories.

An anabeloid is a finite product of connected anabeloids. The connected components can be recovered from the category structure of the anabeloid.

Notice that for the trivial group 1 the anabeloid $B(1)$ is just the category of finite sets. A *base point* of $\mathcal{X}$ is a morphism $B(1) \to \mathcal{X}$. The isomorphism class of its automorphism group is the fundamental group $\Pi_{\mathcal{X}}$ of $\mathcal{X}$. Of course, for $\mathcal{X} = B(G)$ we have $\Pi_{\mathcal{X}} = G$.

Proposition (Mochizuki). Assume $\mathcal{X} = B(G)$ and $\mathcal{Y} = B(H)$ are connected anabeloids. We have an equivalence of categories

$$\mathcal{M}or(\mathcal{X}, \mathcal{Y}) \cong \mathcal{H}om^{out}(G, H).$$

Here on the left is the *category* of morphisms $\mathcal{X} \to \mathcal{Y}$, on the right the category whose objects are continuous homomorphisms $G \to H$, and morphisms $\phi \to \psi$ are elements $h \in H$ such that $\psi(g) = h^{-1}\phi(g)h$ for all $g \in G$.

Passing to isomorphism classes of objects we obtain a bijection of sets

$$\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) \cong \mathrm{Hom}^{out}(G, H).$$

where on the left we consider *isomorphism classes* of morphisms $\mathcal{X} \to \mathcal{Y}$ and on the right outer homomorphisms.

We recover the usual Hom-set $\mathrm{Hom}(G, H)$ by choosing base points of $\mathcal{X}$, $\mathcal{Y}$ and considering base point preserving morphisms.

These equivalences are induced by the obvious map sending a homomorphism $G \to H$ to the induced pullback functor $B(H) \to B(G)$. Checking the equivalences is a nontrivial exercise.

Slimness: A profinite group is **slim** if every open subgroup has trivial centralizer. A morphism $G \to H$ of profinite groups is relatively slim if the image of every open subgroup of G has trivial centralizer in H. By passing to $B(G)$ we transfer these notions to (connected) anabeloids.

Example: If K is a p-adic field or a number field, then Gal_K is slim (consequence of local class field theory, resp. a theorem of F. K. Schmidt). The anabeloid of finite étale covers of a hyperbolic curve over an algebraically closed field of char. 0, or over a p-adic field or number field is slim.

The importance of slimness is that if $\mathcal{X} \to \mathcal{Y}$ is a morphism of connected slim anabeloids whose induced morphism on π_1's has open image, then $\mathrm{Aut}\,(\mathcal{X} \to \mathcal{Y})$ is trivial.

Given an anabeloid $\mathcal{X}$ and an object $S \in \mathcal{X}$, consider the full subcategory $\mathcal{X}_S$ of objects over S. A morphism $\mathcal{Y} \to \mathcal{X}$ is finite étale if it factors as a composite of an isomorphism $\mathcal{Y} \cong \mathcal{X}_S$ and the natural map $\mathcal{X}_S \to \mathcal{X}$ for some S (the latter is given by base change).

Now assume $\mathcal{X}$ is slim, and consider the category $\mathrm{Et}_{\mathcal{X}}$ of finite étale covers of $\mathcal{X}$. This is a priori a 2-category, but by the consequence of slimness recalled above, we may pass to the associated 1-category without losing information (diagrams commute up to unique isomorphism). Fact: the functor $\mathcal{X} \to \mathrm{Et}_{\mathcal{X}}$ given by $S \mapsto (\mathcal{X}_S \to \mathcal{X})$ is an equivalence of categories.

Henceforth we assume our anabeloids are slim, and use the above equivalence.

Now that we have defined finite étale covers, we can define isogenies for anabeloids, and also the categories $\mathrm{Loc}(\mathcal{X})$ and $\overline{\mathrm{Loc}}(\mathcal{X})$ for an anabeloid $\mathcal{X}$. In the first, objects are finite étale $\mathcal{Y} \to \mathcal{X}$, with morphisms finite étale morphisms not necessarily over $\mathcal{X}$. In the second, objects are anabeloids isogenous to $\mathcal{X}$, and morphisms similar.

A core for $\mathcal{X}$ is a terminal object in this category, if it exists.

Note: If $\mathcal{X}$ comes from a hyperbolic k-curve having a k-core, it will not give rise to a core for $\mathcal{X}$, only to a relative notion with respect to the morphism of anabeloids $\mathcal{X} \to B(\mathrm{Gal}_k)$ (the latter is slim over arithmetic fields). We need an absolute notion:

Definition. A quasi-core for $\mathcal{X}$ is a relatively slim morphism of (slim) anabeloids $\mathcal{X} \to \mathcal{Q}$ such that the natural forgetful functor $\mathrm{Loc}_{\mathcal{Q}}(\mathcal{X}) \to \mathrm{Loc}(\mathcal{X})$ is an equivalence. (In $\mathrm{Loc}_{\mathcal{Q}}(\mathcal{X})$ morphisms are over $\mathcal{Q}$ but not necessarily over $\mathcal{X}$.) The quasi-core is faithful if the associated homomorphism $\Pi_{\mathcal{X}} \to \Pi_{\mathcal{Q}}$ is injective.

Fundamental example. Assume K is a number field or a p-adic field, and X a hyperbolic (orbi)curve over K. Assume that $X_{\mathbf{C}}$ is not isogenous to a Shimura curve. Then X has a K-core $X \to C$. The étale fundamental group $\Pi_{\overline{C}}$ of $C_{\overline{K}}$ is isomorphic to the abstract profinite group

$$\widehat{\Pi} = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r | [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_r = 1 \rangle$$

where g is the genus of C and r the number of points at infinity. This group is slim, in particular it has trivial center.

The étale fundamental group Π_C of C sits in an exact sequence

$$1 \to \Pi_{\overline{C}} \to \Pi_C \to \mathrm{Gal}_K \to 1.$$

Choosing an isomorphism $\lambda : \Pi_{\overline{C}} \cong \widehat{\Pi}$, the inner action of Π_C on $\Pi_{\overline{C}}$ induces a homomorphism

$$\Phi : \Pi_C \to \mathrm{Aut}\,(\widehat{\Pi}).$$

The corresponding outer homomorphism is independent of λ.

Denote the anabeloids associated with X, C by $\mathcal{X}$, $\mathcal{C}$, respectively. The composite map

$$\mathcal{X} \to \mathcal{C} \to Q := B(\mathrm{Aut}\,(\widehat{\Pi}))$$

is a quasi-core for $\mathcal{X}$ (this follows from the defining property of the K-core).

Moreover, it is faithful because Φ is injective. This follows from a deep result of Hoshi and Mochizuki (proven earlier for affine curves over number fields by Matsumoto and generalizing the classical result of Belyi for $\mathbf{P}^1 \setminus \{0, 1, \infty\}$): *the outer representation* $\mathrm{Gal}_K \to \mathrm{Out}(\widehat{\Pi})$ *associated to the hyperbolic K-curve C is faithful.*

Thus in the commutative exact diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\overline{C}} & \longrightarrow & \Pi_C & \longrightarrow & \mathrm{Gal}_K & \longrightarrow & 1 \\ & & \cong\big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \widehat{\Pi} & \longrightarrow & \mathrm{Aut}(\widehat{\Pi}) & \longrightarrow & \mathrm{Out}(\widehat{\Pi}) & \longrightarrow & 1 \end{array}$$

the middle arrow is injective.

Application. Since $\widehat{\Pi}$ is topologically finitely generated, it is known that $\mathrm{Aut}\,(\widehat{\Pi})$ is countably topologically generated. Thus there exists a descending sequence

$$\mathrm{Aut}\,(\widehat{\Pi}) = H_0 \supset H_1 \supset H_2 \supset \cdots$$

of open normal subgroups with trivial intersection. The inverse system

$$\mathcal{Q} = \mathcal{Q}_0 \leftarrow \mathcal{Q}_1 \leftarrow \mathcal{Q}_2 \leftarrow \cdots$$

of the anabeloids $\mathcal{Q}_n := B(H_n)$ plays the role of a universal cover of $\mathcal{Q}$ and thus the analogue of the upper half plane $\mathbb{H}$ in the complex case.

Fix a coherent system $(\mathcal{X}_n)$ of connected components of $\mathcal{X} \times_{\mathcal{Q}} \mathcal{Q}_n$. The fact that $\mathcal{Q}$ is a quasi-core implies that every automorphism of $\mathcal{X}_n$ induces a well-defined automorphism of $\mathcal{Q}_n$ *over* $\mathcal{Q}$. Whence an outer homomorphism

$$\lim_{\leftarrow} \mathrm{Aut}\,(\mathcal{X}_n|\mathcal{X}) \to \lim_{\leftarrow} \mathrm{Aut}\,(\mathcal{Q}_n|\mathcal{Q}) =: \Pi_{\mathcal{Q}}$$

which is *injective* as $\mathcal{Q}$ is faithful. Thus it gives rise to a subgroup $\Pi_{\mathcal{X}/\mathcal{Q}} \subset \Pi_{\mathcal{Q}}$ (well-defined up to conjugacy). The construction is functorial in $\mathcal{X}$, thus each element of $\mathrm{Loc}(\mathcal{X})$ and even $\overline{\mathrm{Loc}}(\mathcal{X})$ corresponds to a subgroup of $\Pi_{\mathcal{Q}}$. This gives a canonical construction for the fundamental groups of curves isogenous to X.

The dependence on the choice of the system (H_n) can be eliminated by some group-theoretic considerations.