NOTES ON HOMOLOGICAL ALGEBRA

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Contents

1. Background from category theory

Definition 1.1. A *category* consists of *objects* as well as *morphisms* between pairs of objects; given two objects A, B of a category $\mathcal{C}$, the morphisms from A to B form a set, denoted by $\mathrm{Hom}(A, B)$. (Notice that in contrast to this we do not impose that the objects of the category form a set.) These are subject to the following constraints.

(1) For each object A the set $\mathrm{Hom}(A, A)$ contains a distinguished element id_A, the identity morphism of A.

(2) Given two morphims $\phi \in \mathrm{Hom}(B, C)$ and $\psi \in \mathrm{Hom}(A, B)$, there exists a canonical morphism $\phi \circ \psi \in \mathrm{Hom}(A, C)$, the composition of ϕ and ψ. The composition of morphisms should satisfy two natural axioms:
   - Given $\phi \in \mathrm{Hom}(A, B)$, one has $\phi \circ \mathrm{id}_A = \mathrm{id}_B \circ \phi = \phi$.

- (Associativity rule) For $\lambda \in \mathrm{Hom}(A, B)$, $\psi \in \mathrm{Hom}(B, C)$, $\phi \in \mathrm{Hom}(C, D)$ one has $(\phi \circ \psi) \circ \lambda = \phi \circ (\psi \circ \lambda)$.

A morphism $\phi \in \mathrm{Hom}(A, B)$ is an *isomorphism* if there exists $\psi \in \mathrm{Hom}(B, A)$ with $\psi \circ \phi = \mathrm{id}_A$, $\phi \circ \psi = \mathrm{id}_B$; we denote the set of isomorphisms between A and B by Isom(A, B).

Examples 1.2. In these notes, the main examples we'll consider will be algebraic. Thus we shall consider, for example, the category of groups, abelian groups, rings, or modules over a fixed ring R. In all these examples the morphisms are the homomorphisms between appropriate objects.

Remark 1.3. If the objects themselves form a set, we say that the category is *small*. In this case one can associate an oriented graph to the category by taking objects as vertices and defining an oriented edge between two objects corresponding to each morphism.

In the examples above the categories are not small but if we restrict to some set of objects we obtain small subcategories (in the sense to be defined below).

For small categories it is easy to visualize the contents of the following definition.

Definition 1.4. The *opposite category* $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is "the category with the same objects and arrows reversed"; i.e. for each pair of objects (A, B) of $\mathcal{C}$, there is a canonical bijection between the sets $\mathrm{Hom}(A, B)$ of $\mathcal{C}$ and $\mathrm{Hom}(B, A)$ of $\mathcal{C}^{op}$ preserving the identity morphisms and composition.

Next we consider subcategories.

Definition 1.5. A *subcategory* of a category $\mathcal{C}$ is just a category $\mathcal{D}$ consisting of some objects and some morphisms of $\mathcal{C}$; it is a *full* subcategory if given two objects in $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$, i.e. *all* $\mathcal{C}$-morphisms between A and B are morphisms in $\mathcal{D}$.

Examples 1.6. The category of abelian groups is a full subcategory of the category of groups. Given a ring $R \neq \mathbf{Z}$, the category of R-modules is a subcategory of that of abelian groups, but not a full subcategory.

Now comes the second basic definition of category theory.

Definition 1.7. A *(covariant) functor* F between two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of a rule $A \mapsto F(A)$ on objects and a map on sets of morphisms $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ which sends identity morphisms to identity morphisms and preserves composition. A *contravariant functor* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a functor from $\mathcal{C}_1$ to $\mathcal{C}_2^{op}$.

Examples 1.8. Here are some examples of functors.

(1) The identity functor is the functor $\mathrm{id}_{\mathcal{C}}$ on any category $\mathcal{C}$ which leaves all objects and morphisms fixed.
(2) Other basic examples of functors are obtained by fixing an object A of a category $\mathcal{C}$ and considering the covariant functor $\mathrm{Hom}(A, \_\_)$ (resp. the contravariant functor $\mathrm{Hom}(\_\_, A)$) from $\mathcal{C}$ to the category Sets which sends an object B the set $\mathrm{Hom}(A, B)$ (resp. $\mathrm{Hom}(B, A)$) and a morphism $\phi : B \to C$ to the set-theoretic map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ (resp. $\mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$) induced by composing with ϕ.
(3) There are *forgetful functors* defined by forgetting structure. For instance, associating to an R-module the underlying abelian group and to an R-module homomorphism the underlying group homomorphism defines the forgetful functor from the category of R-modules to that of abelian groups.
(4) On the category Mod_R of R-modules important examples of functors are given by tensor product. Fix an R-module B. The rule

$$A \mapsto A \otimes_R B, \quad (\phi : A_1 \to A_2) \mapsto (\phi \otimes \mathrm{id}_B : A_1 \otimes B \to A_2 \otimes B)$$

defines a functor $\_\_ \otimes_R B : \mathrm{Mod}_R \to \mathrm{Mod}_R$. Similarly, tensoring by a module A on the left gives a functor $A \otimes_R \_\_ : \mathrm{Mod}_R \to \mathrm{Mod}_R$.

Definition 1.9. If F and G are two functors with same domain $\mathcal{C}_1$ and target $\mathcal{C}_2$, a *morphism of functors* Φ between F and G is a collection of morphisms $\Phi_A : F(A) \to G(A)$ in $\mathcal{C}_2$ for each object $A \in \mathcal{C}_1$ such that for every morphism $\phi : A \to B$ in $\mathcal{C}_1$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\Phi_A} & G(A) \\ {\scriptstyle F(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle G(\phi)} \\ F(B) & \xrightarrow{\Phi_B} & G(B) \end{array}$$

commutes. The morphism Φ is an isomorphism if each Φ_A is an isomorphism; in this case we shall write $F \cong G$.

Remark 1.10. Given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ one can define (modulo some set-theoretic difficulties) a new category called the *functor category* of the pair $(\mathcal{C}_1, \mathcal{C}_2)$ whose objects are functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ and whose morphisms are morphisms of functors. Here the composition rule for some Φ and Ψ is induced by the composition of the morphisms Φ_A and Ψ_A for each object A in $\mathcal{C}_1$.

We now turn to categories with additional properties, abstracting some properties of categories of modules over some ring.

Definition 1.11. A category $\mathcal{A}$ is *additive* if the following hold:

- For any two objects A, B the set $\mathrm{Hom}(A, B)$ carries the structure of an abelian group.
- The compositions of morphisms $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ are $\mathbf{Z}$-bilinear maps.
- There is an object $0 \in \mathcal{A}$ that is both initial and final (i.e. for every object $A \in \mathcal{A}$ there is a unique morphism $0 \to A$ and a unique morphism $A \to 0$).
- For any two objects A, B the product $A \times B$ exists (defined by the usual universal property).

In an additive category the *kernel* of a morphism $\phi : A \to B$ is an object $\ker(\phi)$ together with a morphism $\kappa : \ker(\phi) \to A$ such that every morphism $\psi : C \to A$ with $\phi \circ \psi = 0$ factors uniquely as a composite $C \to \ker(\phi) \xrightarrow{\kappa} A$. Similarly, the *cokernel* of ϕ is an object $\mathrm{coker}(\phi)$ together with a morphism $\gamma : B \to \mathrm{coker}(\phi)$ such that every morphism $\psi : B \to C$ with $\psi \circ \phi = 0$ factors uniquely as a composite $B \to \mathrm{coker}(\phi) \xrightarrow{\gamma} C$.

The kernel and the cokernel may not exist for ϕ. When they do, we define the *image* of ϕ as $\mathrm{im}(\phi) := \ker(B \to \mathrm{coker}(\phi))$ and its *coimage* as $\mathrm{coim}(\phi) := \mathrm{coker}(\ker(\phi) \to A)$. Note that by definition there is a canonical morphism $\mathrm{coim}(\phi) \to \mathrm{im}(\phi)$. With these notions exact sequences are defined in the usual way.

Definition 1.12. An additive category $\mathcal{A}$ is *abelian* if every morphism ϕ has a kernel and a cokernel and the canonical morphism $\mathrm{coim}(\phi) \to \mathrm{im}(\phi)$ is an isomorphism.

Basic examples of abelian categories are categories of modules over some (not necessarily commutative) ring. The *Freyd–Mitchell embedding theorem* states that every *small* abelian category can be embedded as a full subcategory in the category modules over a suitable ring R.

Somewhat less straightforward examples are given by sheaves of abelian groups on some topological space. Later we shall encounter additive categories which are not abelian.

2. Categories of modules

A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is *additive* if for any two objects $A, B \in \mathcal{A}$ the induced map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is a group homomorphism. In what follows all functors between additive categories will be understood to be additive.

Definition 2.1. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is *left exact* if for every short exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\mathcal{A}$ the sequence

$$0 \to F(A_1) \to F(A_2) \to F(A_3)$$

is exact; it is *right exact* if

$$F(A_1) \to F(A_2) \to F(A_3) \to 0$$

is exact. We say that F is *exact* if it is both left and right exact.

There are also notions of left and right exactness for contravariant functors G: left exactness is defined by exactness of

$$0 \to G(A_3) \to G(A_2) \to G(A_1)$$

and right exactness by that of

$$G(A_3) \to G(A_2) \to G(A_1) \to 0.$$

Remark 2.2. In the Freyd–Mitchell embedding theorem cited in the previous section the functor realizing the embedding is exact.

Examples 2.3. Fix objects A and B in $\mathcal{A}$.

(1) The functor $\mathrm{Hom}(A, \_\_)$ from $\mathcal{A}$ to the category of abelian groups is left exact but not always right exact.
(2) The contravariant functor $\mathrm{Hom}(\_\_, B)$ is left exact but not always right exact.

Now we specialize to the category Mod_R of modules over a ring R. We shall assume our rings to be commutative with unit. However, everything will hold for noncommutative rings as well, one just has to choose a convention whether one considers left or right modules over a ring R.

We shall study modules satisfying exactness properties for the above two Hom-functors, and also for the tensor product functors $A \otimes_R \_\_ : \mathrm{Mod}_R \to \mathrm{Mod}_R$ and $\_\_ \otimes_R B : \mathrm{Mod}_R \to \mathrm{Mod}_R$ which do not exist in an arbitrary abelian category. They are right exact but not left exact in general. We start with the tensor product.

Definition 2.4. An R-module A is *flat* over R if the functor $A \otimes_R \_\_$ is exact.

Example 2.5. The R-module R is obviously flat. Since tensor products commute with direct sums, free R-modules are also flat. (Recall that a free R-module is by definition an R-module isomorphic to a direct sum of copies of the R-module R.)

In Proposition 5.4 below we'll see that conversely finitely generated flat modules over a Noetherian local ring are free.

We note for later use the following fact:

Proposition 2.6. *An R-module A is flat if and only if the restriction of the functor $A \otimes_R \_\_$ to the full subcategory of finitely generated R-modules is exact.*

Proof. We only have to treat left exactness. Assume $\phi : B_0 \to B$ is an injective map of R-modules, and $\alpha = \sum a_i \otimes b_i$ is an element of $A \otimes_R B_0$ that maps to 0 in $A \otimes B$. To prove that $\alpha = 0$ we may replace B_0 by the finitely generated submodule generated by the b_i. Also, by construction of the tensor product the image of α in $A \otimes_R B$ is 0 if the corresponding element of the free R-module $R[A \times B]$ is a sum of finitely many relations occurring in the definition of $A \otimes_R B$, so we find a finitely generated submodule $\phi(B_0) \subset B^f \subset B$ such that α maps to 0 already in $A \otimes_R B^f$. □

A stronger notion is that of faithful flatness:

Definition 2.7. An R-module A is *faithfully flat* over R if it is flat and for every R-module B one has $B \neq 0$ if and only if $A \otimes_R B \neq 0$.

It is easy to see that faithful flatness is equivalent to the following property: a sequence of R-modules $0 \to B_1 \to B_2 \to B_3 \to 0$ is exact *if and only if* the sequence $0 \to A \otimes_R B_1 \to A \otimes_R B_2 \to A \otimes_R B_3 \to 0$ is exact. Another important characterization is:

Lemma 2.8. *A flat R-module A is faithfully flat if and only if $A \otimes_R R/P \neq 0$ for every maximal ideal $P \subset R$.*

Proof. Necessity is obvious. For sufficiency assume $B \neq 0$ but $A \otimes_R B = 0$. For a nonzero $b \in B$ consider its annihilator $I = \mathrm{Ann}(b) := \{r \in R : rb = 0\}$. Since $b \neq 0$, we have $I \neq R$, so there is a maximal ideal $P \subset R$ with $P \supset I$. Tensoring by A the injective map $R/I \to B$ obtained by sending 1 to b we obtain an injective map $A \otimes_R R/I \hookrightarrow A \otimes_R B = 0$ by flatness of A, so $A \otimes_R R/I = 0$. But $A \otimes_R R/I$ surjects onto $A \otimes_R R/P$, so $A \otimes_R R/P = 0$ as well, contradiction. □

Now we can introduce an important class of (faithfully) flat R-modules:

Proposition 2.9. *If R is Noetherian and $\widehat{R}$ is the completion of R with respect to some ideal $I \subset R$, then $\widehat{R}$ is flat over R. If moreover R is local, then $\widehat{R}$ is faithfully flat over R.*

Proof. First note that for all finitely generated R-modules A we have isomorphisms $\widehat{A} \cong \widehat{R} \otimes_R A$. When $A = R^n$ this is easily checked using the definition of completions. In the general case write A as a cokernel of a suitable morphism $R^m \to R^n$ and use right exactness of completion and of the tensor product. In view of Proposition 2.6 flatness of $\widehat{R}$ now follows as it is known that the functor $A \mapsto \widehat{A}$ is exact on the category of finitely generated modules over Noetherian rings.

Now assume R is local with maximal ideal P. In view of the lemma above we have to check that the tensor product $\widehat{R} \otimes_R R/P \cong \widehat{R}/P\widehat{R}$ is nonzero. In fact, it is known that $\widehat{R}$ is local with maximal ideal $P\widehat{R}$. A 'cheaper' argument is as follows:

by definition of completions we have a natural surjection $\widehat{R} \to R/I$ which we may compose with the natural surjection $R/I \to R/P$ induced by the inclusion $I \subset P$. The composite $\widehat{R} \to R/P$ factors through $\widehat{R}/P\widehat{R}$ which must then be nonzero. □

Another type of important example is the following.

Example 2.10. If $S \subset R$ is a multiplicatively closed subset, the localization R_S is flat over R. In particular, when R is an integral domain, its fraction field is flat over R.

To see this, let $A' \hookrightarrow A$ be an injective morphism of R-modules. We have to show that $A' \otimes_R R_S \to A \otimes_R R_S$ is still injective. A general element of $A' \otimes_R R_S$ is a sum of elements of the form $a' \otimes (r/s')$ with $a' \in A'$, $r \in R$, $s' \in S$. Choosing a common denominator in S and using bilinearity of the tensor product we may rewrite this element in the form $a \otimes (1/s)$ with $a \in A'$, $s \in S$. An element of this form is 0 in $A' \otimes_R R_S$ if and only if $tsa = 0$ for some $t \in S$. But such an equation holds in A' if and only if it holds in A.

Note that R_S is not always faithfully flat over R. For instance, $\mathbf{Q}$ is not faithfully flat over $\mathbf{Z}$ because $A \otimes_{\mathbf{Z}} \mathbf{Q} = 0$ for every torsion abelian group A.

Now to the covariant Hom-functor. The following definition can be made in an arbitrary abelian category:

Definition 2.11. An R-module P is *projective* if the functor $\mathrm{Hom}(P, \_\_) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact.

By left exactness of $\mathrm{Hom}(P, \_\_)$ a module P is projective if and only if the natural map $\mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$ given by $\lambda \to \alpha \circ \lambda$ is surjective for every *surjection* $\alpha : A \to B$.

Lemma 2.12.

(1) *The R-module R is projective.*
(2) *Arbitrary direct sums of projective modules are projective.*

Proof. For the first statement, given an R-homomorphism $\lambda : R \to B$ and a surjection $A \to B$, lift λ to an element of $\mathrm{Hom}(R, A)$ by lifting $\lambda(1)$ to an element of A. The second statement is immediate from the compatibility of Hom-groups with direct sums in the first variable. □

Corollary 2.13. *A free R-module is projective.*

Construction 2.14. Given an R-module A, define a free R-module $F(A)$ by taking direct sum of copies of R indexed by the elements of A. One has a surjection $\pi_A : F(A) \to A$ induced by mapping 1_a to a, where 1_a is the element of $F(A)$ with 1 in the component corresponding to $a \in A$ and 0 elsewhere.

When A is finitely generated by a system $a_1, \dots, a_n$ of generators, one may consider the finitely generated free module $F_{\rm fg}(A)$ defined as a finite direct sum of copies of R indexed by the elements a_i. Sending 1_{a_i} to a_i still defines a surjection $F_{\rm fg}(A) \to A$.

Thus every R-module is the quotient of a free R-module and hence of a projective module. This is expressed by saying that *the category of R-modules has enough projectives*. The full subcategory of finitely generated modules also has enough projectives by the second part of the construction.

Projective modules are in fact direct summands of free modules:

Lemma 2.15. *An R-module P is projective if and only if there exist an R-module A and a free R-module F with $P \oplus A \cong F$.*

By symmetry, A is then also projective.

Proof. For sufficiency, extend a map $\lambda : P \to B$ to F by defining it to be 0 on A and use projectivity of F. For necessity, take F to be the free R-module $F(P)$ associated with P in the above example. We claim that we have an isomorphism as required, with $A = \ker(\pi_P)$. Indeed, as P is projective, we may lift the identity map of P to a map $\pi : P \to F(P)$ with $\pi_P \circ \pi = \mathrm{id}_P$. □

Since free modules are flat, the lemma implies:

Corollary 2.16. *Projective modules are flat.*

Projective modules over local rings are in fact free:

Proposition 2.17. *Let R be a local ring with maximal ideal P and residue field k, and let A be a finitely generated R-module.*

If A is projective, then A is free over R.

Proof. Let $a_1, \dots a_n \in A$ be elements such that their mod PA images form a basis of the k-vector space A/PA. By Nakayama's lemma they generate A, so the map $\phi : R^n \to A$ sending $(r_1, \dots, r_n)$ to $r_1 a_1 + \cdots + r_n a_n$ is surjective and an isomorphism mod P. By projectivity of A we then have $R^n \cong A \oplus B$ where $B = \ker(\phi)$. Since $R^n/PR^n \xrightarrow{\sim} A/PA$, we get $B \subset PR^n$. But then $R^n = A + PR^n$, so $R^n \xrightarrow{\sim} A$, again by Nakayama's lemma. □

Remark 2.18. In fact, Kaplansky proved that the proposition holds without assuming A finitely generated, but the proof is much more involved.

The above proposition yields a characterization of finitely generated projective modules over arbitrary Noetherian rings.

Proposition 2.19. *Let R be a Noetherian ring. A finitely generated R-module A is projective if and only if $A \otimes_R R_P$ is free for all prime ideals $P \subseteq R$.*

In fact, one may restrict to maximal ideals in this statement.

For the proof we need some lemmas.

Lemma 2.20. *Let R be a ring.*

(1) *An R-module A is 0 if and only if $A \otimes_R R_P = 0$ for all maximal ideals P.*
(2) *A morphism $\varphi : A_1 \to A_2$ of R-modules is injective (resp. surjective) if and only if $\varphi \otimes \mathrm{id}_{R_P}$ is (resp. surjective) for all maximal ideals P.*

Proof. For the nontrivial implication of (1) assume $A \neq 0$, and pick a nonzero $a \in A$. The map $R \to A$ sending $r \in R$ to ra shows that the submodule $\langle a \rangle \subset A$ is isomorphic to R/I for some ideal $I \subsetneq R$. Pick a maximal ideal $I \subset P \subset R$. We then have $a \neq 0$ in $A \otimes_R R_P$.

Statement (2) follows by applying (1) to the kernel (resp. cokernel) of φ. □

Lemma 2.21. *Given a finitely presented R-module A, an R-module B and a prime ideal $P \subset R$, we have canonical isomorphisms*

$$\mathrm{Hom}_R(A, B) \otimes_R R_P \xrightarrow{\sim} \mathrm{Hom}_{R_P}(A \otimes_R R_P, B \otimes_R R_P).$$

Proof. We have a natural map $\mathrm{Hom}_R(A, B) \otimes_R R_P \to \mathrm{Hom}_{R_P}(A \otimes_R R_P, B \otimes_R R_P)$ induced by tensoring with R_P. If $A \cong R^n$ for some n, then this map is an isomorphism because the map $\mathrm{Hom}_R(R, B) \otimes_R R_P \xrightarrow{\sim} \mathrm{Hom}_{R_P}(R \otimes_R R_P, B \otimes_R R_P)$ identifies with the identity map of $B \otimes_R R_P$. For the general case write A as a cokernel of some map $R^m \to R^n$ (this is possible as A is finitely presented). We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(A, B) \otimes_R R_P \arrow[r] \arrow[d] & \mathrm{Hom}_R(R^n, B) \otimes_R R_P \arrow[r] \arrow[d] & \mathrm{Hom}_R(R^m, B) \otimes_R R_P \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{R_P}(A \otimes_R R_P, B \otimes_R R_P) \arrow[r] & \mathrm{Hom}_{R_P}(R^n \otimes_R R_P, B \otimes_R R_P) \arrow[r] & \mathrm{Hom}_{R_P}(R^m \otimes_R R_P, B \otimes_R R_P)
\end{tikzcd}$$

whose rows are exact by left exactness of $\mathrm{Hom}(\_\_, B)$ and by flatness of R_P over R. The second and third vertical maps are isomorphisms by the previous case, hence so is the first. □

Proof of Proposition 2.19. The 'only if' part follows from Lemma 2.15 because if A is a direct summand of a free module over R, so is $A \otimes_R R_P$ over R_P. For the 'if' part take an exact sequence $0 \to K \to F \xrightarrow{f} A \to 0$ with F finitely generated and free. We show that this sequence splits. This is equivalent to showing that the map $\mathrm{Hom}_R(A, F) \to \mathrm{Hom}_R(A, A)$ induced by f is surjective. Indeed, if this is the case, then a splitting is given by a preimage of $\mathrm{id}_A \in \mathrm{Hom}_R(A, A)$; conversely, if id_A comes from $\mathrm{Hom}_R(A, F)$, then so does every element of $\mathrm{Hom}_R(A, A)$ by left composition.

By assumption for P maximal the induced map $\mathrm{Hom}_{R_P}(A \otimes_R R_P, F \otimes_R R_P) \to \mathrm{Hom}_{R_P}(A \otimes_R R_P, A \otimes_R R_P)$ is surjective. By the lemma above this is the same as the map $\mathrm{Hom}_R(A, F) \otimes_R R_P \to \mathrm{Hom}_R(A, A) \otimes_R R_P$ (note that A is finitely presented because R is Noetherian)), so we conclude from part (2) of Lemma 2.20. $\square$

We now consider the dual notion of injective modules.

Definition 2.22. An R-module Q is *injective* if the functor $\mathrm{Hom}(\_\_, Q) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact.

By right exactness of $\mathrm{Hom}(\_\_, Q)$ a module Q is injective if and only if given an injective map $\alpha : A \hookrightarrow B$, every homomorphism $\lambda_A : A \to Q$ extends to a homomorphism $\lambda_B : B \to Q$ with $\lambda_A = \lambda_B \circ \alpha$.

Remark 2.23. Arbitrary direct products of injective modules are injective. This follows from compatibility of the functor $\mathrm{Hom}(A, \_\_)$ with direct products.

Lemma 2.24 (Baer's criterion)**.** *An R-module Q is injective if and only if for every ideal $I \hookrightarrow R$ and every R-module homomorphism $\lambda_I : I \to Q$ there is an extension $\lambda_R : R \to Q$.*

Proof. Only the 'if' part requires proof. Assume given an inclusion $A \hookrightarrow B$ and a map $\lambda : A \to Q$. Consider pairs (A', λ') where $A \subset A' \subset B$ is an R-submodule and $\lambda' : A' \to Q$ extends λ. Inclusion maps $A' \hookrightarrow A''$ induce a natural partial ordering on the set of such pairs and the condition of Zorn's lemma is satisfied. Let $(\widetilde{A}, \widetilde{\lambda})$ be a maximal pair. If $\widetilde{A} = B$, we are done. Suppose $\widetilde{A} \neq B$, and pick $b \in B \setminus \widetilde{A}$. The set $I := \{r \in R : rb \in \widetilde{A}\}$ is an ideal in R equipped with a natural map $\lambda_I : I \to \widetilde{A}$ given by $r \mapsto rb$. By assumption the composite map $\widetilde{\lambda} \circ \lambda_I : I \to Q$ extends to a map $\lambda_R : R \to Q$. On the submodule $\widetilde{A} \cap \langle b \rangle \subset B$ the map $\widetilde{\lambda}$ coincides with the map $\lambda_b : \langle b \rangle \to Q, rb \mapsto \lambda_R(r)$. Hence $\widetilde{\lambda}$ and $rb \mapsto \lambda_R(r)$ patch together to a map from $\widetilde{A} + \langle b \rangle \subset B$ to Q, contradicting the maximality of $\widetilde{A}$. $\square$

Recall that an abelian group A is *divisible* if for all $n \in \mathbf{Z}$ the map $a \mapsto na$ is surjective on A. Basic examples of divisible abelian groups are $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$.

Corollary 2.25. *An abelian group Q is injective if and only if it is divisible.*

Proof. For 'only if' fix $a \in Q$ and define a homomorphism $n\mathbf{Z} \to Q$ by sending n to a. By injectivity it extends to a homomorphism $\mathbf{Z} \to Q$. The image of 1 will be an element $b \in Q$ with $nb = a$. Conversely, since every ideal of $\mathbf{Z}$ is of the form $n\mathbf{Z}$, reversing the argument gives that the condition in Baer's criterion is satisfied. $\square$

This enables us to construct injective modules over an arbitrary ring R. Note first that given an abelian group T and a ring R, the set of abelian group homomorphisms $\mathrm{Hom}_{\mathbf{Z}}(R,T)$ carries a natural R-module structure by composing maps $R \to T$ by the multiplication-by r-map $R \to R$ for $r \in R$.

Lemma 2.26. *For an R-module A the natural map of abelian groups*

$$\mathrm{Hom}_{\mathbf{Z}}(A,T) \to \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbf{Z}}(R,T))$$

given by $\phi \mapsto (a \mapsto (r \mapsto \phi(ra))$ for $r \in R$, $a \in A$ is an isomorphism. Moreover, making A vary induces an isomorphism of contravariant functors

$$\mathrm{Hom}_{\mathbf{Z}}(\_\_,T) \xrightarrow{\sim} \mathrm{Hom}_R(\_\_, \mathrm{Hom}_{\mathbf{Z}}(R,T)).$$

Proof. An inverse map sends $\rho \in \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbf{Z}}(R,T))$ to $a \mapsto \rho(a)(1)$ for $a \in A$. Functoriality follows from the construction. □

Corollary 2.27. *If Q is an injective abelian group, then $\mathrm{Hom}_{\mathbf{Z}}(R,Q)$ is an injective R-module.*

Proof. Assume given an injection $\iota : A \hookrightarrow B$. We have to show surjectivity of the map $\mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbf{Z}}(R,Q)) \to \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbf{Z}}(R,Q))$. By the lemma it identifies with the natural map $\mathrm{Hom}_{\mathbf{Z}}(B,Q) \to \mathrm{Hom}_{\mathbf{Z}}(A,Q)$ induced by ι which is surjective by injectivity of Q. □

Now we can prove that the *category of R-modules has enough injectives.*

Proposition 2.28. *Every R-module A can be embedded in an injective R-module.*

Proof. Set $Q := \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})$. This is an injective R-module by the previous corollary. Define now a module $I(A)$ as the direct product of copies of Q indexed by the set $\mathrm{Hom}_R(A,Q)$. This is still an injective R-module by Remark 2.23. Define a map $A \to I(A)$ by sending $a \in A$ to $\phi(a)$ in the component indexed by $\phi \in \mathrm{Hom}_R(A,Q)$. To see that this map is injective, note first that $\mathrm{Hom}_R(A,Q) \cong \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})$ by the previous lemma. Thus it will suffice to construct for each nonzero $a \in A$ a group homomorphism $\bar{\phi} : A \to \mathbf{Q}/\mathbf{Z}$ with $\bar{\phi}(a) \neq 0$, for then the corresponding $\phi \in \mathrm{Hom}_R(A,Q)$ will satisfy $\phi(a) \neq 0$. Let $\langle a \rangle \subset A$ be the $\mathbf{Z}$-submodule of A generated by a. Define a group homomorphism $\langle a \rangle \to \mathbf{Q}/\mathbf{Z}$ by sending a to any nonzero element of $\mathbf{Q}/\mathbf{Z}$ if a has infinite order and to a nonzero element of order dividing n if a has finite order n. By divisibility of $\mathbf{Q}/\mathbf{Z}$ this map extends to a homomorphism $\bar{\phi} : A \to \mathbf{Q}/\mathbf{Z}$ as required. □

3. Complexes and resolutions

We begin with some constructions that work in an arbitrary abelian category.

Definition 3.1. A *(cohomological) complex* $A^\bullet$ in an abelian category $\mathcal{A}$ is a sequence of morphisms

$$\dots\dots \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \xrightarrow{d^{i+2}} \dots$$

for all $i \in \mathbf{Z}$, satisfying $d^{i+1} \circ d^i = 0$ for all i.

We shall also use the convention $A_{-i} := A^i$, giving rise to the *homological indexing* of the complex.

We introduce the notations

$$Z^i(A^\bullet) := \ker(d^i), \quad B^i(A^\bullet) := \mathrm{Im}\,(d^{i-1}) \quad \text{and} \quad H^i(A^\bullet) := Z^i(A^\bullet)/B^i(A^\bullet).$$

The complex $A^\bullet$ is said to be *acyclic* or *exact* if $H^i(A^\bullet) = 0$ for all i.

A *morphism of complexes* $\phi : A^\bullet \to B^\bullet$ is a collection of homomorphisms $\phi^i : A^i \to B^i$ for all i such that the diagrams

$$\begin{array}{ccc} A^i & \longrightarrow & A^{i+1} \\ {\scriptstyle\phi^i}\big\downarrow & & \big\downarrow{\scriptstyle\phi^{i+1}} \\ B^i & \longrightarrow & B^{i+1} \end{array}$$

commute for all i. Thus complexes form a category in which morphisms are defined as above; we shall denote it by $C(\mathcal{A})$. The reader will check that this category is again abelian.

By its defining property, a morphism of complexes $\phi : A^\bullet \to B^\bullet$ induces maps $H^i(\phi) : H^i(A^\bullet) \to H^i(B^\bullet)$ for all i. We say that ϕ is a *quasi-isomorphism* if the $H^i(\phi)$ are isomorphisms for all i.

An important source for quasi-isomorphisms is the following.

Definition 3.2. Two morphisms of complexes $\phi, \psi : A^\bullet \to B^\bullet$ are *(chain) homotopic* if there exist maps $k^i : A^i \to B^{i-1}$ for all i satisfying

$$\phi^i - \psi^i = k^{i+1} \circ d^i_A + d^{i-1}_B \circ k^i \tag{1}$$

for all i.

Two complexes $A^\bullet$ and $B^\bullet$ are *homotopy equivalent* if there exist morphisms of complexes $\phi : A^\bullet \to B^\bullet$ and $\rho : B^\bullet \to A^\bullet$ such that $\phi \circ \rho$ is homotopic to the identity map of $B^\bullet$ and $\rho \circ \phi$ is homotopic to the identity map of $A^\bullet$.

The following statement follows from the definitions:

Lemma 3.3. *If ϕ and ψ are homotopic morphisms $A^\bullet \to B^\bullet$, then $H^i(\phi) = H^i(\psi)$ for all i.*

In particular, when ϕ induces a homotopy equivalence of complexes, then ϕ is a quasi-isomorphism.

Remark 3.4. Historically one of the first examples of a (homological) complex of abelian groups was the singular complex $S_\bullet(X)$ associated with a topological space X; its homology groups are by definition the (singular) homology groups of X. The assignment $X \mapsto S_\bullet(X)$ induces a functor from the category of topological spaces (with continuous maps as morphisms) to the category of complexes of abelian groups. It is a basic result in algebraic topology that homotopic continuous maps from a space X to a space Y induce homotopic morphisms of complexes $S_\bullet(X) \to S_\bullet(Y)$ and hence homotopy equivalent topological spaces give rise to homotopy equivalent singular complexes. This is the origin of the use of homotopical morphisms of complexes in homological algebra.

A *short exact sequence of complexes* is a short exact sequence in the category $C(\mathcal{A})$. In other words, it is a sequence of morphisms of complexes

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

such that the sequences

$$0 \to A^i \to B^i \to C^i \to 0$$

are exact for all i. Now we have the following basic fact.

Proposition 3.5. *Given a short exact sequence*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of complexes, there is a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \xrightarrow{\partial} H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet) \to \dots$$

The map ∂ is usually called the *connecting homomorphism* or the *(co)boundary map*.

For the proof of the proposition we need the following equally basic lemma.

Lemma 3.6 (The Snake Lemma). *Given a commutative diagram*

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

with exact rows, there is an exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma).$$

Proof. It is easy to give a proof in a category of R-modules. The construction of all maps in the sequence is then immediate, except for the map $\partial : \ker(\gamma) \to \operatorname{coker}(\alpha)$. For this, lift $c \in \ker(\gamma)$ to $b \in B$. By commutativity of the right square, the element $\beta(b)$ maps to 0 in C', hence it comes from a unique $a' \in A'$. Define $\partial(c)$ as the image of a' in $\operatorname{coker}(\alpha)$. Two choices of b differ by an element $a \in A$ which maps to 0 in $\operatorname{coker}(\alpha)$, so ∂ is well-defined. Checking exactness is left as an exercise to the readers.

In a general abelian category take the smallest abelian subcategory containing all morphisms in the diagram. It is a small subcategory, so we may apply the Freyd–Mitchell embedding theorem to it. Since the embedding functor is exact, we deduce the required exact sequence from the case of module categories. □

Proof of Proposition 3.5. Applying the Snake Lemma to the diagram

$$\begin{array}{ccccccc}
 & & A^i/B^i(A^\bullet) & \longrightarrow & B^i/B^i(B^\bullet) & \longrightarrow & C^i/B^i(C^\bullet) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & Z^{i+1}(A^\bullet) & \longrightarrow & Z^{i+1}(B^\bullet) & \longrightarrow & Z^{i+1}(C^\bullet) & &
\end{array}$$

yields a long exact sequence

$$H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet) \to H^{i+1}(C^\bullet),$$

and the proposition is obtained by splicing these sequences together. □

Corollary 3.7. *Assume given a commutative diagram of morphisms of complexes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\
 & & {\scriptstyle\phi_A}\Big\downarrow & & {\scriptstyle\phi_B}\Big\downarrow & & {\scriptstyle\phi_C}\Big\downarrow & & \\
0 & \longrightarrow & A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C'^\bullet & \longrightarrow & 0
\end{array}$$

with exact rows. If any two of the vertical maps are quasi-isomorphisms, then so is the third one.

Proof. Apply the five lemma to the associated commutative diagram of long exact sequences. □

Now we assume $\mathcal{A}$ *has enough projectives* (e.g. it is a category of modules over a ring). As a consequence, every object A has a *projective resolution* $P_\bullet \to A$, i.e. there is an acyclic complex of the form

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

(note the homological indexing!) with P_i projective. Such a resolution can be obtained inductively: first take a surjection $p_0 : P_0 \to A$ with P_i projective. Once P_i

and $p_i : P_i \to P_{i-1}$ have been defined (with the convention $P_{-1} = A$), one defines P_{i+1} and p_{i+1} by applying the same construction to $\ker(p_i)$ in place of A.

Remark 3.8. A projective resolution can be interpreted as a quasi-isomorphism between the complex

$$\cdots \to P_2 \to P_1 \to P_0 \to 0 \to 0 \to \cdots$$

and the complex

$$\cdots \to 0 \to 0 \to A \to 0 \to 0$$

in which A is the only nonzero term and it is placed in degree 0. Indeed, we have a morphism of complexes given by the map $P_0 \to A$ in degree 0 and the zero map elsewhere; it is a quasi-isomorphism because both complexes have trivial homology outside degree 0 and there it equals A. This almost tautological observation will be useful later.

Now the basic fact concerning projective resolutions is:

Lemma 3.9. *Assume given a diagram*

$$\begin{tikzcd}
\dots \arrow[r] & P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
\dots \arrow[r] & B_2 \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] & B_0 \arrow[r, "b_0"] & B \arrow[r] & 0
\end{tikzcd}$$

where the upper row is a complex with the P_i projective and the lower row is an acyclic complex. Then α extends to a morphism of complexes given by the diagram:

$$\begin{tikzcd}
\dots \arrow[r] & P_2 \arrow[r, "p_2"] \arrow[d, "\alpha_2"] & P_1 \arrow[r, "p_1"] \arrow[d, "\alpha_1"] & P_0 \arrow[r, "p_0"] \arrow[d, "\alpha_0"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
\dots \arrow[r] & B_2 \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] & B_0 \arrow[r, "b_0"] & B \arrow[r] & 0
\end{tikzcd}$$

Moreover, any two such extensions are chain homotopic.

Proof. To construct α_i, assume that the α_j are already defined for $j < i$, with the convention $\alpha_{-1} = \alpha$. Observe that $\mathrm{Im}\,(\alpha_{i-1} \circ p_i) \subset \mathrm{Im}\,(b_i)$; this is immediate for $i = 0$ and follows from $b_{i-1} \circ \alpha_{i-1} \circ p_i = \alpha_{i-2} \circ p_{i-1} \circ p_i = 0$ for $i > 0$ by exactness of the lower row. Hence by the projectivity of P_i we may define α_i as a preimage in $\mathrm{Hom}(P_i, B_i)$ of the map $\alpha_{i-1} \circ p_i : P_i \to \mathrm{Im}\,(b_i)$. For the second statement, suppose $\beta_i : P_i \to B_i$ define another extension. Define $k_{-1} = 0$ and assume k_j defined for $j < i$ satisfying $\alpha_j - \beta_j = k_{j-1} \circ p_j + b_{j+1} \circ k_j$. This implies $\mathrm{Im}\,(\alpha_i - \beta_i - (k_{i-1} \circ p_i)) \subset \mathrm{Im}\,(b_{i+1})$ because

$$b_i \circ (\alpha_i - \beta_i - (k_{i-1} \circ p_i)) = (\alpha_{i-1} - \beta_{i-1}) \circ p_i - b_i \circ k_{i-1} \circ p_i = k_{i-2} \circ p_{i-1} \circ p_i = 0,$$

so, again using the projectivity of P_i, we may define k_i as a preimage of $\alpha_i - \beta_i - (k_{i-1} \circ p_i) \in \mathrm{Hom}(P_i, \mathrm{Im}\,(b_{i+1}))$ in $\mathrm{Hom}(P_i, B_{i+1})$. $\square$

Corollary 3.10. *Any two projective resolutions of an object A are homotopy equivalent.*

Proof. Given two projective resolutions $P_\bullet \to A$ and $P'_\bullet \to A$, the identity map of A lifts to morphisms of complexes $\phi: P_\bullet \to P'_\bullet$ and $\phi': P'_\bullet \to P_\bullet$ by the lemma above. By the second statement of the lemma $\phi \circ \phi' : P'_\bullet \to P'_\bullet$ is chain homotopic to the identity map of $P'_\bullet$ and similarly for $\phi' \circ \phi : P_\bullet \to P_\bullet$. □

For a category $\mathcal{A}$ that *has enough injectives* the preceding arguments dualize. Using the fact that every object A embeds in an injective object we construct inductively *injective resolutions* $A \to Q^\bullet$, i.e. acyclic complexes of the form

$$0 \to A \to Q^0 \to Q^1 \to Q^2 \to \cdots$$

with the Q^i injective. The analogue of the previous lemma holds, with the same proof (performed in the opposite category of $\mathcal{A}$):

Lemma 3.11. *Assume given a diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \xrightarrow{a^0} & A^0 & \xrightarrow{a^1} & A^1 & \xrightarrow{a^2} & A^2 & \longrightarrow & \ldots \\ & & \Big\downarrow{\scriptstyle \alpha} & & & & & & & & \\ 0 & \longrightarrow & B & \xrightarrow{q^0} & Q^0 & \xrightarrow{q^1} & Q^1 & \xrightarrow{q^2} & Q^2 & \longrightarrow & \ldots \end{array}$$

where the lower row is a complex with the Q^i injective and the upper row is an acyclic complex. Then α extends to a morphism of complexes given by the diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \xrightarrow{a^0} & A^0 & \xrightarrow{a^1} & A^1 & \xrightarrow{a^2} & A^2 & \longrightarrow & \ldots \\ & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha_0} & & \Big\downarrow{\scriptstyle \alpha_1} & & \Big\downarrow{\scriptstyle \alpha_2} & & \\ 0 & \longrightarrow & B & \xrightarrow{q^0} & Q^0 & \xrightarrow{q^1} & Q^1 & \xrightarrow{q^2} & Q^2 & \longrightarrow & \ldots \end{array}$$

Moreover, any two such extensions are chain homotopic. In particular, any two injective resolutions of A are homotopy equivalent.

4. Derived functors

Derived functors remedy the defect of exactness of left or right exact functors.

Construction 4.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F: \mathcal{A} \to \mathcal{B}$ an additive functor. Assume that $\mathcal{A}$ has enough projectives. Then the *left derived functors L_iF* of F are defined as follows. Given an object A in $\mathcal{A}$, choose a projective resolution $P_\bullet \to A$ and consider the complex $F(P_\bullet)$, then set $L_iF(A) := H_i(F(P_\bullet))$. Given a morphism $\alpha : A \to B$ in $\mathcal{A}$, choose projective resolutions $P^A_\bullet \to A$, $P^B_\bullet \to B$. By Lemma 3.9 the map α induces a morphism of complexes $\alpha_\bullet : P^A_\bullet \to P^B_\bullet$. Define $L_iF(\alpha) := H_i(F(\alpha_\bullet))$.

Dually, when $\mathcal{A}$ has enough injectives, the *right derived functors* R^iF of F are defined by choosing an injective resolution $A \to Q^\bullet$ for an object A, and setting $R^iF(A) := H^i(F(Q^\bullet))$. Given a morphism $\alpha : A \to B$ in $\mathcal{A}$, the morphism $R^iF(\alpha)$ is defined by lifting α to a morphism of injective resolutions using Lemma 3.11, and then taking the i-th cohomology.

Lemma 4.2. *The definition of $L_iF(A)$ does not depend on the choice of the projective resolution $P_\bullet$, and that of $L_iF(\alpha)$ on the choice of the lifting $\alpha_\bullet$ of α.*

Similar statements hold for the right derived functors R^iF.

Proof. We do the case of L_iF. If $P_\bullet \to A$, $P'_\bullet \to A$ are two projective resolutions, they are homotopy equivalent by by Corollary 3.10. Applying the functor F we get that the complexes $F(P'_\bullet)$ and $F(P'_\bullet)$ are also homotopy equivalent via $F(\phi)$ and $F(\phi')$. It follows that $F(\phi)$ induces canonical quasi-isomorphisms $F(P_\bullet) \to F(P'_\bullet)$. The well-definedness of $L_iF(\alpha)$ follows from the second statement of Lemma 3.9. □

Proposition 4.3. *Assume that $\mathcal{A}$ has enough projectives and moreover F is a right exact functor. Then $L_0(F) \cong F$, and given a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules, there is an associated long exact sequence of the form*

$$\cdots \to L_iF(A) \to L_iF(B) \to L_iF(C) \to L_{i-1}F(A) \to \cdots$$

ending with $F(C) \to 0$.

Similarly, when $\mathcal{A}$ has enough injectives and F is left exact, we have $R^0(F) \cong F$, and a short exact sequence $0 \to A \to B \to C \to 0$ induces a long exact sequence of the form

$$\cdots \to R^iF(A) \to R^iF(B) \to R^iF(C) \to R^{i+1}F(A) \to \cdots$$

starting with $0 \to F(A)$.

The proof uses a lemma.

Lemma 4.4 (Horseshoe Lemma)**.** *Assume given a short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ and projective resolutions $P^A_\bullet \to A$, $P^C_\bullet \to C$. There exists a projective resolution $P^B_\bullet \to B$ fitting in a short exact sequence of complexes

$$0 \to P^A_\bullet \to P^B_\bullet \to P^C_\bullet \to 0$$

and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^A_\bullet & \longrightarrow & P^B_\bullet & \longrightarrow & P^C_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

A similar statement holds if we have injective resolutions $A \to Q_A^\bullet$, $C \to Q_C^\bullet$.

Proof. Notice first that any short exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$ of projective modules splits as a direct sum $P_2 \cong P_1 \oplus P_3$ (lift the identity map of P_3 to a map $P_3 \to P_2$). So if the lemma is true, we must have $P_i^B \cong P_i^A \oplus P_i^C$ for all i. We therefore set $P_i^B := P_i^A \oplus P_i^C$ and construct the maps in the required short exact sequence of resolutions by induction on i. First, by projectivity of P_0^C the map $p_C : P_0^C \to C$ lifts to a map $P_0^C \to B$. Taking the sum of this map with the composite map $P_0^A \to A \to B$ defines a map $P_0^A \oplus P_0^C \to B$, i.e. a map $p_B : P_0^B \to B$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0^A & \longrightarrow & P_0^B & \longrightarrow & P_0^C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle p_A} & & \big\downarrow{\scriptstyle p_B} & & \big\downarrow{\scriptstyle p_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

commute. Using the Snake Lemma we see that the surjectivity of p_A and p_C implies that of p_B and moreover the sequence $0 \to \ker(p_A) \to \ker(p_B) \to \ker(p_C) \to 0$ is exact. Now we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1^A & \longrightarrow & P_1^B & \longrightarrow & P_1^C & \longrightarrow & 0 \\ & & \big\downarrow & & & & \big\downarrow & & \\ 0 & \longrightarrow & \ker(p_A) & \longrightarrow & \ker(p_B) & \longrightarrow & \ker(p_C) & \longrightarrow & 0 \end{array}$$

with surjective vertical maps, so by repeating the above argument we get a surjective map $P_1^B \to \ker(p_B)$ making the diagram commute. Continuing the procedure we obtain the required short exact sequence of resolutions. $\square$

Proof of Proposition 4.3. The statements $L_0(F) \cong F$ and $R^0(F) \cong F$ under the stated exactness assumptions follow from the definitions. We now derive the long exact for left derived functors, the other one being similar. Apply the construction of the Horseshoe Lemma to get an exact sequence $0 \to P_\bullet^A \to P_\bullet^B \to P_\bullet^C \to 0$ of projective resolutions. As already remarked, here in fact $P_i^B \cong P_i^A \oplus P_i^C$ for all i, so that $F(P_i^B) \cong F(P_i^A) \oplus F(P_i^C)$ by additivity of F. Thus we have a short exact sequence of complexes $0 \to F(P_\bullet^A) \to F(P_\bullet^B) \to F(P_\bullet^C) \to 0$ to which we apply Proposition 3.5. $\square$

Remark 4.5.
(1) For a projective object P we have $L_iF(P) = 0$ for $i > 0$ as we may take $0 \to P \to P \to 0$ as a projective resolution. Similarly, $R^iF(Q) = 0$ for $i > 0$ when Q is injective. This gives rise to an important technique called *dimension shifting* which we explain for left derived functors. Given an object A we may choose an exact sequence $0 \to K \to P \to A \to 0$ with P projective. The long exact sequence then

induces isomorphisms $L_iF(A) \xrightarrow{\sim} L_{i-1}F(K)$ for $i > 1$. In this way, if we have to prove a property of L_iF for all A and all $i > 0$, we may reduce to the case $i = 1$ using induction.

(2) There is an additional functoriality property of derived functors that is often useful: given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

of short exact sequences, the diagram

$$\begin{array}{ccc} L_iF(C) & \longrightarrow & L_{i-1}F(A) \\ \downarrow & & \downarrow \\ L_iF(C') & \longrightarrow & L_{i-1}F(A') \end{array}$$

of boundary maps in the associated log exact sequences commutes for all i, and similarly for right derived functors. We omit the verification; it can also be derived from the construction of total derived functors we'll encounter later.

We now come to fundamental examples for the category of modules over a ring.

Examples 4.6. Let A be an R-module.

(1) The functor $A \otimes_R \_\_$ is right exact. Its i-th left derived functor is denoted by $\mathrm{Tor}_i^R(A, \_\_)$. Thus for every R-module B we have an isomorphism $\mathrm{Tor}_0^R(A, B) \cong A \otimes_R B$ and every short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$ of R-modules induces a long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(A, B_1) \to \mathrm{Tor}_1^R(A, B_2) \to \mathrm{Tor}_1^R(A, B_3) \to A \otimes_R B_1 \to A \otimes_R B_2 \to A \otimes_R B_3 \to 0.$$

(2) The functor $\mathrm{Hom}_R(A, \_\_)$ is left exact. Its i-th right derived functor is denoted by $\mathrm{Ext}^i_R(A, \_\_)$. Thus for every R-module B we have an isomorphism $\mathrm{Ext}^0_R(A, B) \cong \mathrm{Hom}_R(A, B)$ and every short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$ of R-modules induces a long exact sequence

$$0 \to \mathrm{Hom}_R(A, B_1) \to \mathrm{Hom}_R(A, B_2) \to \mathrm{Hom}_R(A, B_3) \to \mathrm{Ext}^1_R(A, B_1) \to \mathrm{Ext}^1_R(A, B_2) \to \cdots$$

Of course, these facts hold more generally in any abelian category having enough injectives.

One can define derived functors of contravariant functors by the same method as for covariant ones; the only difference is that left derived functors are defined via injective resolutions and right derived functors via projective ones. The basic example is:

Example 4.7. Let B be an R-module. The contravariant functor $\mathrm{Hom}_R(\_\_, B)$ is left exact. Its i-th right derived functor, denoted by $\mathrm{Ext}^i_R(\_\_, B)$, is defined by taking a projective resolution $P_\bullet \to A$ of an R-module A and setting $\mathrm{Ext}^i_R(A, B) := H^i(\mathrm{Hom}_R(P_\bullet, B))$. We have an isomorphism $\mathrm{Ext}^0_R(A, B) \cong \mathrm{Hom}_R(A, B)$ and every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ of R-modules induces a long exact sequence

$$0 \to \mathrm{Hom}_R(A_3, B) \to \mathrm{Hom}_R(A_2, B) \to \mathrm{Hom}_R(A_1, B) \to \mathrm{Ext}^1_R(A_3, B) \to \mathrm{Ext}^1_R(A_2, B) \to \cdots$$

Again this works more generally in any abelian category having enough injectives.

Now an important question arises: we have defined the groups $\mathrm{Ext}^i_R(A, B)$ in two ways, via a projective resolution of A and an injective resolution of B. Do the two methods give the same result? Similarly, we have defined the groups $\mathrm{Tor}_i(A, B)$ via a projective resolution of B; does using a projective resolution of A yield the same groups? The answer is yes in both cases - we'll see it in the section on total derived functors.

5. Ext and Tor

Now that we have Ext functors at our disposal, we can give another characterization of projective modules.

Proposition 5.1. *The following are equivalent for an R-module A.*

(1) *A is projective.*
(2) $\mathrm{Ext}^i_R(A, B) = 0$ *for all $i > 0$ and all R-modules B.*
(3) $\mathrm{Ext}^1_R(A, B) = 0$ *for all R-modules B.*

Proof. The implication $(1) \Rightarrow (2)$ is a special case of Remark 4.5 (1), $(2) \Rightarrow (3)$ is obvious, and $(3) \Rightarrow (1)$ follows from the long exact sequence of Ext. $\square$

For injective modules we have a similar characterization, but it can be sharpened using Baer's criterion.

Proposition 5.2. *The following are equivalent for an R-module B.*

(1) *B is injective.*
(2) $\mathrm{Ext}^i_R(A, B) = 0$ *for all $i > 0$ and all R-modules A.*
(3) $\mathrm{Ext}^1_R(A, B) = 0$ *for all R-modules A.*
(4) $\mathrm{Ext}^1_R(R/I, B) = 0$ *for all ideals $I \subset R$.*

Proof. The equivalence of (1)–(3) is proven as above and $(3) \Rightarrow (4)$ is obvious. Now consider the exact sequence $0 \to I \to R \to R/I \to 0$ for an ideal $I \subset R$ and apply the functor $\mathrm{Hom}_R(\_\_, B)$. The associated long exact sequence together with assumption (4) shows that the map $\mathrm{Hom}_R(R, B) \to \mathrm{Hom}_R(I, B)$ is surjective, so (1) holds by Baer's criterion. □

There is a similar characterization for flat modules as well.

Proposition 5.3. *The following are equivalent for an R-module A.*

(1) *A is flat.*
(2) $\mathrm{Tor}_i^R(A, B) = 0$ *for all $i > 0$ and all R-modules B.*
(3) $\mathrm{Tor}_1^R(A, B) = 0$ *for all R-modules B.*
(4) $\mathrm{Tor}_1^R(R/I, A) = 0$ *for every ideal $I \subset R$.*

Proof. To prove $(1) \Rightarrow (2)$ we use dimension shifting. Take an exact sequence $0 \to K \to P \to B \to 0$ with P projective. The long exact sequence of Tor gives an exact sequence

$$\mathrm{Tor}_1^R(A, P) \to \mathrm{Tor}_1^R(A, B) \to A \otimes_R K \to A \otimes_R P$$

Here $\mathrm{Tor}_1^R(A, P) = 0$ because P is projective, hence $\mathrm{Tor}_1^R(A, B) = 0$ as tensoring by A is left exact by assumption. In view of $\mathrm{Tor}_i(A, P) = 0$ for $i > 0$ the continuation of the sequence gives isomorphisms

$$\mathrm{Tor}_i^R(A, B) \xrightarrow{\sim} \mathrm{Tor}_{i-1}^R(A, K)$$

for all $i > 1$, whence (2) by induction on i.

The implications $(2) \Rightarrow (3) \Rightarrow (4)$ being obvious, only $(4) \Rightarrow (1)$ remains. Assume $\phi : B_0 \to B$ is an injective map of R-modules. To prove that $\phi \otimes \mathrm{id}_R : A \otimes_R B_0 \to A \otimes_R B$ is also injective we may assume using Proposition 2.6 that B_0 and B are both finitely generated. In this case we find $t_1, \dots, t_r \in B$ so that $B = \langle B_0, t_1, \dots, t_r \rangle$. Setting $B_j := \langle B_0, t_1, \dots, t_j \rangle$ for all $1 \le j \le r$ we obtain a finite filtration $B_0 \subset B_1 \subset \cdots \subset B_r = B$ such that $B_j/B_{j-1} \cong \langle t_j \rangle \cong R/I_j$ for the ideal $I_j \subset R$ annihilating t_j. But then tensoring the exact sequence $0 \to B_{j-1} \to B_j \to R/I_j \to 0$ by A gives an exact sequence

$$\mathrm{Tor}_1^R(R/I_j, A) \to A \otimes_R B_{j-1} \to A \otimes_R B_j$$

where $\mathrm{Tor}_1^R(R/I_j, A) = 0$ by assumption. Therefore $A \otimes_R B_0 \to A \otimes_R B$ is injective, being the composite of the injective maps $A \otimes_R B_{j-1} \to A \otimes_R B_j$. □

Using Proposition 5.3 it is easy to prove a structure theorem for finitely generated flat modules over Noetherian local rings.

Proposition 5.4. *Let R be a Noetherian local ring with maximal ideal P and residue field k, and let A be a finitely generated R-module. If A is flat over R, or more generally $\mathrm{Tor}_1(A,k)=0$, then A is free over R.*

Proof. Let $a_1,\dots a_n \in A$ be elements such that their mod PA images form a basis of the k-vector space A/PA. By Nakayama's lemma they generate A, so the map $\phi : R^n \to A$ sending $(r_1,\dots,r_n)$ to $r_1a_1+\cdots+r_na_n$ is surjective, giving rise to an exact sequence $0\to B\to R^r\to A\to 0$. Now tensor this sequence by k over R. Since $\mathrm{Tor}_1(A,k)=0$, the long exact sequence of Tor implies the exactness of $0\to B/PB\to R^r/PR^r\to A/PA\to 0$, whence $B=PB$. Since R is Noetherian, B is finitely generated, hence 0 by Nakayama's lemma. □

Finally we explain the origin of the names of the functors Tor and Ext. For Tor the name comes from 'torsion':

Proposition 5.5. *Let R be a ring and A an R-module. If $r\in R$ is a non-zerodivisor, then*

$$\mathrm{Tor}_1(R/(r),A)\cong\{a\in A\mid ra=0\}.$$

The module on the right hand side is called the r-torsion in A. The module A is called torsion free if it has no r-torsion for any r. In the case $R=\mathbf{Z}$ and $n\in\mathbf{Z}$ we get back the notion of n-torsion in an abelian group. It can be shown that in this case the whole torsion subgroup is isomorphic to $\mathrm{Tor}_1(\mathbf{Q}/\mathbf{Z},A)$.

Proof. Consider the exact sequence $0\to R\xrightarrow{r}R\to R/(r)\to 0$. Since R is projective as an R-module, part of the associated long exact Tor-sequence gives an exact sequence

$$0=\mathrm{Tor}_1^R(R,A)\to\mathrm{Tor}_1^R(R/(r),A)\to A\xrightarrow{r}A$$

whence the statement follows. □

Corollary 5.6. *Over a principal ideal domain a module is torsion free if and only if it is flat.*

Proof. If R is a principal ideal domain and $I\subset R$ is an ideal, then $I=(r)$ for some $r\in R$ and therefore $\mathrm{Tor}_1^R(R/I,A)=\mathrm{Tor}_1^R(R/(r),A)$. The vanishing of this group for all $r\in R$ is equivalent to A being torsion free by the proposition and to A being flat by Proposition 5.3. □

The Ext functor received its name from its relation to extensions. An *extension* of an R-module C by A is an R-module B fitting in a short exact sequence

$$0\to A\to B\xrightarrow{p}C\to 0.$$

The extension is *split* if there is a map $i : C \to B$ with $p \circ i = \mathrm{id}_C$. In this case B is isomorphic to the direct sum $A \oplus C$.

Two extensions B and B' are *equivalent* if there is an R-module map $\phi : B \to B'$ fitting in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

The Snake Lemma shows that in this case ϕ must be an isomorphism, whence it follows that we have indeed defined an equivalence relation. Denote by $\mathrm{Ext}(C, A)$ the set of equivalence classes of extensions of C by A.

Construction 5.7. We construct a map $\mathrm{Ext}(C, A) \to \mathrm{Ext}^1_R(C, A)$ as follows. Take a projective resolution $P_\bullet \to C$. By Lemma 3.9 the diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & C & \longrightarrow & 0 \\ & & & & & & & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ \dots & \longrightarrow & 0 & \xrightarrow{b_2} & A & \xrightarrow{b_1} & B & \xrightarrow{b_0} & C & \longrightarrow & 0 \end{array}$$

can be filled in as

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \alpha_1} & & \big\downarrow{\scriptstyle \alpha_0} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ \dots & \longrightarrow & 0 & \xrightarrow{b_2} & A & \xrightarrow{b_1} & B & \xrightarrow{b_0} & C & \longrightarrow & 0 \end{array}$$

where $\alpha_1 \circ p_2 = 0$. This shows that $\alpha_1 \in \mathrm{Hom}_R(P_1, A)$ is contained in $Z^1(\mathrm{Hom}(P_1, A))$, whence a class $e \in \mathrm{Ext}^1_R(C, A)$. Since any two projective resolutions of C are chain homotopy equivalent by Corollary 3.10, the class e does not depend on the choice of $P_\bullet$. Finally, equivalent extensions give rise to the same class $e \in \mathrm{Ext}^1_R(C, A)$ by construction.

In case of a split extension the splitting $i : C \to B$ induces a commutative diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle i \circ p_0} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ \dots & \longrightarrow & 0 & \xrightarrow{b_2} & A & \xrightarrow{b_1} & B & \xrightarrow{b_0} & C & \longrightarrow & 0 \end{array}$$

so that the associated class is 0.

Remark 5.8. There is another way to construct the extension class e: apply the functor $\mathrm{Hom}_R(C, \_)$ to the exact sequence $0 \to A \to B \to C \to 0$. The resulting long exact sequence gives rise to a coboundary map $\mathrm{Hom}_R(C, C) \to \mathrm{Ext}^1_R(C, A)$. Define e as the image of the identity map of C by this map. One can check that it depends only on the extension class of B and the resulting map $\mathrm{Ext}(C, A) \to \mathrm{Ext}^1_R(C, A)$ is the same as the one constructed above.

Proposition 5.9. *The map* $\mathrm{Ext}(C, A) \to \mathrm{Ext}^1_R(C, A)$ *constructed above is a bijection sending the class of the split extension* $A \oplus C$ *to 0.*

The proof uses the *pushout* construction: given an exact sequence of R-modules $0 \to A \to B \to C \to 0$ and an R-module map $\phi : A \to A'$, define an R-module X as the quotient of $A' \oplus B$ by the submodule of elements of the form $(\phi(a), -a)$ for $a \in A$. The natural projection $A' \oplus B \to B$ induces a map $X \to C$ sitting in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow{\scriptstyle\mathrm{id}} & & \\ 0 & \longrightarrow & A' & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

Moreover, X has the following universal property: for any X' sitting in a diagram of the above type there is an R-module map $X \to X'$ inducing an equivalence of extensions of C by A'. All this is straightforward to verify.

Sketch of proof of Proposition 5.9. We have already noted that the split extension class goes to 0. We construct an inverse map $\mathrm{Ext}^1_R(C, A) \to \mathrm{Ext}(C, A)$ as follows. Choose projective resolution $P_\bullet \to C$ as above. A class in $\mathrm{Ext}^1_R(C, A)$ is then represented by a homomorphism $\phi : P_1/\mathrm{Im}\,(p_2) \to A$. Now form the pushout of the extension $0 \to P_1/\mathrm{Im}\,(p_2) \to P_0 \to C \to 0$ by ϕ and take the associated class in $\mathrm{Ext}(C, A)$. As in the above construction, one verifies using Lemma 3.9 that choosing another projective resolution gives rise to the same extension class. It follows from the constructions that the two maps are inverse to each other; we leave details to the reader. □

Remark 5.10. It is possible to define an abelian group structure on $\mathrm{Ext}(C, A)$ so that the above bijection becomes an isomorphism of abelian groups. Besides pushout, this also uses the analogous pullback construction: given an exact sequence of R-modules $0 \to A \to B \xrightarrow{p} C \to 0$ and an R-module map $\psi : C' \to C$, define an R-module Y as the submodule of $B \oplus C'$ given by $\{(b, c') : p(b) = \psi(c')\}$. The inclusion $A \to B$ induces an inclusion $A \to Y$ sitting in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & Y & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{p} & C & \longrightarrow & 0. \end{array}$$

Here Y has a similar universal property as the pushout.

Now assume $0 \to A \to B \to C \to 0$ and $0 \to A \to B' \to C \to 0$ represent two classes in $\mathrm{Ext}(C, A)$. Form first the direct sum extension

$$0 \to A \oplus A \to B \oplus B' \to C \oplus C \to 0,$$

then take pushout by the map $A \oplus A \to A$, $(a_1, a_2) \mapsto a_1 + a_2$, and finally take the pullback of the resulting extension of $C \oplus C$ by A by the diagonal map $C \to C \oplus C$. The resulting extension is the *Baer sum* of the extensions given by B and B'. It can be checked that the construction respects the equivalence relation on extensions and gives $\mathrm{Ext}(C, A)$ the structure of an abelian group with zero element $A \oplus C$ so that the map $\mathrm{Ext}(C, A) \to \mathrm{Ext}^1_R(C, A)$ is an isomorphism. (Note: it is enough to check that the map $\mathrm{Ext}(C, A) \to \mathrm{Ext}^1_R(C, A)$ respects addition, then the group axioms for $\mathrm{Ext}(C, A)$ follow from those in $\mathrm{Ext}^1_R(C, A)$.)

Remarks 5.11. 1. The above description of Ext^1 via extensions works more generally in an abelian category having enough projectives, using the same arguments.

2. There is a generalization of the above construction to higher Ext groups due to Yoneda. Elements of the Yoneda Ext groups $\mathrm{YExt}^n(C, A)$ are represented by n-fold extensions

$$0 \to A \to B_1 \to B_2 \to \cdots \to B_n \to C \to 0$$

subject to a certain equivalence relation. We'll discuss this construction later, in the context of derived categories.

6. Homological dimension

In this section and the next we employ the notation A for rings and M, N for A-modules.

Definition 6.1. Let A be a ring, M is an A-module. We say that M has a projective resolution of length i if there exists an exact sequence

$$0 \to P_i \to \cdots \to P_0 \to M \to 0$$

with all P_j projective.

The *projective dimension* $\mathrm{pd}(M)$ of M is defined as the smallest i for which M has a projective resolution of length i; it may be infinite. The *global dimension* of A is

$$\mathrm{gldim}(A) := \sup\{\mathrm{pd}(M) \mid M \text{ is an } A\text{-module}\}.$$

It can be infinite as well.

Proposition 6.2. *The following are equivalent for an A-module M:*

(1) $\mathrm{pd}(M) \leq d$,

(2) $\mathrm{Ext}^i_A(M, N) = 0$ *for all A-modules N and $i > d$,*
(3) $\mathrm{Ext}^{d+1}_A(M, N) = 0$ *for all A-modules N,*
(4) *If* $0 \to M_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0$ *is exact and the P_i are projective, then M_d is projective.*

Proof. The implications $(4) \Rightarrow (1)$ and $(2) \Rightarrow (3)$ are obvious. $(1) \Rightarrow (2)$ follows because we may calculate the Ext functors using a projective resolution of length $\leq d$. To prove $(3) \Rightarrow (4)$, we split the exact sequence of (4) in short exact sequences of the form $0 \to M_i \to P_{i-1} \to M_{i-1} \to 0$ (here $M_0 = M$). Since the P_i are projective, the associated long exact sequences for Ext give isomorphisms $\mathrm{Ext}^{d+2-i}_A(M_{i-1}, N) \cong \mathrm{Ext}^{d+1-i}_A(M_i, N)$ for all N and all $0 \leq i \leq d$. Then (3) implies $\mathrm{Ext}^1_A(M_d, N) = 0$ for all N, so M_d is projective by Proposition 5.1. □

Corollary 6.3. *The global dimension of A is the smallest (possibly infinite) d such that $\mathrm{Ext}^{d+1}_A(M, N) = 0$ for all A-modules M, N.*

Example 6.4. The global dimension of a field is 0. The global dimension of $\mathbf{Z}$ is 1. Indeed, given an abelian group B, we may embed it in an injective abelian group Q. For abelian groups being injective is the same as being divisible, whence we get that the quotient Q/B is also injective. This means that B has an injective resolution of length 2, whence $\mathrm{Ext}^2_{\mathbf{Z}}(M, B) = 0$ for every abelian group M. (Alternatively, we could have deduced $\mathrm{gldim}(\mathbf{Z}) = 1$ from the fact that any subgroup of a free abelian group is free.) We shall see a vast generalization of this fact in Theorem 7.4 below.

Remarks 6.5.
1. One can define the injective dimension of a module as the length of the shortest possible injective resolution and prove an analogue of Proposition 6.2 for injective dimension. This shows that the global dimension of A is also the supremum of injective dimensions of modules because the previous corollary can be reproven using injective resolutions.
2. Quite generally, one can define the homological dimension of an abelian category $\mathcal{A}$ as the smallest d such that $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0$ for all $i > d$ and all objects A, B in $\mathcal{A}$. (Here the Ext-groups are defined, for instance, using the Yoneda method.) The above notion is the special case of module categories.

The following proposition allows us to restrict to finitely generated modules.

Lemma 6.6. *Let A be a ring and $i \geq 0$ an integer. The following are equivalent:*

(1) $\mathrm{Ext}^i_A(M, N) = 0$ *for all A-modules M, N.*
(2) $\mathrm{Ext}^i_A(M, N) = 0$ *for all A-modules N and all finitely generated A-modules M.*
(3) $\mathrm{Ext}^i_A(A/I, N) = 0$ *for all A-modules N and ideals $I \subset A$.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ being obvious, we show $(3) \Rightarrow (1)$. Take an injective resolution $0 \to N \to Q^\bullet$ of N, and truncate it as

$$0 \to N \to Q^0 \to Q^1 \to \cdots \to Q^{i-2} \to N^{i-1} \to 0$$

By a similar dimension-shifting argument as in the previous proof we have an isomorphism $\mathrm{Ext}^i_A(M,N) \cong \mathrm{Ext}^1_A(M,N^{i-1})$, so that $\mathrm{Ext}^i_A(M,N) = 0$ for all M if and only if $\mathrm{Ext}^1_A(M,N^{i-1}) = 0$ for all M. By Proposition 5.2 this is equivalent to N^{i-1} being injective, and also to $\mathrm{Ext}^1_A(A/I,N^{i-1}) = 0$ for all ideals $I \subset A$. This in turn is equivalent to saying that $\mathrm{Ext}^i_A(A/I,N) = 0$ for all i, again by dimension shifting. □

Corollary 6.7. $\mathrm{gldim}(A) = \sup\{\mathrm{pd}(M) \mid M$ *is a finitely generated A-module*$\}$.

In the local case projective dimension can also be calculated by Tor.

Proposition 6.8. *Let A be a Noetherian local ring with maximal ideal P and M a finitely generated A-module. Then $\mathrm{pd}(M) \le d$ if and only if $\mathrm{Tor}^A_{d+1}(M,k) = 0$, where $k = A/P$.*

Proof. The 'only if' part follows by calculating Tor by means of a projective resolution of length $\le d$. We prove the 'if' part by induction on d. The case $d = 0$ follows from Proposition 5.4. For the inductive step use the fact that M is finitely generated to obtain an exact sequence $0 \to N \to A^n \to M \to 0$ with some n. Here N is also finitely generated because A is Noetherian. The associated long exact Tor-sequence implies $\mathrm{Tor}^A_d(N,k) \cong \mathrm{Tor}^A_{d+1}(M,k)$ for $d > 0$, so $\mathrm{pd}(M) \le \mathrm{pd}(N) + 1 \le d$ by induction (for the first inequality note that a projective resolution of N can be extended by A^n to obtain a projective resolution of M). □

We can now prove a characterization of global dimension for Noetherian local rings which involves a single module.

Corollary 6.9. *If A is a Noetherian local ring with residue field k, then*

$$\mathrm{gldim}(A) = \mathrm{pd}(k) = \max\{d : \mathrm{Tor}^A_d(k,k) \neq 0\}.$$

Proof. The second equality follows from Proposition 6.8 applied with $M = k$. To prove the first one, note that by Corollary 6.7 and Proposition 6.8 we have $\mathrm{gldim}(A) \le d$ if and only if $\mathrm{Tor}^A_{d+1}(M,k) = 0$ for all finitely generated M over A. If $\mathrm{pd}(k) \le d$, then $\mathrm{Tor}^A_{d+1}(M,k) = 0$ follows by using a projective resolution of length $\le d$. Conversely, if $\mathrm{Tor}^A_{d+1}(M,k) = 0$ for all finitely generated M, then in particular this holds for $M = k$, whence $\mathrm{pd}(k) \le d$ by Proposition 6.8. □

We now state one of the most important results about homological dimension.

Theorem 6.10. (Serre) *A Noetherian local ring A is regular if and only if* $\mathrm{gldim}(A) < \infty$. *In this case* $\mathrm{gldim}(A) = \dim(A)$.

We shall give several proofs of Serre's theorem in these notes. The proof given in this section will use induction along regular sequences. In Section 8 we shall give a second proof of one implication and prove a refined statement using the Koszul complex. Finally, in Section 15 we shall present a recent proof that uses derived categories.

We begin with some auxiliary statements.

Proposition 6.11. *Let A be any ring, $x \in A$ a non-zerodivisor and M an $A/(x)$-module such that* $\mathrm{pd}_{A/(x)}(M) < \infty$. *Then* $\mathrm{pd}_A(M) = \mathrm{pd}_{A/(x)}(M) + 1$.

Proof. We proceed by induction on $\mathrm{pd}_{A/(x)}(M)$. If it is 0, then M is projective over $A/(x)$. Since x is a non-zerodivisor, we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow A/(x) \longrightarrow 0\ .$$

This is a projective resolution of $A/(x)$ over A hence $\mathrm{pd}_A(A/(x)) \leq 1$. Now $\mathrm{pd}_A(A/(x)) = 0$ would mean that $A/(x)$ is projective over A, hence a direct summand of a free A-module F. That is impossible because x is a non-zerodivisor in A, hence in F but a zero-divisor on $A/(x)$. So $\mathrm{pd}_A(A/(x)) = 1$ and therefore $\mathrm{pd}_A(F) = 1$ for any free $A/(x)$-module F. This also implies $\mathrm{pd}_A(M) = 1$ since M is a direct summand of a free $A/(x)$-module.

For the inductive step assume $\mathrm{pd}_{A/(x)}(M) > 0$ and take an exact sequence of $A/(x)$-modules

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

where P is projective over $A/(x)$. We have two associated long exact Ext-sequences of the form

$$\mathrm{Ext}^i(P,N) \longrightarrow \mathrm{Ext}^i(K,N) \longrightarrow \mathrm{Ext}^{i+1}(M,N) \longrightarrow \mathrm{Ext}^{i+1}(P,N)\ , \tag{2}$$

one over $A/(x)$ and one over A. Over $A/(x)$ we have $\mathrm{Ext}^i(P,N) = 0$ for all $i > 0$ hence $\mathrm{Ext}^i_{A/(x)}(K,N) \cong \mathrm{Ext}^{i+1}_{A/(x)}(M,N)$ for all $i > 0$. This implies

$$\mathrm{pd}_{A/(x)}(M) = \mathrm{pd}_{A/(x)}(K) + 1. \tag{3}$$

By induction, we then have

$$\mathrm{pd}_A(K) = \mathrm{pd}_{A/(x)}(K) + 1. \tag{4}$$

On the other hand, over A we have $\mathrm{Ext}^i(P,N) = 0$ for all $i > 1$ as we have proven $\mathrm{pd}_A(P) = 1$ above. Hence $\mathrm{Ext}^i_A(K,N) \cong \mathrm{Ext}^{i+1}_A(M,N)$ for all $i > 1$. This gives

$$\mathrm{pd}_A(M) = \mathrm{pd}_A(K) + 1$$

provided that $\mathrm{pd}_A(M) > 1$. This proves the proposition for the case $\mathrm{pd}_A(M) > 1$.

To conclude, we show that $\mathrm{pd}_{A/(x)}(M) > 0$ implies $\mathrm{pd}_A(M) > 1$. Assume this is not the case, i.e. $\mathrm{pd}_A(M) = 1$. This implies $\mathrm{Ext}^i_A(M, N) = 0$ for $i > 1$ and all N. Since P is projective over $A/(x)$, we know it has projective dimension 1 over A, so $\mathrm{Ext}^i_A(P, N) = 0$ for $i > 1$ and all N as well. From the long exact sequence (2) we get $\mathrm{Ext}^2_A(K, N) = 0$ for all N, and therefore $\mathrm{pd}_A(K) \leq 1$. Here $\mathrm{pd}_A(K) = \mathrm{pd}_{A/(x)}(M)$ by equations (3) and (4). So we have to exclude the case $\mathrm{pd}_A(M) = \mathrm{pd}_{A/(x)}(M) = 1$.

Choose an exact sequence of A-modules

$$0 \longrightarrow C \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F free. Then C is projective by Proposition 6.2 since $\mathrm{pd}_A(M) = 1$. So this is, in fact, a projective resolution of M. Tensoring the sequence with $A/(x)$ yields

$$\mathrm{Tor}^A_1(F, A/(x)) \longrightarrow \mathrm{Tor}^A_1(M, A/(x)) \longrightarrow C/xC \longrightarrow F/xF \longrightarrow M/xM \longrightarrow 0$$

where the first term is 0 because F is free and $M/xM = M$ since M is an $A/(x)$-module. Since C/xC and F/xF are already projective over $A/(x)$ and $\mathrm{pd}_{A/(x)}(M) = 1$, we get that $\mathrm{Tor}^A_1(M, A/(x))$ is projective over $A/(x)$ by Proposition 6.2 (applied with $d = 2 > 1$). By Proposition 5.5 we have $\mathrm{Tor}^A_1(M, A/(x)) = \{m \in M \mid xm = 0\} = M$, and therefore M is projective over $A/(x)$, a contradiction. □

Combining the proposition with Corollary 6.9 gives

Corollary 6.12. *If A is a Noetherian local ring with maximal ideal P, $x \in P$ is a non-zerodivisor and* $\mathrm{gldim}(A/(x)) < \infty$, *then*

$$\mathrm{gldim}(A) = \mathrm{gldim}(A/(x)) + 1.$$

We now prove a similar transition statement for A-modules.

Proposition 6.13. *Let A be a ring, M be an A-module and x a non-zerodivisor on both A and M. Then*

$$\mathrm{pd}_A(M) \geq \mathrm{pd}_{A/(x)}(M/xM)$$

If moreover A is a Noetherian local ring with maximal ideal P, M is finitely generated and $x \in P$, then equality holds.

Proof. We may assume $d := \mathrm{pd}_A(M) < \infty$. We proceed by induction on d. If $d = 0$, then M is projective over A and then so is M/xM over $A/(x)$. For $d > 0$, choose an exact sequence of A-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0 \tag{5}$$

with F free. Then as in the previous proof $\mathrm{pd}_A(K) = d-1$ and hence $\mathrm{pd}_{A/(x)}(K/xK) \leq d-1$ by induction (note that since x is a non-zerodivisor on A, the same holds for F and hence K). Tensoring the above sequence by $A/(x)$ we get an exact sequence

$$\mathrm{Tor}_1^A(M, A/(x)) \longrightarrow K/xK \longrightarrow F/xF \longrightarrow M/xM \longrightarrow 0$$

where $\mathrm{Tor}_1^A(M, A/(x)) = \{m \in M \mid xm = 0\} = 0$ by Proposition 5.5. Therefore either $\mathrm{pd}(M/xM) = 0$ and the inequality holds trivially, or the argument with the long exact sequence of Ext gives $\mathrm{pd}_{A/(x)}(M/xM) = \mathrm{pd}_{A/(x)}(K/xK)+1 \leq d$. The first statement is proven.

We prove the second statement by induction on $n = \mathrm{pd}_{A/(x)}(M/xM)$ starting with $n = 0$. In this case M/xM is projective over $A/(x)$, hence free by Proposition 2.15. We claim that M is also free over A, which will prove the case $n = 0$. Let $m_1, \dots, m_r$ be a free generating system of M/xM over $A/(x)$. By Nakayama's lemma, it is also a generating system over A since $x \in P$. Now, assume that $a_1m_1 + \cdots + a_rm_r = 0$ for some $a_i \in A$. We know that $a_i \in (x)$ since modulo (x) there is no nontrivial relation. Therefore we find $a_i' \in A$ with $a_i = a_i'x$ for all i and may rewrite the relation as $(a_1'm_1 + \cdots + a_r'm_r)x = 0$. Since x is a non-zerodivisor on M, this implies $a_1'm_1 + \cdots + a_r'm_r = 0$. But then $a_i' \in (x)$ and so, after repeating the argument infinitely many times, finally obtain $a_i \in \cap_n(x^n) \subset \cap_n P^n$ for all i. This gives $a_i = 0$ by Krull's Intersection Theorem.

For the inductive step assume $n > 0$. From the proof of the first statement we already know $\mathrm{pd}_{A/(x)}(M/xM) = \mathrm{pd}_{A/(x)}(K/xK) + 1$, and from exact sequence (5) we get $\mathrm{pd}_A(M) = \mathrm{pd}_A(K) + 1$ since $\mathrm{pd}_A(M) \geq n > 0$ by the first statement. We conclude by applying the inductive hypothesis to K. □

Before starting the proof of Serre's theorem we need to recall some basic facts about associated primes.

Facts 6.14. An *associated prime* in a ring A is a prime ideal $P \subset A$ such that $P = \{x \in A : \; ax = 0\}$ for some $a \in A$. Not every ideal of this form is a prime ideal but every such ideal is contained in an associated prime. Consequently the union of all associated primes is the set of zero-divisors in A. When A is Noetherian, there are only finitely many associated primes in A.

Proof of Theorem 6.10. Assume first A is regular. We proceed by induction on $d := \dim(A)$. If $d = 0$ then A is a field, so every A-module is free and the global dimension is 0. If $d > 0$, then let x be a non-zerodivisor in the maximal ideal P (take an element in a regular sequence). We know that $A/(x)$ is a regular local ring of dimension $d - 1$. On the other hand $\mathrm{gldim}(A/(x)) = \mathrm{gldim}(A) - 1$ by Corollary 6.12

provided that $\operatorname{gldim}(A/(x)) < \infty$, but that is true by induction. By induction we also know that $\dim(A/(x)) = \operatorname{gldim}(A/(x))$, so $\operatorname{gldim}(A) < \infty$ and $\operatorname{gldim}(A) = \dim(A)$ follow.

For the converse we use induction on $\operatorname{gldim}(A)$. If it is zero, then all A-modules are projective, hence the finitely generated ones are free by Proposition 2.15. In particular, the module $k := A/P$ is free which is only possible if $A = k$, so A is regular of dimension 0.

If $\operatorname{gldim}(A) =: d > 0$, we prove first that there exists a non-zerodivisor $x \in P \setminus P^2$. For this it will be enough to show that P is not an associated prime of A, because then the existence of x will follow from the Prime Avoidance Lemma applied to P^2 and the associated primes of A. Suppose P is the annihilator of some $x \in A$. Sending 1 to x induces a homomorphism $A \to A$ with kernel P; let C be its cokernel. We thus have an exact sequence of A-modules

$$0 \to k \to A \to C \to 0.$$

Part of the associated long exact Tor-sequence reads

$$\operatorname{Tor}^A_{d+1}(C, k) \to \operatorname{Tor}^A_d(k, k) \to \operatorname{Tor}^A_d(A, k)$$

where $\operatorname{Tor}^A_d(A, k) = 0$ because A is a free A-module and $\operatorname{Tor}^A_d(k, k) \neq 0$ by Corollary 6.9. But then $\operatorname{Tor}^A_{d+1}(C, k) \neq 0$, contradicting the assumption $\operatorname{gldim}(A) = d$.

So let $x \in P \setminus P^2$ be a non-zerodivisor. Assume for a moment that we know that $\operatorname{gldim}(A/(x)) < \infty$. Then by Corollary 6.12 we have $\operatorname{gldim}(A/(x)) = \operatorname{gldim}(A) - 1$, so by induction $A/(x)$ is regular. Lifting a regular sequence generating P mod (x) and adding x we obtain a regular sequence generating P. This proves that A is regular.

We still have to justify that $\operatorname{gldim}(A/(x)) < \infty$ if $\operatorname{gldim}(A) < \infty$. In view of Proposition 6.8 we have to prove $\operatorname{pd}_{A/(x)}(k) < \infty$. Using the exact sequence

$$0 \longrightarrow P/(x) \longrightarrow A/(x) \longrightarrow k \longrightarrow 0$$

of $A/(x)$-modules we reduce to proving $\operatorname{pd}_{A/(x)}(P/(x)) < \infty$. By the second part of Proposition 6.13,

$$\operatorname{pd}_{A/(x)}(P/xP) = \operatorname{pd}_A(P) < \infty$$

But $P/(x)$ is not the same as P/xP. So to finish the proof we shall show that the exact sequence

$$0 \longrightarrow (x)/xP \longrightarrow P/xP \longrightarrow P/(x) \longrightarrow 0$$

splits. This will suffice, since then $P/(x)$, being a direct summand of P/xP, will also have finite projective dimension (use the Ext criterion provided by Proposition 6.2).

As $x \in P \setminus P^2$ there exist $x_2, \dots, x_r \in P$ such that $x, x_2, \dots, x_r$ modulo P^2 is a basis of P/P^2. Then $(x) \cap ((x_2, \dots, x_r) + P^2) \subseteq xP$. Indeed, if not, then there would exist

$y \in (x_2, \ldots, x_r) + P^2$ such that $y = xu$ where $u \in A \setminus P$ is a unit. However, then $x = u^{-1}y \in (x_2, \ldots, x_r) + P^2$, contradicting the choice of $(x_2, \ldots, x_r)$. Now consider the sequence of maps

$$P/(x) \overset{=}{\to} ((x)+(x_2,\ldots,x_r)+P^2)/(x) \overset{\cong}{\to} ((x_2,\ldots,x_r)+P^2)/((x)\cap((x_2,\ldots,x_r)+P^2)) \to$$

$$\to P/xP \to P/(x).$$

The composition is the identity as one can check, and we get the required splitting. □

Remark 6.15. Notice that if we knew that the statement of Corollary 6.12 holds without the assumption $\mathrm{gldim}(A/(x)) < \infty$, the whole last section of the above proof (and hence also Proposition 6.13) would be unnecessary.

This is what we shall prove in Section 15: more precisely, we shall construct a direct sum decomposition

$$\mathrm{Tor}_i^A(k,k) \cong \mathrm{Tor}_i^{A/(x)}(k,k) \oplus \mathrm{Tor}_{i-1}^{A/(x)}(k,k)$$

assuming only that A is local with residue field k (but assuming $x \notin P^2$ which is harmless), from which the required statement follows in the Noetherian case by Corollary 6.9.

7. Applications of Serre's theorem

We now discuss structural results for regular rings whose proof is enabled, or at least greatly simplified, by homological methods. We begin with the following statement.

Proposition 7.1. *Let A be a Noetherian ring. Then*

$$\mathrm{gldim}(A) = \sup\{\mathrm{gldim}(A_Q) : Q \subset A \text{ is a prime ideal}\}$$

$$= \sup\{\mathrm{gldim}(A_Q) : Q \subset A \text{ is a maximal ideal}\}.$$

The proof uses a base change property for Ext groups.

Lemma 7.2. *Let A be a Noetherian ring, B a flat A-algebra, M a finitely generated A-module and N an arbitrary A-module.*

We have isomorphisms

$$\mathrm{Ext}^i_A(M,N) \otimes_A B \cong \mathrm{Ext}^i_B(M \otimes_A B, N \otimes_A B)$$

for all $i \geq 0$.

Proof. First we treat the case $i = 0$ and M free. Tensoring a homomorphism $M \to N$ by B gives a map $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$ which factors through a map $\mathrm{Hom}_A(M, N) \otimes_A B \to \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$ since the target is a B-module. For $M = A$ this map identifies with the identity map of $N \otimes_A B$ and hence is an isomorphism. Using compatibility of the Hom and tensor product functors with finite direct sums we obtain an isomorphism when M is free.

Since M is finitely generated and A is Noetherian there exists a resolution $P_\bullet \to M$ with the P_i finitely generated and free. (Indeed, there is a surjection $p : P_0 \twoheadrightarrow M$ with a finitely generated free A-module P_0; since A is Noetherian, the kernel K of p is again finitely generated and we may repeat the process starting with K.) By flatness of B we have

$$\mathrm{Ext}^i_A(M, N) \otimes_A B = H^i(\mathrm{Hom}_A(P_\bullet, N)) \otimes_A B \cong H^i(\mathrm{Hom}_A(P_\bullet, N) \otimes_A B)$$

and by the previous paragraph the latter group identifies with

$$H^i(\mathrm{Hom}_B(P_\bullet \otimes_A B, N \otimes_A B)) = \mathrm{Ext}^i_B(M \otimes_A B, N \otimes_A B).$$

□

Proof of Proposition 7.1. First we prove that $\mathrm{gldim}(A_Q) \leq \mathrm{gldim}(A)$ for every prime ideal $Q \subset A$. This is obvious when $\mathrm{gldim}(A)$ is infinite, so we may assume it is a finite number d. Then the A-module A/Q has a projective resolution $P_\bullet \to A/Q$ of length $\leq d$. Here for each i the A_Q-module $P_i \otimes_A A_Q$ is projective (indeed, if P_i is a direct summand of a free module, so is $P_i \otimes_A A_Q$). But then by flatness of A_Q over A the complex $P_\bullet \otimes_A A_Q$ is a projective resolution of the residue field $A_Q/QA_Q \cong (A/Q) \otimes_A A_Q$ over A_Q, and we conclude by Corollary 6.9.

On the other hand, suppose $\mathrm{gldim}(A) = d$ for some d. Then there are A-modules M, N such that $\mathrm{Ext}^d_A(M, N) \neq 0$; by Lemma 6.6 we may assume M is finitely generated. By Lemma 2.20 (1) we find a maximal ideal Q such that $\mathrm{Ext}^d_A(M, N) \otimes_A A_Q \neq 0$. Since A_Q is flat over A, Lemma 7.2 (1) gives $\mathrm{Ext}^d_{A_Q}(M \otimes_A A_Q, N \otimes_A A_Q) \neq 0$, so that $\mathrm{gldim}(A_Q) \geq d$ also holds. The same argument shows that $\mathrm{gldim}(A) = \infty$ implies that for any d we can find a maximal ideal Q with $\mathrm{gldim}(A_Q) \geq d$. □

Combining the proposition with Theorem 6.10, we immediately get:

Corollary 7.3. *Let A be a regular local ring and $Q \subset A$ a prime ideal. Then A_Q is also regular.*

Recall now that a Noetherian ring is *regular* if all of its localizations by maximal ideals are regular local rings. By Corollary 7.3 this is the same as requiring that all localizations by prime ideals are regular local rings. Now combining Theorem 6.10 with Proposition 7.1 we obtain:

Corollary 7.4. *If A is a Noetherian ring of finite Krull dimension d, then A is regular if and only if* $\mathrm{gldim}(A) = d$*.*

In particular, since polynomial rings over fields are regular, we have:

Corollary 7.5. (Hilbert's Syzygy Theorem) *If k is a field, then* $\mathrm{gldim}(k[x_1, \ldots, x_d]) = d$*.*

Remark 7.6. In fact, over $k[x_1, \ldots, x_d]$ every finitely generated projective module is free. This was a conjecture of Serre, solved independently by Quillen and Suslin. Consequently, every finitely generated module over $k[t_1, \ldots, t_d]$ has a finite free resolution.

The last classical result about regular rings is:

Theorem 7.7. (Auslander – Buchsbaum) *A regular local ring is a unique factorization domain.*

For the proof we need several auxiliary statements.

Lemma 7.8. *A Noetherian integral domain A is a unique factorization domain (UFD) if and only if every height 1 prime ideal in it is principal.*

For the proof recall the following basic criterion for unique factorization: a domain A is a UFD if and only if the principal ideals satisfy the ascending chain condition (this is automatic for A Noetherian) and every irreducible element is a prime. Here $p \in A$ is called irreducible if it cannot be written as a product of two non-units and a prime if (p) is a prime ideal.

Proof. If A is a unique factorization domain, every height 1 prime ideal P contains a prime element p (take a prime divisor of some nonzero $a \in P$), so that there is an inclusion $(p) \subseteq P$ of prime ideals which must be an equality since $\mathrm{ht}(P) = 1$. Conversely, if every height 1 prime ideal is principal and $p \in A$ is an irreducible element, take a minimal prime ideal P containing p. Since A is a domain, the Hauptidealsatz gives $\mathrm{ht}(P) = 1$. By assumption we then have $P = (a)$ for some $a \in A$ which must therefore divide p. As p is irreducible, we get $P = (p)$. □

Remark 7.9. The criterion of the lemma has an interesting geometric interpretation: for a local ring of some variety at a point P it means that every codimension 1 subvariety can be defined, at least locally around a point, by a single equation, or in other words by cutting with a hypersurface. Therefore the theorem will imply that this always holds around smooth points.

The next lemma is:

Lemma 7.10. (Nagata) If A is a Noetherian integral domain such that A_x is a unique factorization domain for some prime element $x \in A$, then A is a unique factorization domain.

Here A_x denotes the localization of A by the subset $\{1, x, x^2, x^3, \dots\}$.

Proof. We use the criterion of the previous lemma. Take a height 1 prime ideal $P \subseteq A$. If $x \in P$, then $P = (x)$ since (x) is a prime ideal and $\mathrm{ht}(P) = 1$, so we are done. So assume $x \notin P$. In this case the lemma shows that there exists $p \in P$ such that $PA_x = pA_x$. We may assume that $p \notin (x) \cap P$. Indeed, if $p = ax$ for some $a \in A$, then $a \in P$ as P is a prime ideal and $x \notin P$. If $a \in (x)$ we repeat the process, obtaining an ascending chain of prime ideals $(p) \subsetneq (a) \subsetneq (a_1) \subsetneq (a_2) \subsetneq \cdots$ which must stop at some ideal (a_i) as A is Noetherian. Here a_i is not contained in (x) but $pA_x = a_i A_x$, so we may replace p by a_i.

We now show $P = (p)$. So far we know that for all $y \in P$ there exists $a \in A$ and $m, n > 0$ such that $y/x^n = p(a/x^m)$ since A is a domain. This means that $x^k y \in (p)$ for big enough k, so it is enough to show that $xy \in (p)$ implies $y \in (p)$. If $xy = ap$, then $a \in (x)$ as (x) is a prime ideal and $p \notin (x)$. Therefore $a = bx$ for some b, hence $xy = ap = bxp$ and finally $y = bp$ because A is a domain. $\square$

Next a lemma which is basically linear algebra.

Lemma 7.11. (Kaplansky) *If A is an integral domain and $I_1, \dots, I_n, J_1, \dots, J_n \subseteq A$ ideals such that*

$$\bigoplus_{i=1}^{n} I_i \cong \bigoplus_{i=1}^{n} J_i$$

as A-modules, then $I_1 \cdot \dots \cdot I_n \cong J_1 \cdot \dots \cdot J_n$ as A-modules.

The lemma is easiest to prove using exterior products, about which we recall some basics.

Facts 7.12. Let A be a ring, M an A-module and $n \geq 0$. The n-th exterior power (or wedge power) of M is defined by

$$\Lambda^n M := M^{\otimes n} / \langle m_1 \otimes \cdots \otimes m_n \mid \exists 1 \leq i < j \leq n : \, m_i = m_j \rangle$$

where $\Lambda^0 M = A$ and $\Lambda^1 M = M$. We denote the image of $m_1 \otimes \cdots \otimes m_n$ in $\Lambda^n M$ by $m_1 \wedge \cdots \wedge m_n$. The following properties hold:

1. The A-module $\Lambda^n M$ is characterized by the following universal property: for all A-modules N and all n-linear maps $\varphi : M \times \cdots \times M \to N$ such that $\varphi(m_1, \dots, m_n) = 0$ if $m_i = m_j$ for some $i \neq j$ there exists a factorization $M \times \cdots \times M \to \Lambda^n M \to N$ where the first map is the natural surjection.
2. Every A-module map $M \to N$ induces maps $\Lambda^n M \to \Lambda^n N$ for all $n \geq 0$.

(3) There are natural associative product maps $\Lambda^n M \times \Lambda^m M \to \Lambda^{n+m} M$. These two properties follow from the corresponding properties of the tensor product.

(4) If B is an A-algebra, there are canonical isomorphisms

$$\Lambda^n(M \otimes_A B) \cong (\Lambda^i M) \otimes_A B.$$

Indeed, one checks that the right hand side verifies the universal property characterizing the left hand side.

(5) In $\Lambda^n M$ we have the relations for all i:

$$m_1 \wedge \cdots \wedge m_i \wedge m_{i+1} \wedge \cdots \wedge m_n = -\, m_1 \wedge \cdots \wedge m_{i+1} \wedge m_i \wedge \cdots \wedge m_n$$

(6) If $M \cong A^r$ is free with basis $e_1, \ldots, e_r$, then $\Lambda^n M$ is free with basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \le i_1 < i_2 < \cdots < i_n \le r\};$$

in particular, for $r = n$ it is free of rank 1. If $v_1, \ldots, v_r$ are r elements in A^r, then $v_1 \wedge \cdots \wedge v_r = \det(a_{ij}) e_1 \wedge \cdots \wedge e_r$, where $[a_{ij}]$ is the matrix of the linear map $M \to M$ given by $e_i \mapsto v_i$ for $i = 1, \ldots, r$.

(7) For all A-modules M, N we have isomorphisms

$$\Lambda^n(M \oplus N) \cong \bigoplus_{i+j=n} \Lambda^i M \otimes \Lambda^j N.$$

(The isomorphism is induced by the maps $(m_1 \wedge \cdots \wedge m_i) \otimes (n_1 \wedge \cdots \wedge n_j) \mapsto m_1 \wedge \cdots \wedge m_i \wedge n_1 \wedge \cdots \wedge n_j$ for $m_i \in M$, $n_j \in N$. That this map is indeed an isomorphism is easy to verify in the case when M and N are free modules using the previous fact, and that will be the only case we'll need. For general M and N the argument is a bit more involved.)

Proof of Lemma 7.11. Let K be the fraction field of A, and put $M := I_1 \oplus \cdots \oplus I_n$. Then $M \otimes_A K \cong K^n$ since $I \otimes_A K = K$ for all ideals I. Therefore by property (4) above $\Lambda^n M \otimes_A K \cong \Lambda^n(M \otimes_A K) \cong K$ which, composed with the map $\Lambda^n M \to \Lambda^n M \otimes_A K$ given by $m \mapsto m \otimes 1$, gives a map $\phi : \Lambda^n M \to K$. We now describe $\mathrm{Im}(\phi) \subset K$. Let $e_1, \ldots, e_n$ be the standard basis of K^n coming from the isomorphism $M \otimes_A K \cong K^n$. Use property (6) above to write a generator $m_1 \wedge \cdots \wedge m_n$ of $\Lambda^n M$ as

$$m_1 \wedge \cdots \wedge m_n = \det(a_{ij}) e_1 \wedge \cdots \wedge e_n \in \Lambda^n K^n$$

with $a_{ij} \in I_i$ for all i, so that $\phi(m_1 \wedge \cdots \wedge m_n) = \det(a_{ij}) \in K$. Since $I_1 \cdot \cdots \cdot I_n = \langle \det(a_{ij}) \mid a_{ij} \in I_i \rangle \subseteq K$, we get $\mathrm{Im}(\phi) = I_1 \cdot \cdots \cdot I_n$. The same argument gives $\mathrm{Im}(\phi) = J_1 \cdot \cdots \cdot J_n$, whence the lemma. $\square$

Finally, a homological input.

Lemma 7.13. (Serre) *If A is a ring and P is a projective A-module such that there exists a finite free resolution of length n, then P is stably free, i.e. there exist free modules F and F' such that $P \oplus F' \cong F$. If moreover A is Noetherian and P is finitely generated, then we may find finitely generated F and F'.*

Proof. Pick a resolution

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \dots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} P \longrightarrow 0$$

where we may choose finitely generated F_i when A is Noetherian and P is finitely generated. As P is projective, the map φ_0 has a retraction, so $F_0 \cong P \oplus \mathrm{Im}(\varphi_1)$ and $\mathrm{Im}(\varphi_1)$ is projective. We can iterate this, obtaining $F_i \cong \mathrm{Im}(\varphi_i) \oplus \mathrm{Im}(\varphi_{i+1})$ for all i, hence finally

$$P \oplus F' := P \oplus \bigoplus_{i=1}^{n} \mathrm{Im}(\varphi_i) \cong P \oplus \bigoplus_{i \text{ odd}} F_i \cong \bigoplus_{i \text{ even}} F_i =: F$$

so the statement holds. □

Proof of Theorem 7.7. : Let A be a regular local domain of dimension d. We proceed by induction on $\dim(A)$, the case $\dim(A) = 0$ being clear. Pick an $x \in P \backslash P^2$. It is known that $A/(x)$ is again regular and local,hence an integral domain. This means that (x) is a prime ideal, so by Lemmas 7.8 and 7.10 it is enough to prove that every prime ideal $Q \subseteq A_x$ of height 1 is principal.

If M is a maximal ideal of A_x, then $(A_x)_M$ is the localization of A by the prime ideal $M \cap A$, hence it is also regular by Corollary 7.3. Here $\dim(A_x)_M < \dim A$ because $x \notin M$, so $M \cap A$ is not maximal. By induction $(A_x)_M$ is then a unique factorization domain. Hence $Q(A_x)_M$ is a principal ideal, since either $Q \subset M$ and then $Q(A_x)_M$ is still of height 1, or else $Q(A_x)_M = (1)$. In other words, $Q(A_x)_M$ is a free module of rank 1 over $(A_x)_M$. This being true for all maximal ideals $M \subset A_x$, we conclude from Proposition 2.19 that Q is projective as an A_x-module.

On the other hand, we know that $Q \cap A \subset A$ is a prime ideal satisfying $Q = (Q \cap A)A_x$. From Theorem 6.10 we also know that $Q \cap A$ considered as a finitely generated A-module has a finite free resolution. But then Q has a finite free resolution as well, since we can tensor the resolution of $Q \cap A$ with A_x. Therefore by Lemma 7.13 there exist m and n such that $Q \oplus (A_x)^m \cong (A_x)^n$. Here $m = n - 1$ because tensoring with $(A_x)_M$ gives $Q(A_x)_M \oplus (A_x)_M^m \cong (A_x)_M^n$ where we have seen above that $Q(A_x)_M$ is a free module of rank 1. Hence we can conclude by Lemma 7.11 applied with $I_1 = Q$, $I_i = A_x$ for $i = 2, \dots, n$ and $J_i = A_x$ for all $i = 1, \dots, n$. □

8. The Koszul complex

We now introduce a technical tool that is very useful for the study of regular sequences.

Definition 8.1. Let A be a commutative ring, M an A-module and $f : M \to A$ an A-linear map. The *Koszul complex* $K(f)$ of f is defined as

$$\dots \longrightarrow \Lambda^n M \xrightarrow{\mathrm{d}_f^{n-1}} \Lambda^{n-1} M \xrightarrow{\mathrm{d}_f^{n-2}} \dots \longrightarrow \Lambda^2 M \xrightarrow{\mathrm{d}_f^1} M \xrightarrow{\mathrm{d}_f^0} A\ ,$$

where $\mathrm{d}_f^0 = f$ and

$$\mathrm{d}_f^{n-1}(m_1 \wedge \dots \wedge m_n) = \sum_{i=1}^n (-1)^{i+1} \cdot f(m_i) \cdot m_1 \wedge \dots \wedge m_{i-1} \wedge m_{i+1} \wedge \dots \wedge m_n$$

Note that the map d_f^{n-1} exists by the universal property of the wedge product. It is straightforward to check that $\mathrm{d}_f^{n-1} \circ \mathrm{d}_f^n = 0$.

Remark 8.2. It follows from the definition that for $x \in \Lambda^i M$ and $y \in \Lambda^j M$ we have

$$\mathrm{d}_f^{i+j-1}(x \wedge y) = \mathrm{d}_f^{i-1}(x) \wedge y + (-1)^i x \wedge \mathrm{d}_f^{j-1}(y) \tag{6}$$

If we view the direct sum of the $\Lambda^n M$ as a graded A-algebra with multiplication induced by the wedge product, the d_f^{n-1} give it the structure of a *differential graded algebra*: a graded A-algebra equipped with an A-module endomorphism d sending the degree n part to the degree $n-1$ part and satisfying the compatibility above with respect to the multiplicative structure.

Example 8.3. Consider the case $M = A$. Every A-module homomorphism $f : A \to A$ is given by multiplication by the element $x := f(1)$. The Koszul complex of $K(f)$ is of the form $A \xrightarrow{x} A$, with $H_0(K(f)) \cong A/xA$. Moreover, $H_1(K(f)) \cong \ker(A \xrightarrow{x} A)$, so that $H_1(K(f)) = 0$ if and only if x is a non-zerodivisor.

Tensoring by a general A-module M we obtain

$$H_0(M \otimes_A K(f)) \cong M/xM, \quad H_1(M \otimes_A K(f)) \cong \ker(M \xrightarrow{x} M).$$

We shall be particularly interested in the case when $M = A^r$ is a free A-module of finite rank. In this case the Koszul complex of a map $A^r \to A$ has a particularly simple form which we now proceed to determine. We first need the notion of tensor products of complexes.

Definition 8.4. Let $C_\bullet$ and $D_\bullet$ two chain complexes of A-modules concentrated in nonnegative degrees in the homological numbering. Their tensor product $(C \otimes D)\bullet$ is the complex whose degree n term is given by

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes_A D_j$$

and the differential by the formula

$$d_n^{C\otimes D}(x \otimes y) = d_i^C(x) \otimes y + (-1)^i x \otimes d_j^D(y). \tag{7}$$

Remarks 8.5.
(1) One defines the tensor product of two cohomological complexes concentrated in nonnegative degrees in the same way.
(2) We note for later use that there are natural maps

$$H_i(C_\bullet) \otimes_A H_j(D_\bullet) \to H_{i+j}((C \otimes_A D)_\bullet)$$

defined as follows. If $x \in \ker(d_i^C)$ and $y \in \ker(d_j^D)$, then $x \otimes y$ defines an element in $\ker(d_{i+j}^{C\otimes D})$. If moreover $x = d_{i+1}^C(x')$ for some $x' \in C_{i+1}$, then $d_{i+j+1}^{C\otimes D}(x' \otimes y) = x \otimes y$, so we have a map $H_i(C_\bullet) \otimes_A \ker(d_j^D) \to \ker(d_{i+j}^{C\otimes D})$. By a similar argument it factors through the image of d_{j+1}^D to give a map on homology as stated.

Example 8.6. Let $f_1, f_2 : A \to A$ be two A-module homomorphisms. Then $K(f_1) \otimes K(f_2)$ is the complex

$$A \xrightarrow{\mathrm{d}_f^1} A \oplus A \xrightarrow{\mathrm{d}_f^0} A \tag{8}$$

where the differential d_f^0 is given by $(x, y) \mapsto f_1(x) + f_2(y)$ and d_f^1 by $x \otimes y \mapsto (f_1(x)y, -f_2(y)x)$. Here we have identified $A \otimes_A A$ with A via the multiplication map $x \otimes y \mapsto xy$.

Now consider the A-module map $(f_1, f_2) : A \oplus A \to A$. Since we have canonical isomorphisms $\Lambda^2(A \oplus A) \cong A \otimes_A A \cong A$ (see below or apply Fact 7.12 (6)), the associated Koszul complex $K(f_1, f_2)$ has the shape (8). Moreover, one checks that the differentials are the same as those described above, so we obtain an isomorphism $K(f_1) \otimes K(f_2) \cong K(f_1, f_2)$.

More generally, we have:

Proposition 8.7. *Given A-modules M, N and A-module maps $f_1 : M \to A$, $f_2 : N \to A$, set $f = (f_1, f_2) : M \oplus N \to A$.*

There is a canonical isomorphism $K(f) \cong K(f_1) \otimes K(f_2)$.

Proof. The corresponding terms of $K(f)$ and $K(f_1) \otimes K(f_2)$ are canonically isomorphic by Fact 7.12 (7). To show that the differentials are the same, notice that they are the same in degree 0, and in both cases they can be built out of d_0 using the formula $d(x \wedge y) = d(x) \wedge y + (-1)^i x \wedge dy$. □

Now consider $f = (f_1, \dots, f_r) : A^r \to A$ and set $x_i := f_i(1)$. We then have $f(a_1, \dots, a_r) = \Sigma_i a_i x_i$. Introduce the notation

$$K(\underline{x}) = K(x_1, \dots, x_r) := K(f).$$

By Fact 7.12 (6) it is a complex of free A-modules of length r. The previous proposition gives:

Corollary 8.8. *With notation as above we have an isomorphism of complexes*

$$K(\underline{x}) \cong K(x_1) \otimes \cdots \otimes K(x_r).$$

Now we come to the main result of this section.

Theorem 8.9. *If $x_1, \dots, x_r$ is a regular sequence in A, then $K(\underline{x})$ is acyclic in degrees > 0 and therefore defines a finite free resolution of $A/(x_1, \dots, x_r)$.*

For the proof of the theorem we need:

Lemma 8.10. *If $C_\bullet$ is any complex of A-modules and $x \in A$, there exists an exact sequence of complexes*

$$0 \longrightarrow C_\bullet \longrightarrow C_\bullet \otimes_A K(x) \longrightarrow C_\bullet[-1] \longrightarrow 0 \tag{9}$$

where $(C_\bullet[-1])_i = C_{i-1}$. Moreover, in the corresponding long exact sequence

$$\dots \longrightarrow H_i(C_\bullet) \longrightarrow H_i(C_\bullet \otimes_A K(x)) \longrightarrow H_{i-1}(C_\bullet) \longrightarrow H_{i-1}(C_\bullet) \longrightarrow \dots$$

the map $H_{i-1}(C_\bullet) \to H_{i-1}(C_\bullet)$ is multiplication by $(-1)^{i-1}x$.

Proof. We know that $K(x) = A \xrightarrow{x} A$. Thus in the complex $C_\bullet \otimes K(x)$ the degree i term is

$$(C_\bullet \otimes K(x))_i = (C_i \otimes_A A) \oplus (C_{i-1} \otimes_A A) \cong C_i \oplus C_{i-1}$$

with the differential $C_i \oplus C_{i-1} \to C_{i-1} \oplus C_{i-2}$ given by

$$\begin{bmatrix} \partial & (-1)^{i-1}x \\ 0 & \partial \end{bmatrix}$$

where ∂ is the differential of $C_\bullet$. This differential is the middle vertical map in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_i \arrow[r] \arrow[d, "\partial"] & C_i \oplus C_{i-1} \arrow[r] \arrow[d] & C_{i-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{i-1} \arrow[r] & C_{i-1} \oplus C_{i-2} \arrow[r] & C_{i-2} \arrow[r] & 0
\end{tikzcd}$$

whose rows assemble to the exact sequence of complexes (9). To compute the connecting homomorphism in the long exact sequence, we applying the Snake Lemma to the above diagram: take $\alpha \in \mathrm{Ker}(C_{i-1} \to C_{i-2})$, lift it to $(0, \alpha) \in C_i \oplus C_{i-1}$, map this element to $((-1)^{i-1}x\alpha, 0) \in C_{i-1} \oplus C_{i-2}$ by applying the matrix above and finally take the component in C_{i-1}. It is $(-1)^{i-1}x\alpha$ as stated. □

The long exact sequence of the lemma gives:

Corollary 8.11. *There exists an exact sequence*

$$0 \to H_i(C_\bullet)/xH_i(C_\bullet) \to H_i(C_\bullet \otimes_A K(x)) \to \mathrm{Ker}(H_{i-1}(C_\bullet) \xrightarrow{x} H_{i-1}(C_\bullet)) \to 0.$$

Proof of Theorem 8.9. We proceed by induction on r, the case $r = 1$ being Example 8.3. Moreover, applying the results of Example 8.3 to $M = H_{i-1}(C_\bullet)$ and $M = H_i(C_\bullet)$ we may rewrite the exact sequence of Corollary 8.11 as

$$0 \to H_0(H_i(C_\bullet) \otimes_A K(x)) \to H_i(C_\bullet \otimes_A K(x)) \to H_1(H_{i-1}(C_\bullet) \otimes_A K(x)) \to 0.$$

For the inductive step, set $C_\bullet = K(x_1, \dots, x_{r-1})$, yielding $K(\underline{x}) \cong C_\bullet \otimes_A K(x_r)$ in view of Proposition 8.7. By the inductive hypothesis $H_i(C_\bullet) = H_{i-1}(C_\bullet) = 0$ for all $i > 1$, so the above short exact sequence applied with $x = x_r$ gives $H_i(K(\underline{x})) = 0$ for all $i > 1$. We still need to compute $H_1(K(\underline{x}))$. Since $H_1(C_\bullet) = 0$, the above short exact sequence reduces to an isomorphism $H_1(K(\underline{x})) \cong H_1(H_0(C_\bullet) \otimes_A K(x_r))$, where $H_0(C_\bullet) \cong A/(x_1, \dots, x_{r-1})$. But then by Example 8.3

$$H_1(K(\underline{x})) \cong \mathrm{Ker}(A/(x_1, \dots, x_{r-1}) \xrightarrow{x_r} A/(x_1, \dots, x_{r-1})) \tag{10}$$

which is 0 since $\underline{x}$ is a regular sequence. □

When A is a Noetherian local ring, the converse of Theorem 8.9 also holds. In fact, the following is true:

Proposition 8.12. *If a sequence $\underline{x} = (x_1, \dots, x_r)$ contained in the maximal ideal P of a Noetherian local ring A satisfies $H_1(K(\underline{x})) = 0$, then it is a regular sequence.*

Proof. The case $r = 1$ is again Example 8.3, and for $r > 1$ we can use induction on r. Apply Corollary 8.11 with $C_\bullet := K(x_1, \dots, x_{r-1})$. Since $C_\bullet \otimes K(x_r) \cong K(\underline{x})$, the assumption $H_1(K(\underline{x})) = 0$ gives $H_1(C_\bullet)/x_r H_1(C_\bullet) = 0$. But $x_r \in P$, hence $H_1(C_\bullet) = 0$ by Nakayama's lemma. Therefore by induction $x_1, \dots, x_{r-1}$ is a regular sequence, and moreover x_r is a non-zerodivisor modulo $(x_1, \dots, x_{r-1})$ by the vanishing of $H_1(K(\underline{x}))$ and the isomorphism (10). □

Remark 8.13. The above proposition yields another proof of the fact that in a Noetherian local ring every permutation of a regular sequence is regular.

We shall use the theorem through the corollary:

Corollary 8.14. *If $I = (x_1, \dots, x_r)$ with the x_i forming a regular sequence, then*

$$\mathrm{Tor}_i^A(A/I, M) \cong H_i(K(\underline{x}) \otimes_A M)$$
$$\mathrm{Ext}_A^i(A/I, M) \cong H^i(\mathrm{Hom}(K(\underline{x}), M))$$

for all A-modules M.

Application 8.15. The corollary makes it possible to give a quick proof of one half of Serre's theorem: *If A is a regular local ring of dimension d, then A has global dimension d.*

Indeed, note first that $\mathrm{gldim}(A) \leq d$ already follows from Theorem 8.9 and Corollary 6.9. To get equality, apply the first statement of the corollary with I the maximal ideal of A and M its residue field k. It follows that $\mathrm{Tor}_i^A(k,k)$ is just the degree i term of $K(\underline{x}) \otimes_A k$ for all i; indeed, the differentials $K(\underline{x}) \otimes_A k$ are all 0 since the x_i map to 0 in k. But by construction of the Koszul complex the degree i term of $K(\underline{x}) \otimes_A k$ is a k-vector space of dimension $\binom{d}{i}$; in particular it is nonzero for $i = d$ and 0 for $i > d$.

We can use the above proof to give a 'numerical' criterion for a Noetherian local ring to be regular.

Corollary 8.16. *Let A be a Noetherian local ring with maximal ideal P and residue field k, and set $r := \dim_k P/P^2$. The ring A is regular if and only if $\mathrm{Tor}_i^A(k,k)$ is a k-vector space of dimension $\binom{r}{i}$ for all i.*

Proof. When A is regular of Krull dimension d, we have $r = d$ and we have seen the conclusion above. Conversely, if the dimension of $\mathrm{Tor}_i^A(k,k)$ is as in the statement, it is 0 for $i > r$, and we conclude from Corollary 6.9 and Serre's theorem. $\square$

Remark 8.17. In the situation of the corollary there is a natural way to construct an isomorphism $\mathrm{Tor}_1^A(k,k) \cong k^r$, generalizing the argument in the regular case. This is done using *minimal resolutions:* a free resolution $F_\bullet \to M$ of a finitely generated A-module M is minimal if each F_i is finitely generated and $Z_i(F_\bullet) \subset PF_i$ for all i. It follows from the defining property that the differentials of the complex $F_\bullet \otimes_A k$ are all 0. One can construct $F_\bullet$ inductively. First one takes a k-basis $x_1, \dots, x_n$ of M/PM, sets $F_0 = A^n$ and defines the map $F_0 \to M$ by lifting the obvious map $F_0 \to M/PM$. In the inductive step the same procedure is applied to $Z_i(F_\bullet)$ in place of M. For $M = k$ one starts with $F_0 = A$ and then proceeds with a map $A^r \to A$ lifting the natural map $A^r \to P/P^2 \cong k^r$. Thus indeed $\mathrm{Tor}_1^A(k,k) \cong H_1(F_\bullet \otimes_A k) \cong k^r$.

To sum up: $\mathrm{Tor}_1^A(k,k)$ always has the 'correct' dimension, it is some of the higher Tor's that differ in the non-regular case.

The isomorphisms $\mathrm{Tor}_i^A(k,k) \cong \Lambda^i \mathrm{Tor}_1^A(k,k)$ of Application 8.15 are part of an even stronger statement. Quite generally, for a module M over a ring A one can equip the direct sum

$$\Lambda^\bullet(M) := \bigoplus_{i=0}^{\infty} \Lambda^i(M)$$

with a product structure induced by the product maps of Fact 7.12 (3). The resulting A-algebra is the *exterior algebra* of M.

For A regular and $M = k$ we can consider the direct sum

$$\mathrm{Tor}_\bullet^A(k,k) := \bigoplus_{i=0}^{\infty} \mathrm{Tor}_i^A(k,k)$$

which, as an A-module (or k-vector space) identifies with $\Lambda^\bullet \mathrm{Tor}_1^A(k,k)$ by the above. On the other hand, the k-vector space $\mathrm{Tor}_\bullet^A(k,k)$ already carries a product structure which is compatible with the wedge product structure on $\Lambda^\bullet \mathrm{Tor}_1^A(k,k)$. We now explain the details. We shall need the easy lemma:

Lemma 8.18. *If P_1, P_2 are projective A-modules, then so is $P_1 \otimes_A P_2$.*

Proof. We have to show that $\mathrm{Hom}_A(P_1 \otimes_A P_2, \_\_)$ is an exact functor. But

$$\mathrm{Hom}_A(P_1 \otimes_A P_2, \_\_) \cong \mathrm{Hom}_A(P_1, \mathrm{Hom}_A(P_2, \_\_))$$

where the right hand side is a composition of two exact functors by assumption. □

Construction 8.19 (Internal product for Tor)**.** Let A be a commutative ring and R an A-algebra. We construct an associative A-linear multiplication

$$\mathrm{Tor}_i^A(R,R) \times \mathrm{Tor}_j^A(R,R) \to \mathrm{Tor}_{i+j}^A(R,R)$$

for all $i, j \geq 0$ called the *internal product*.

It will be enough to construct maps

$$\mathrm{Tor}_i^A(M_1,N_1) \times \mathrm{Tor}_j^A(M_2,N_2) \to \mathrm{Tor}_{i+j}^A(M_1 \otimes_A M_2, N_1 \otimes_A N_2) \tag{11}$$

for all A-modules M_1, M_2, N_1, N_2 (the *external product*). Indeed, setting $M_1 = M_2 = N_1 = N_2 = R$ and applying the multiplication map $R \otimes_A R \to R$ in both variables we then obtain the internal product.

Choose projective resolutions $P_\bullet^1 \to M_1$, $P_\bullet^2 \to M_2$, $P_\bullet \to M_1 \otimes_A M_2$. Recall that the groups $\mathrm{Tor}_i^A(M_1,N_1)$ and $\mathrm{Tor}_j^A(M_2,N_2)$ are computed by tensoring $P_\bullet^1$ by N_1 and $P_\bullet^2$ by N_2, respectively, and then taking homology. On the other hand, the tensor product complex $(P^1 \otimes_S P^2)_\bullet$ has projective terms by Lemmas 8.18 and 2.12 (2). Moreover, the maps $P_0^1 \to M_1$, $P_0^2 \to M_2$ induce a map $(P^1 \otimes_S P^2)_\bullet \to M_1 \otimes_A M_2$, so by Lemma 3.9 the identity map of $M_1 \otimes_A M_2$ induces a morphism of complexes $(P^1 \otimes_A P^2)_\bullet \to P_\bullet$. It follows that we have a morphism of complexes

(12) $(P^1_\bullet \otimes_A N_1) \otimes_A (P^2_\bullet \otimes_A N_2) \cong (P^1 \otimes_A P^2)_\bullet \otimes_A (N_1 \otimes_A N_2) \to P_\bullet \otimes_A (N_1 \otimes_A N_2)$

On the other hand, by Remark 8.5 (2) we have a natural map

$$\mathrm{Tor}_i^A(M_1, N_1) \times \mathrm{Tor}_j^A(M_2, N_2) \to H_{i+j}((P^1_\bullet \otimes_A N_1) \otimes_A (P^2_\bullet \otimes_A N_2))$$

whence the external product (11) arises by composition with the map induced by (12) on H_{i+j}.

The above product has the following property (called *graded-commutativity*):

Proposition 8.20. *For $a \in \mathrm{Tor}_i^A(R, R)$ and $b \in \mathrm{Tor}_j^A(R, R)$ denote by $a \cdot b \in \mathrm{Tor}_{i+j}^A(R, R)$ their internal product. Then*

$$a \cdot b = (-1)^{ij} b \cdot a.$$

For the proof we need:

Lemma 8.21. *Let $C_\bullet$ be a homological complex. Taking $a \in C_i$ and $b \in C_j$ and sending $a \otimes b$ to $(-1)^{ij} b \otimes a$ induces a morphism of complexes $\tau : (C \otimes C)_\bullet \to (C \otimes C)_\bullet$.*

Proof. We have

$$\tau(d(a \otimes b)) = \tau(da \otimes b + (-1)^i a \otimes db) = (-1)^{(i-1)j} b \otimes da + (-1)^{i+i(j-1)} db \otimes a,$$

whereas

$$d(\tau(a \otimes b)) = d((-1)^{ij} b \otimes a) = (-1)^{ij} db \otimes a + (-1)^{ij+j} b \otimes da.$$

The two are equal since $(-1)^{ij} = (-1)^{i+i(j-1)}$ and $(-1)^{ij+j} = (-1)^{(i+1)j} = (-1)^{(i-1)j}$. $\square$

Proof of Proposition 8.20. In the construction of the internal product take $P^1_\bullet = P^2_\bullet =: C_\bullet$. Then the map $\tau : (C \otimes C)_\bullet \to (C \otimes C)_\bullet$ of the lemma gives a morphism of complexes $(P^2 \otimes_A P^1)_\bullet \to (P^1 \otimes_A P^2)_\bullet$ which, composed with a morphism $(P^1 \otimes_A P^2)_\bullet \to P_\bullet$ given by Lemma 3.9, gives a morphism of complexes $(P^2 \otimes_A P^1)_\bullet \to P_\bullet$. By construction, the first morphism computes $a \otimes b$ and the second one $(-1)^{ij} b \otimes a$. $\square$

Proposition 8.22. *Let A be a ring, $I \subset A$ an ideal generated by a regular sequence $x_1, \ldots, x_r$ and $R := A/I$. Then we have an isomorphism of graded R-algebras*

$$\Lambda^\bullet \mathrm{Tor}_1^A(R, R) \xrightarrow{\sim} \mathrm{Tor}_\bullet^A(R, R)$$

induced by the identity in degree 1 and the internal product on $\mathrm{Tor}_\bullet^A(R, R)$.

In particular, if A is a regular local ring with residue field k, we have an isomorphism of graded k-algebras

$$\Lambda^\bullet \mathrm{Tor}_1^A(k,k) \xrightarrow{\sim} \mathrm{Tor}_\bullet^A(k,k).$$

Proof. By Corollary 8.14 and the same argument as in Application 8.15 we have isomorphisms $\mathrm{Tor}_i^A(R,R) \cong H_i(K(\underline{x}) \otimes_A R) \cong \Lambda^i(R^r)$, where $K(\underline{x})$ is the associated Koszul complex. In particular, for $i=1$ we get $H_1(K(\underline{x}) \otimes_A R) \cong R^r$, which yields isomorphisms $\Lambda^i \mathrm{Tor}_1^A(R,R) \xrightarrow{\sim} \mathrm{Tor}_i^A(R,R)$ for all i.

It remains to check that the wedge product maps

$$\Lambda^i \mathrm{Tor}_1^A(R,R) \times \Lambda^j \mathrm{Tor}_1^A(R,R) \to \Lambda^{i+j} \mathrm{Tor}_1^A(R,R) \tag{13}$$

become identified with the internal product maps

$$\mathrm{Tor}_i^A(R,R) \times \mathrm{Tor}_j^A(R,R) \to \mathrm{Tor}_{i+j}^A(R,R)$$

via the above isomorphism. Taking $P_\bullet^1 = P_\bullet^2 = K(\underline{x})$ in the construction of the internal product above, we have to consider the map $K(\underline{x}) \otimes_A K(\underline{x}) \to K(\underline{x})$ lifting the multiplication map $R \otimes_A R \to R$ whose existence is stipulated by Lemma 3.9. Such a morphism of complexes is given in degree n by the sum of the wedge product maps

$$\bigoplus_{i+j=n} \Lambda^i(A^r) \otimes \Lambda^j(A^r) \to \Lambda^n(A^r);$$

that they induce a morphism of complexes follows from comparing formulas (6) and (7). That this morphism of complexes induces the internal product on Tor follows from the uniqueness statement of Lemma 3.9; that it induces the map (13) results from the construction. $\square$

9. Cohomology of groups

We now briefly consider group cohomology, a case where cohomology groups defined via derived functors have a concrete description. As an application, we shall prove a classical theorem of Schur and Zassenhaus from the theory of finite groups.

Let G be a group. By a *(left) G-module* we shall mean an abelian group A equipped with a left action by G, i.e. a map $G \times A \to A, (\sigma, a) \mapsto \sigma a$ satisfying $\sigma(a_1 + a_2) = \sigma a_1 + \sigma a_2$ and $(\sigma_1\sigma_2)a = (\sigma_1(\sigma_2 a))$. Notice that this equivalent to giving a left module over the integral group ring $\mathbf{Z}[G]$: indeed, for elements $\sum n_\sigma \sigma \in \mathbf{Z}[G]$ and $a \in A$ we may define $(\sum n_\sigma \sigma)a := \sum n_\sigma \sigma(a)$ and conversely, a $\mathbf{Z}[G]$-module structure implies in particular the existence of "multiplication-by-σ" maps on A for all $\sigma \in G$. A morphism of G-modules is a map of abelian groups $A_1 \to A_2$ compatible with the G-action; this also extends uniquely to a map of $\mathbf{Z}[G]$-modules. Thus we can identify the category of left G-modules with the category of left $\mathbf{Z}[G]$-modules.

We say that A is a *trivial* G-module if G acts trivially on A, i.e. $\sigma a = a$ for all $\sigma \in G$ and $a \in A$. The abelian group $\mathrm{Hom}_G(\mathbf{Z}, A)$ of G-module homomorphisms from the trivial G-module $\mathbf{Z}$ to a G-module A (or else the group $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, A)$) identifies with the subgroup A^G of G-invariant elements in A.

Definition 9.1. Let G be a group and A a G-module. We define the *i-th cohomology group of A with coefficients in G* as

$$H^i(G, A) := \mathrm{Ext}^i_{\mathbf{Z}[G]}(\mathbf{Z}, A)$$

where $\mathbf{Z}$ is, as before, equipped with the trivial G-action.

Thus the $H^i(G, \_\_)$ are covariant functors from the category of G-modules to that of abelian groups. Notice that here we are dealing with Ext groups over non-commutative rings but, as remarked above, everything is defined in the same way as in the commutative case. By the above discussion we may also define $H^i(G, \_\_)$ as the i-th right derived functor of the functor $A \mapsto A^G$.

The cohomology groups $H^i(G, A)$ thus satisfy $H^0(G, A) = A^G$ for all G-modules A, and given a short exact sequence

$$0 \to A \to B \to C \to 0$$

of G-modules, there exists an infinite long exact sequence

$$\dots \to H^i(G, A) \to H^i(G, B) \to H^i(G, C) \to H^{i+1}(G, A) \to \dots$$

of abelian groups, starting from $i = 0$.

Remark 9.2. Similarly, the groups $H_i(G, A) := \mathrm{Tor}_i^{\mathbf{Z}[G]}(\mathbf{Z}, A)$ define a homology theory for groups. It is somewhat less useful in applications than group cohomology.

To calculate the groups $H^i(G, A)$ explicitly, one uses concrete projective resolutions. The most useful of these is the following one, inspired by simplicial constructions in topology.

Construction 9.3 (The standard resolution)**.** Consider for each $i \geq 0$ the $\mathbf{Z}[G]$-module $\mathbf{Z}[G^{i+1}]$, where G^{i+1} is the $(i+1)$-fold direct power of G and the action of G is determined by $\sigma(\sigma_0, \dots, \sigma_i) = (\sigma\sigma_0, \dots, \sigma\sigma_i)$. These are free $\mathbf{Z}[G]$-modules, generated by the free $\mathbf{Z}[G]$-basis formed by the elements $(1, \sigma_1, \dots, \sigma_i) \in G^{i+1}$. For $i > 0$ define G-homomorphisms $\delta_i : \mathbf{Z}[G^{i+1}] \to \mathbf{Z}[G^i]$ by $\delta_i = \sum_j (-1)^j s^i_j$, where $s^i_j : \mathbf{Z}[G^{i+1}] \to \mathbf{Z}[G^i]$ is the map determined by sending

$$(\sigma_0, \dots, \sigma_i) \mapsto (\sigma_0, \dots, \sigma_{j-1}, \sigma_{j+1}, \dots, \sigma_i).$$

In this way, we get a projective resolution

$$\cdots \to \mathbf{Z}[G^3] \xrightarrow{\delta_2} \mathbf{Z}[G^2] \xrightarrow{\delta_1} \mathbf{Z}[G] \xrightarrow{\delta_0} \mathbf{Z} \to 0,$$

where δ_0 sends each σ_i to 1. This resolution is called the *standard resolution* of $\mathbf{Z}$. To see that the sequence is indeed exact, an immediate calculation shows first that $\delta_i \circ \delta_{i+1} = 0$ for all i. Then fix $\sigma \in G$ and define $h^i : \mathbf{Z}[G^{i+1}] \to \mathbf{Z}[G^{i+2}]$ by sending $(\sigma_0, \ldots, \sigma_i)$ to $(\sigma, \sigma_0, \ldots, \sigma_i)$. Another calculation shows $\delta_{i+1} \circ h^i + h^{i-1} \circ \delta_i = \mathrm{id}_{\mathbf{Z}[G^{i+1}]}$, i.e. the h^i define a chain homotopy of the identity map of the standard resolution with the zero map, whence the claim.

For a G-module A, one calls the elements of $\mathrm{Hom}_G(\mathbf{Z}[G^{i+1}], A)$ *i-cochains*, whereas those of $Z^{i+1}(\mathrm{Hom}_G(\mathbf{Z}[G^\bullet], A))$ and $B^{i+1}(\mathrm{Hom}_G(\mathbf{Z}[G^\bullet], A))$ *i-cocycles* and *i-coboundaries*, respectively. We shall denote these respective groups by $C^i(G, A)$, $Z^i(G, A)$ and $B^i(G, A)$. The cohomology groups $H^i(G, A)$ then arise as the groups $H^{i+1}(\mathrm{Hom}_G(\mathbf{Z}[G^\bullet], A))$.

For calculations, another expression is very useful.

Construction 9.4 (Inhomogeneous cochains)**.** In $\mathbf{Z}[G^{i+1}]$ consider the particular basis elements

$$[\sigma_1, \ldots, \sigma_i] := (1, \sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1 \ldots \sigma_i).$$

From the definition of the G-action on $\mathbf{Z}[G^{i+1}]$ we get that $\mathbf{Z}[G^{i+1}]$ is none but the free $\mathbf{Z}[G]$-module generated by the elements $[\sigma_1, \ldots, \sigma_i]$. A calculation shows that on these elements the differentials δ_i are expressed by

$$\delta_i([\sigma_1, \ldots, \sigma_i]) = \sigma_1[\sigma_2, \ldots, \sigma_i] + \sum_{j=1}^{i} (-1)^j [\sigma_1, \ldots, \sigma_j\sigma_{j+1}, \ldots, \sigma_i] +$$

$$+(-1)^{i+1}[\sigma_1, \ldots, \sigma_{i-1}]. \tag{14}$$

Therefore we may identify i-cochains with functions $[\sigma_1, \ldots, \sigma_i] \to a_{\sigma_1, \ldots, \sigma_i}$ and compute the maps $\delta_{i-1}^* : C^{i-1}(G, A) \to C^i(G, A)$ by the formula

$$a_{\sigma_1, \ldots, \sigma_{i-1}} \mapsto \sigma_1 a_{\sigma_2, \ldots, \sigma_i} + \sum_{j=1}^{i} (-1)^j a_{\sigma_1, \ldots, \sigma_j\sigma_{j+1}, \ldots, \sigma_i} + (-1)^{i+1} a_{\sigma_1, \ldots, \sigma_{i-1}}.$$

The functions $a_{\sigma_1, \ldots, \sigma_i}$ are called *inhomogeneous cochains.*

Here is how to calculate the groups $H^i(G, A)$ in low dimensions by means of inhomogeneous cochains.

Examples 9.5.

(1) A 1-cocycle is given by a function $\sigma \mapsto a_\sigma$ satisfying $a_{\sigma_1\sigma_2} = \sigma_1 a_{\sigma_2} + a_{\sigma_1}$. It is a 1-coboundary if and only if it is of the form $\sigma \mapsto \sigma a - a$ for some $a \in A$. Note that in the special case when G acts trivially on A, i.e. $\sigma(a) = a$ for all $a \in A$, we have $Z^1(G, A) = \mathrm{Hom}(G, A)$ and $B^1(G, A) = 0$, so finally $H^1(G, A) = \mathrm{Hom}(G, A)$.

(2) A 2-cocycle is given by a function $(\sigma_1, \sigma_2) \mapsto a_{\sigma_1,\sigma_2}$ satisfying

$$\sigma_1 a_{\sigma_2,\sigma_3} - a_{\sigma_1\sigma_2,\sigma_3} + a_{\sigma_1,\sigma_2\sigma_3} - a_{\sigma_1,\sigma_2} = 0.$$

It is a 2-coboundary, i.e. an element of $\mathrm{Im}\,(\partial^{1*})$ if it is of the form $\sigma_1 a_{\sigma_2} - a_{\sigma_1\sigma_2} + a_{\sigma_1}$ for some 1-cochain $\sigma \mapsto a_\sigma$.

Example 9.6. For some questions (e.g. as in the example of group extensions below) it is convenient to work with *normalized cochains*. These are obtained by considering the free resolution

$$\cdots \to L_2 \xrightarrow{\delta_2^n} L_1 \xrightarrow{\delta_1^n} L_0 \xrightarrow{\delta_0^n} \mathbf{Z} \to 0,$$

where L_i is the free G-submodule of $\mathbf{Z}[G^{i+1}]$ generated by those $[\sigma_1, \dots \sigma_i]$ where none of the σ_j is 1. The morphisms δ_i^n are defined by the same formulae as for the δ_i in (14), except that if we happen to have $\sigma_j\sigma_{j+1} = 1$ for some j in $[\sigma_1, \dots \sigma_i]$, we set the term involving $\sigma_j\sigma_{j+1}$ on the right-hand side to 0. This indeed defines a map $L_i \to L_{i-1}$, and a calculation shows that we again have $\ker(\delta_i^n) = \mathrm{Im}\,(\delta_{i+1}^n)$. So we have obtained a free resolution of **Z** and may use it for computing the cohomology of a G-module A. Elements in $\mathrm{Hom}_G(L_i, A)$ may be identified with inhomogeneous i-cochains $a_{\sigma_1,\dots,\sigma_i}$ which have the value 0 whenever one of the σ_j equals 1.

Construction 9.7 (Group extensions)**.** An important example of 2-cocycles arising 'in nature' comes from the theory of group extensions. Consider an exact sequence of groups $0 \to A \to E \to G \xrightarrow{\pi} 1$, with A abelian. The conjugation action of E on A passes to the quotient in G and gives A the structure of a G-module. Now associate with E a 2-cocycle as follows. Choose a *normalized set-theoretic section* of π, i.e. a map $s : G \to E$ with $s(1) = 1$ and $\pi \circ s = \mathrm{id}_G$. For elements $\sigma_1, \sigma_2 \in G$ the element $a_{\sigma_1,\sigma_2} := s(\sigma_1)s(\sigma_2)s(\sigma_1\sigma_2)^{-1}$ maps to 1 in G, and therefore defines an element of A. An immediate calculation shows that $(\sigma_1, \sigma_2) \mapsto a_{\sigma_1,\sigma_2}$ is a 2-cocycle of G with values in A, which is in fact normalized, i.e. satisfies $a_{1,\sigma} = a_{\sigma,1} = 1$ for all $\sigma \in G$. If $t : G \to E$ is another normalized section, then for each $\sigma \in G$ we find $a_\sigma \in A$ such that $t(\sigma) = a_\sigma s(\sigma)$; note that $a_1 = 1$. So

$$t(\sigma_1)t(\sigma_2)t(\sigma_1\sigma_2)^{-1} = a_{\sigma_1}s(\sigma_1)a_{\sigma_2}s(\sigma_2)s(\sigma_1\sigma_2)^{-1}a_{\sigma_1\sigma_2}^{-1} =$$

$$= a_{\sigma_1}(s(\sigma_1)a_{\sigma_2}s(\sigma_1)^{-1})(s(\sigma_1)s(\sigma_2)s(\sigma_1\sigma_2)^{-1})a_{\sigma_1\sigma_2}^{-1} = a_{\sigma_1,\sigma_2}(\sigma_1(a_{\sigma_2})a_{\sigma_1\sigma_2}^{-1}a_{\sigma_1}) \tag{15}$$

by commutativity of A and the definition of the G-action on A. This means that $t(\sigma_1)t(\sigma_2)t(\sigma_1\sigma_2)^{-1}$ differs from a_{σ_1,σ_2} by a 2-coboundary, i.e. the cohomology class of a_{σ_1,σ_2} does not depend on the choice of the section. In this way one associates with E a class $c(E) \in H^2(G, A)$. Furthermore, we see that in the case when there is a section s which is a group homomorphism, i.e. the extension E splits as a semidirect product of G by A, we have $c(E) = 0$. Conversely, we see from the calculation in (15) that if $c(E) = 0$ and $s : G \to E$ is a normalized section, then we may modify s by a normalized 1-cochain so that the resulting section becomes multiplicative.

In fact, once we fix a G-action on A, we have a non-commutative analogue of the theory of module extensions. Consider the set $\mathrm{Ext}(G, A)$ of equivalence classes of extensions E of G by A inducing the given action of G on A and declare two extensions E and E' equivalent if there is an isomorphism $\lambda : E \xrightarrow{\sim} E'$ inducing a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \lambda} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Proposition 9.8. *The map $E \mapsto c(E)$ preserves the above equivalence relation and induces a bijection between $\mathrm{Ext}(G, A)$ and $H^2(G, A)$.*

Sketch of proof. The first statement follows because composing a section $s : G \to E$ with λ gives rise to the same 2-cocycle $G \to A$ as s. Thus we have an induced map $\mathrm{Ext}(G, A) \to H^2(G, A)$. The inverse map is constructed as follows: one represents a class in $H^2(G, A)$ by a *normalized* cocycle a_{σ_1,σ_2} and defines a group E with underlying set $A \times G$ and group law $(a_1, \sigma_1) \cdot (a_2, \sigma_2) := (a_1 + \sigma_1(a_2) + a_{\sigma_1,\sigma_2}, \sigma_1\sigma_2)$, where we have written the group law of A additively. The cocycle relation implies that this product is associative, and the fact that a_{σ_1,σ_2} is normalized implies that $(0, 1)$ is a unit element. The element $(-\sigma^{-1}(a) - \sigma^{-1}(a_{\sigma,\sigma^{-1}}), \sigma^{-1})$ yields a two-sided inverse for (a, σ) (note that the cocycle relation implies $\sigma^{-1}(a_{\sigma,\sigma^{-1}}) = a_{\sigma^{-1},\sigma}$). Therefore E is indeed a group, and the map $a \mapsto (a, 1)$ realizes A as a subgroup which is the kernel of the surjective homomorphism $E \to G$ given by $(a, \sigma) \mapsto \sigma$. The map $\sigma \mapsto (0, \sigma)$ is a normalized section whose associated 2-cocycle is exactly a_{σ_1,σ_2}. Finally, if we change the 2-cocycle a_{σ_1,σ_2} by adding the normalized 2-coboundary $\sigma_1(a_{\sigma_2}) - a_{\sigma_1\sigma_2} + a_{\sigma_1}$ and denote the resulting extension by E', the map $(a, \sigma) \mapsto (a + a_\sigma, \sigma)$ defines a homomorphism $E \to E'$ inducing the identity on A and G (note that $a_1 = 0$ as a_σ is normalized), so that E and E' are equivalent. $\square$

Let H be a subgroup of G and A an H-module. Then $\mathbf{Z}[G]$ with its canonical G-action is an H-module as well, and we can associate with A the G-module

$$M_H^G(A) := \operatorname{Hom}_H(\mathbf{Z}[G], A)$$

where the action of G on an H-homomorphism $\phi : \mathbf{Z}[G] \to A$ is given by $(\sigma\phi)(g) := \phi(g\sigma)$ for a basis element g of $\mathbf{Z}[G]$. One sees that $\sigma\phi$ is indeed an H-homomorphism.

Lemma 9.9. *Assume moreover given a G-module M. We have a canonical isomorphism*

$$\operatorname{Hom}_G(M, \operatorname{Hom}_H(\mathbf{Z}[G], A)) \xrightarrow{\sim} \operatorname{Hom}_H(M, A)$$

induced by mapping a G-homomorphism $m \to \phi_m$ in the left-hand side group to the H-homomorphism $m \mapsto \phi_m(1)$.

Proof. Given an H-homomorphism $\lambda : M \to A$, consider the map $m \mapsto \lambda_m$, where $\lambda_m \in \operatorname{Hom}_H(\mathbf{Z}[G], A)$ is the map determined by $g \mapsto \lambda(gm)$. The kind reader will check that we get an element of $\operatorname{Hom}_G(M, \operatorname{Hom}_H(\mathbf{Z}[G], A))$ in this way, and that the two constructions are inverse to each other. □

Now apply the lemma to the terms of a projective $\mathbf{Z}[G]$-resolution $P_\bullet$ of $\mathbf{Z}$. Note that this is also a resolution by projective H-modules because $\mathbf{Z}[G]$ is free as a $\mathbf{Z}[H]$-module (a system of coset representatives yields a basis). Passing to cohomology groups, we get:

Corollary 9.10 (Shapiro's Lemma)**.** *Given a subgroup H of G and an H-module A, there are canonical isomorphisms*

$$H^i(G, M_H^G(A)) \xrightarrow{\sim} H^i(H, A)$$

for all $i \geq 0$.

The case when $H = \{1\}$ is particularly important. In this case an H-module A is just an abelian group; we denote $M_H^G(A)$ simply by $M^G(A)$ and call it the *co-induced* module associated with A.

Corollary 9.11. *The group $H^i(G, M^G(A))$ is trivial for all $i > 0$.*

Proof. In this case the right hand side in Shapiro's lemma is trivial (e.g. because $0 \to \mathbf{Z} \to \mathbf{Z} \to 0$ gives a projective resolution of $\mathbf{Z}$). □

Using Shapiro's lemma we may define two basic maps relating the cohomology of a group to that of a subgroup.

Construction 9.12 (Restriction maps)**.** Let G be a group, A a G-module and H a subgroup of G. There are natural maps of G-modules

$$A \xrightarrow{\sim} \operatorname{Hom}_G(\mathbf{Z}[G], A) \to \operatorname{Hom}_H(\mathbf{Z}[G], A) = M_H^G(A),$$

the first one given by mapping $a \in A$ to the unique G-homomorphism sending 1 to a and the second by considering a G-homomorphism as an H-homomorphism. Taking cohomology and applying Shapiro's lemma we thus get maps

$$\mathrm{Res}: \; H^i(G, A) \to H^i(H, A)$$

for all $i \geq 0$, called *restriction maps.* One sees that for $i = 0$ we get the natural inclusion $A^G \to A^H$.

When the subgroup H has finite index, there is a natural map in the opposite direction.

Construction 9.13 (Corestriction maps)**.** Let H be a subgroup of G of finite index n and let A be a G-module.

Given an H-homomorphism $\phi: \mathbf{Z}[G] \to A$, define a new map $\mathbf{Z}[G] \to A$ by the assignment

$$\phi_H^G: \; x \mapsto \sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} x),$$

where $\rho_1, \ldots, \rho_n$ is a system of left coset representatives for H in G. This is manifestly a group homomorphism which does not depend on the choice of the ρ_j; indeed, if we replace the system of representatives (ρ_j) by another system $(\rho_j \tau_j)$ with some $\tau_j \in H$, we get $\rho_j \tau_j \phi(\tau_j^{-1} \rho_j^{-1} x) = \rho_j \phi(\rho_j^{-1} x)$ for all j, the map ϕ being an H-homomorphism. Furthermore, the map ϕ_H^G is also a G-homomorphism, because we have for all $\sigma \in G$

$$\sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} \sigma x) = \sigma \left(\sum_{j=1}^{n} (\sigma^{-1} \rho_j) \phi((\sigma^{-1} \rho_j)^{-1} x) \right) = \sigma \left(\sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} x) \right),$$

as the $\sigma^{-1} \rho_j$ form another system of left coset representatives.

The assignment $\phi \mapsto \phi_H^G$ thus defines a well-defined map

$$\mathrm{Hom}_H(\mathbf{Z}[G], A) \to \mathrm{Hom}_G(\mathbf{Z}[G], A) \cong A,$$

so by taking cohomology and applying Shapiro's lemma we get maps

$$\mathrm{Cor}: \; H^i(H, A) \to H^i(G, A)$$

for all $i \geq 0$, called *corestriction maps.*

An immediate consequence of the preceding constructions is the following basic fact.

Proposition 9.14. *Let G be a group, H a subgroup of finite index n in G and A a G-module. Then the composite maps*

$$\mathrm{Cor} \circ \mathrm{Res}: \; H^i(G, A) \to H^i(G, A)$$

are given by multiplication by n for all $i \geq 0$.

Proof. Indeed, if $\phi : \mathbf{Z}[G] \to A$ is a G-homomorphism, then for all $x \in \mathbf{Z}[G]$ we have $\phi_H^G(x) = \sum \rho_j \phi(\rho_j^{-1} x) = \sum \rho_j \rho_j^{-1} \phi(x) = n\phi(x)$. □

In the case $H = \{1\}$ we get:

Corollary 9.15. *Let G be a finite group of order n. Then the elements of $H^i(G, A)$ have finite order dividing n for all G-modules A and integers $i > 0$.*

Corollary 9.16 (Schur–Zassenhaus)**.** *Let $0 \to A \to E \to G \to 1$ be an extension of finite groups, with A abelian. If the orders of A and G are coprime, the extension is split.*

Proof. By Example 9.7 the conclusion is equivalent to $H^2(G, A) = 0$. If A has order m, then $mH^2(G, A) = 0$ (since $m\mathrm{Hom}_G(P_i, A) = \mathrm{Hom}_G(P_i, mA) = 0$ for the terms of a projective resolution $P_\bullet \to \mathbf{Z}$). On the other hand, if G has order n, then $nH^2(G, A) = 0$ by the previous corollary. Since by assumption $(n, m) = 1$, the vanishing of $H^2(G, A)$ follows. □

Remark 9.17. The Schur–Zassenhaus theorem holds without assuming A abelian. Here is how to reduce the case of general A to abelian A (after Wikipedia).

We prove by induction on the order of E that it contains a subgroup of order n; this subgroup must then map isomorphically onto G by the assumption $(n, m) = 1$. Assume first A is solvable. Then it has an abelian characteristic subgroup $A' \subset A$ (e.g. the last nontrivial subgroup in the commutator series). Then A' is normal in E, so we may consider the quotient E/A' which is an extension of G by A/A'. By induction it contains a subgroup of H order n; let H' be its preimage in E. Then $H' \subset E$ is an extension of H by A', and so by Corollary 9.16 contains a subgroup of order n.

Now if there is a p-Sylow subgroup of A whose normalizer N in E is strictly smaller than E, then by the Frattini argument[1] $E = AN$, whence $N/N \cap A \cong E/A \cong G$ has order n. Applying induction to the extension $1 \to N \cap A \to N \to N/N \cap A \to 1$ we get that $N \subset E$ contains a subgroup of order n. Finally, if all p-Sylow subgroups of A are normal in E (hence in A), then A is their direct product, hence solvable, and we are done by the solvable case.

10. Hochschild homology and cohomology

Hochschild (co)homology is a generalization of group (co)homology theory for modules over arbitrary rings defined via a simplicial method.

[1] Here is the argument: if P is a p-Sylow of A and $g \in E$, then gPg^{-1} is another p-Sylow of A since A is normal in E. But then by the conjugacy of Sylow subgroups $gPg^{-1} = aPa^{-1}$ for some $a \in A$. Hence $a^{-1}g \in N$, i.e. $g \in AN$.

Let R be a not necessarily commutative ring with unit. An $R - R$ *bimodule* is an abelian group M that has a left and a right module structure over R satisfying $(rm)s = r(ms)$ for $r, s \in R$, $m \in M$. When R is commutative, an $R - R$ bimodule is the same thing as an R-module if we define right multiplication via $mr := rm$.

Construction 10.1. Suppose that the ring R is an A-algebra for a commutative ring A. All tensor products in this construction will be taken over A. For an $R - R$ bimodule M set

$$M_i := M \otimes R^{\otimes i}$$

for $i \geq 0$, with $M_0 = M$. Define maps $\delta_i^j : M_i \to M_{i-1}$ by

$$\delta_i^j(m \otimes r_1 \otimes \cdots \otimes r_i) = \begin{cases} mr_1 \otimes r_2 \otimes \cdots \otimes r_i & j = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_j r_{j+1} \otimes \cdots \otimes r_i & 0 < j < i \\ r_i m \otimes r_2 \otimes \cdots \otimes r_{i-1} & j = i. \end{cases}$$

A calculation shows that together with the maps $\delta_i := \sum_j (-1)^j \delta_i^j$ the M_i form a homological complex $M_\bullet$. The group $HH_i(R, M) := H_i(M_\bullet)$ is the *i-th Hochschild homology group of R with coefficients in M.*

Similarly, define

$$M^i := \mathrm{Hom}_A(R^{\otimes i}, M)$$

for $i \geq 0$, with $M^0 = M$. For $i > 0$ the elements of M^i can be identified with A-multilinear maps $R^{\times i} \to M$. Now define maps $\partial_j^i : M^i \to M^{i+1}$ by sending a map $f : R^{\times i} \to M$ to $\partial_j^i f$ satisfying

$$\partial_j^i(f)(r_1, r_2, \cdots, r_{i+1}) = \begin{cases} r_1 f(r_2, r_3, \ldots, r_{i+1}) & j = 0 \\ f(r_1, \ldots, r_j r_{j+1}, \ldots, r_{i+1}) & 0 < j < i + 1 \\ f(r_1, \ldots, r_i) r_{i+1} & j = i + 1. \end{cases}$$

One again checks that together with the maps $\partial^i := \sum_j (-1)^j \partial_j^i$ the M^i form a co-homological complex $M^\bullet$. The group $HH^i(R, M) := H^i(M^\bullet)$ is the *i-th Hochschild cohomology group of R with coefficients in M.*

As usual, we shall employ the terminology *i-cycles* for $\ker(\delta_i)$ and *i-boundaries* for $\mathrm{im}(\delta_{i+1})$. Similarly, we have cocycles and coboundaries in the cohomological setting.

Example 10.2.

(1) A Hochschild 1-cocycle is an A-linear map $f : R \to M$ satisfying $r_1 f(r_2) - f(r_1 r_2) + f(r_1) r_2 = 0$ for all $r_1, r_2 \in R$. It is a 1-coboundary if there is $m \in M$ with $f(r) = rm - mr$ for all r.

(2) A Hochschild 2-cocycle is an A-bilinear map $f:\ R\times R\to M$ satisfying

$$r_1f(r_2,r_3)-f(r_1r_2,r_3)+f(r_1,r_2r_3)-f(r_1,r_2)r_3=0. \tag{16}$$

It is a 2-coboundary if it is of the form

$$f(r_1,r_2)=r_1g(r_2)-g(r_1r_2)+g(r_1)r_2 \tag{17}$$

for an A-linear map $R\to M$.

Example 10.3. In the case $A=\mathbf{Z}$, $R=\mathbf{Z}[G]$ for a group G we may equip any left $\mathbf{Z}[G]$-module M with a bimodule structure by declaring the right action of G to be trivial. In this case the complex $M^\bullet$ identifies with the complex $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[G^{\bullet+1}],M)$ with $\mathbf{Z}[G^{\bullet+1}]$ the standard resolution of $\mathbf{Z}$. The differentials in $M^\bullet$ translate to those used for calculating group cohomology via inhomogeneous cochains in Construction 9.4. Thus in this case we recover group cohomology with coefficients in M.

Similarly, the Hochschild homology groups in this case translate to the group homology groups mentioned before.

Under restrictive assumptions we may give more conceptual definitions for Hochschild (co)homology. Introduce the *opposite ring* R^{op} in which addition is the same but multiplication is given by $(r_1,r_2)\mapsto r_2r_1$. Left R-modules may be identified with right R^{op}-modules, and similarly for right modules. If we set $S:=R\otimes_A R^{\mathrm{op}}$, then $R-R$ bimodules give rise to left S-modules via the group law induced by $(r_1\otimes r_2)m:=r_1mr_2$ and right S-modules via $m(r_1\otimes r_2):=r_2mr_1$. In particular, R itself becomes an $S-S$-bimodule. Of course, in the case when R is commutative, $R=R^{\mathrm{op}}$ and $S=R\otimes R$.

Proposition 10.4. *Assume R is projective as an A-module and set $S:=R\otimes_A R^{\mathrm{op}}$. For every $R-R$ bimodule M we have canonical isomorphisms*

$$HH_i(R,M)\cong \mathrm{Tor}_i^S(M,R),\qquad HH^i(R,M)\cong \mathrm{Ext}^i_S(R,M).$$

Proof. Consider the Hochschild complex $M_\bullet$ for $M=R^{\otimes 2}$. The degree i term is $R^{\otimes i+2}$. Define an S-module structure on $R^{\otimes i+2}$ via

$$(r\otimes r')(r_1\otimes r_2\otimes\cdots\otimes r_{i+2})=rr_1\otimes r_2\otimes\cdots\otimes r_{i+1}\otimes r_{i+2}r'.$$

Since $R^{\otimes i}$ is projective over A by Lemma 8.18, $R^{\otimes i+2}$ is projective over S. On the other hand, setting $k_i(r_1\otimes r_2\otimes\cdots\otimes r_{i+2})=1\otimes r_1\otimes r_2\otimes\cdots\otimes r_{i+2}$ defines a chain homotopy of the identity map of $(R^{\otimes 2})_\bullet$ with 0 as in the special case of group (co)homology. So $(R^{\otimes 2})_\bullet$ is acyclic in degrees >0 and defines a projective resolution of the S-module R via the multiplication map $R\otimes_A R\to R$. We may thus compute $\mathrm{Tor}_i^S(M,R)\cong HH_i(M\otimes_S(R^{\otimes 2})_\bullet)$. On the other hand, $M\otimes_S(R^{\otimes 2})_\bullet\cong M_\bullet$ via the isomorphisms $M \otimes_{R\otimes_A R^{\rm op}} R^{\otimes i+2} \cong M \otimes_A R^{\otimes i}$, whence the statement for homology. The proof for cohomology is similar, using $\mathrm{Hom}_S(R^{\otimes i+2}, M) \cong \mathrm{Hom}_A(R^{\otimes i}, M)$. $\square$

As a first application of this description, we prove a localization statement that will serve later.

Proposition 10.5. *Let R be a commutative A-algebra that is projective as an A-module. If M is an R-module and $T \subset R$ is a multiplicatively closed subset, then*

$$HH_i(R_T, M \otimes_R R_T) \cong HH_i(R, M) \otimes_R R_T$$

for all $i \geq 0$.

The proof uses some general properties of the Tor functor.

Lemma 10.6. *Let B be a commutative ring, C a flat B-algebra and M a B-module.*

(1) *For every B-module N we have canonical isomorphisms*

$$\mathrm{Tor}_i^B(M, N \otimes_B C) \cong \mathrm{Tor}_i^B(M, N) \otimes_B C.$$

(2) *For every C-module N_C we have canonical isomorphisms*

$$\mathrm{Tor}_i^B(M, N_C) \cong \mathrm{Tor}_i^C(M \otimes_B C, N_C).$$

for all $i \geq 0$.

Proof. For (1) take a projective resolution $P_\bullet \to M$. Then by flatness of C over B

$$\mathrm{Tor}_i^B(M, N \otimes_B C) \cong H_i(P_\bullet \otimes_B (N \otimes_B C)) \cong H_i(P_\bullet \otimes_B N) \otimes_B C \cong \mathrm{Tor}_i^B(M, N) \otimes_B C.$$

For (2) the same projective resolution $P_\bullet \to M$ gives

$$\mathrm{Tor}_i^B(M, N_C) \cong H_i(P_\bullet \otimes_B N_C) \cong H_i(P_\bullet \otimes_B C) \otimes_C N_C) \cong \mathrm{Tor}_i^C(M \otimes_B C, N_C)$$

as $P_\bullet \otimes_B C \to M \otimes_B C$ is a projective resolution by flatness of C over B. $\square$

Proof of Proposition 10.5. As R_T is flat over R, so is $R_T \otimes_A R$ over $R \otimes_A R$ and finally $(R_T \otimes_A R) \otimes_R R_T \cong R_T \otimes_A R_T$ over $(R \otimes_A R) \otimes_R R \cong R \otimes_A R$. We view M as an R-module via the multiplication map $R \otimes_A R \to R$. Lemma 10.6 (1) thus gives

$$\mathrm{Tor}_i^{R\otimes_A R}(R, M \otimes_{R\otimes_A R} (R_T \otimes_A R_T) \cong \mathrm{Tor}_i^{R\otimes_A R}(R, M) \otimes_{R\otimes_A R} (R_T \otimes_A R_T)$$

which we may rewrite as

$$\mathrm{Tor}_i^{R\otimes_A R}(R, M \otimes_R R_T) \cong \mathrm{Tor}_i^{R\otimes_A R}(R, M) \otimes_R R_T \cong HH_i(R, M) \otimes_R R_T$$

using Proposition 10.4. On the other hand, Lemma 10.6 (2) gives

$$\begin{aligned}\mathrm{Tor}_i^{R\otimes_A R}(R, M \otimes_R R_T) &\cong \mathrm{Tor}_i^{R_T\otimes_A R_T}(R \otimes_{R\otimes_A R} (R_T \otimes_A R_T), M \otimes_R R_T) \\ &\cong \mathrm{Tor}_i^{R_T\otimes_A R_T}(R_T, M \otimes_R R_T)\end{aligned}$$

which is $HH_i(R_T, M \otimes_R R_T)$, again by Proposition 10.4. $\square$

Hochschild 2-cocyles are related to extensions as in the special case of group cohomology.

Definition 10.7. Let M be an $R-R$-bimodule. A *Hochschild extension* of R by M is an A-algebra E together with a *split* exact sequence of A-modules

$$0 \to M \to E \xrightarrow{\pi} R \to 0$$

such that $\pi : E \to R$ is a ring homomorphism and M viewed as a two-sided ideal in E satisfies $M^2 = 0$.

Two Hochschild extensions E, E' of R by M are called *equivalent* if there is an A-algebra homomorphism $E \to E'$ inducing the identity on M and R.

Note that if $M \subset E$ is a two-sided ideal satisfying $M^2 = 0$, then M has an $R-R$ bimodule structure over E/M. Also, the splitting condition is automatically satisfied when R is projective over A (e.g. when A is a field).

Construction 10.8. Let $0 \to M \to E \xrightarrow{\pi} R \to 0$ be a Hochschild extension of R by M, and let $s : R \to E$ be an A-linear section of π. For all $r_1, r_2 \in R$ the formula

$$f(r_1, r_2) := s(r_1)s(r_2) - s(r_1 r_2) \in M$$

defines an A-bilinear function $f : R \times R \to M$ which is in fact a 2-cocycle. Indeed, if we identify R with $s(R)$, then $E \cong M \oplus R$ as an A-module and the multiplication on E is given by

$$(m_1, r_1)(m_2, r_2) = (r_1 m_2 + m_1 r_2 + f(r_1, r_2), r_1 r_2) \tag{18}$$

because $M^2 = 0$. The A-submodule $s(R) \subset E$ is identified with the submodule of elements of the form $(0, r)$. Now for $r_1, r_2, r_3 \in R$ we have $(0, r_1)(0, r_2) = (f(r_1, r_2), r_1 r_2)$, so

$$[(0, r_1)(0, r_2)](0, r_3) = (f(r_1, r_2), r_1 r_2)(0, r_3) = (f(r_1, r_2)r_3 + f(r_1 r_2, r_3), r_1 r_2 r_3)$$

and

$$(0, r_1)[(0, r_2)(0, r_3)] = (0, r_1)[(f(r_2, r_3), r_2 r_3)] = (r_1 f(r_2, r_3) + f(r_1, r_2 r_3), r_1 r_2 r_3).$$

These two elements are equal by associativity of the multiplication in E, so the cocycle relation (16) holds as claimed. Now if $t : R \to E$ is another section of π giving rise to the 2-cocyle $g(r_1, r_2) := t(r_1)t(r_2) - t(r_1 r_2)$, then

$$f(r_1, r_2) - g(r_1, r_2) = t(r_1 r_2) - t(r_1)t(r_2) - s(r_1 r_2) + s(r_1)s(r_2) =$$
$$= t(r_1)[s(r_2) - t(r_2)] - [s(r_1 r_2) - t(r_1 r_2)] + [(s(r_1) - t(r_1)]s(r_2)$$

which is a 2-coboundary according to (17).

Thus our Hochschild extension $0 \to M \to E \xrightarrow{\pi} R \to 0$ gives rise to a well-defined cohomology class $c(E) \in HH^2(R, M)$. Note that in the case the extension is trivial, i.e. π has a section that is also multiplicative, we have $c(E) = 0$.

Proposition 10.9. *The assignment $E \mapsto c(E)$ induces a bijection between equivalence classes of Hochschild extensions of R by M and elements of the Hochschild cohomology group $HH^2(R, M)$.*

Sketch of proof. Equivalent Hochschild extensions give rise to the same cohomology class because composing a section $s : R \to E$ with the map $E \to E'$ inducing the equivalence of extensions yields the same 2-cocycle. Given a class in $HH^2(R, M)$ represented by a 2-cocycle $f(r_1, r_2)$, we define a ring structure on the A-module direct sum $M \oplus R$ via the formula (18). One checks that we indeed obtain a ring in which M is an ideal of square zero. By construction the associated 2-cocycle is $f(r_1, r_2)$. Finally, a similar argument as in the case of group extensions shows that cohomologous cocycles give rise to equivalent extensions. □

Now assume R is commutative. In this case we infer from formula (18) that the extension E is a commutative ring if and only if the 2-cocycle $f(r_1, r_2)$ satisfies $f(r_1, r_2) = f(r_2, r_1)$; such 2-cocycles are called *symmetric*. Classes of symmetric 2-cocycles form a subgroup in $HH^2(R, M)$; let us denote it by $HH^2_s(R, M)$. The following corollary is immediate:

Corollary 10.10. *If R is commutative, the assignment $E \mapsto c(E)$ induces a bijection between equivalence classes of commutative Hochschild extensions of R by M and elements of the Hochschild cohomology group $HH^2_s(R, M)$.*

To conclude this topic, recall that a commutative A-algebra R is called *formally smooth* if it satisfies the following property: given a commutative diagram

(19)
$$\begin{array}{ccc} R & \xrightarrow{\bar\lambda} & B/M \\ \uparrow & & \uparrow \\ A & \xrightarrow{\mu} & B \end{array}$$

with a commutative ring B and an ideal $M \subset B$ satisfying $M^2 = 0$, the map $\bar\lambda$ lifts to a map $\lambda : R \to B$ making the diagram commute.

Corollary 10.11. *Let R be a commutative A-algebra that is projective as an A-module. Then R is formally smooth over A if and only if $HH^2_s(R, M) = 0$ for all R-modules M.*

Proof. If R is formally smooth and B is a commutative Hochschild extension of R by M, then the formal smoothness property shows that the extension $0 \to M \to B \to$

$R \to 0$ has a multiplicative splitting, so its class in $HH^2_s(B, M)$ is 0. Now apply the previous corollary.

Conversely, given a diagram as above, take the pullback of the A-module extension $0 \to M \to B \to B/M \to 0$ by $\bar{\lambda} : R \to B/M$ as in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & R & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \bar{\lambda}} & & \\ 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & B/M & \longrightarrow & 0. \end{array}$$

Then E inherits a commutative ring structure from B and R in which M is still an ideal of square zero and the projection $\pi : E \to R$ is a ring homomorphism. Moreover, since R is projective over A, the extension $0 \to M \to E \to R \to 0$ splits as an A-module extension, hence is a Hochschild extension. Now by the previous corollary $HH^2_s(R, M) = 0$ implies that there is a multiplicative splitting $s : R \to E$ of π. Composing s with the map $E \to B$ gives the required lifting $\lambda : R \to B$. □

In the rest of this section all rings considered will be commutative. We shall prove the Hochschild–Kostant–Rosenberg theorem which asserts that if k is a field and R a localization of a finitely generated k-algebra which is formally smooth over k, there are isomorphisms

$$HH_i(R, R) \cong \Omega^i_{R/k}$$

where $\Omega^i_{R/k} := \Lambda^i \Omega^1_{R/k}$ is the module of differental i-forms over R.

In fact, more is true. We shall see that the direct sum

$$HH_\bullet(R, R) := \bigoplus_{i=0}^{\infty} HH_i(R, R)$$

has a multiplicative structure making it a (non-commutative) graded R-algebra. On the other hand,

$$\Omega^\bullet_{R/k} := \bigoplus_{i=0}^{i} \Omega^i_{R/k}$$

has a graded R-algebra structure induced by the wedge product (see Facts 7.12). We shall in fact construct an isomorphism

$$HH_\bullet(R, R) \cong \Omega^\bullet_{R/k}$$

of graded R-algebras.

We begin with a baby case.

Lemma 10.12. *Let R be a commutative A-algebra. We have a canonical isomorphism of R-modules*

$$HH_1(R, R) \cong \Omega^1_{R/A}.$$

Proof. The degree 1 term of the Hochschild complex for R is $R \otimes_A R$. Since R is commutative, the outgoing differential $\delta_1 : r_1 \otimes r_2 \mapsto r_1 r_2 - r_2 r_1$ is 0, and

$$HH_1(R,R) \cong R \otimes_A R / \langle rr_1 \otimes r_2 - r \otimes r_1 r_2 + r_2 r \otimes r_1 \rangle.$$

Now define a surjective map $\chi : R \otimes_A R \to \Omega^1_{R/A}$ by $r_1 \otimes r_2 \mapsto r_1 dr_2$. We then have

$$\chi(rr_1 \otimes r_2 - r \otimes r_1 r_2 + r_2 r \otimes r_1) = r(r_1 dr_2 - d(r_1 r_2) + r_2 dr_1) = 0$$

by the Leibniz rule, so χ induces an isomorphism $HH_1(R,R) \xrightarrow{\sim} \Omega^1_{R/A}$ as required. □

Let now k be a field and R a k-algebra. Recall from Proposition 10.4 that we then have isomorphisms $HH_i(R,R) \cong \mathrm{Tor}_i^S(R,R)$ for all i, where $S = R \otimes_k R$. Therefore $HH_\bullet(R,R)$ is an R-algebra equipped with an R-linear multiplication introduced in Construction 8.19. If the characteristic of k is not 2, then Proposition 8.20 implies that this multiplication is alternating, i.e. satisfies $a \otimes a = 0$ for $a \in HH_1(R,R)$. It then follows from the universal property of the wedge product that the R-module map $\Omega^1_{R/k} \to HH_1(R,R)$ coming from Lemma 10.12 induces a map of graded R-algebras

$$\Omega^\bullet_{R/k} \to HH_\bullet(R,R). \tag{20}$$

It can be proven that the multiplication on $HH_\bullet(R,R)$ is alternating even in characteristic 2, so the above map exists in arbitrary characteristic. We omit the somewhat involved proof in the characteristic 2 case.

Theorem 10.13 (Hochschild–Kostant–Rosenberg)**.** *Assume that R is a localization of a finitely generated k-algebra and moreover R is formally smooth over k. Then the map (20) is an isomorphism of graded R-algebras.*

We shall need the following easy lemma.

Lemma 10.14. *Let S be a regular local ring and $I \subset S$ an ideal such that S/I is also a regular local ring. Then I is generated by a regular sequence.*

Proof. Let Q be the maximal ideal of S and $\kappa := S/Q$. From the isomorphism of κ-vector spaces $Q/(I+Q^2) \cong (Q/I)/(Q/I)^2$ we infer that we may find a κ-basis of Q/Q^2 by completing a κ-basis of $(Q/I)/(Q/I)^2$ with a κ-basis $\bar{s}_1, \dots, \bar{s}_r$ of $(I+Q^2)/Q^2$. Choosing preimages $s_i \in I$ of the $\bar{s}_i$ we obtain $I = (s_1, \dots, s_r)$ from Nakayama's lemma. By construction the s_i form part of a minimal system of generators of Q, hence they form a regular sequence by the theory of regular local rings. □

Proof of Theorem 10.13. By a basic property of formal smoothness, if R is formally smooth over k, then so is $S = R \otimes_k R$. Also, every localization of R or S is still formally smooth over k. On the other hand, R is Noetherian by assumption and so is S. Indeed, since R is a localization of some finitely generated k-algebra $k[a_1, \dots, a_m]$, it follows that $S = R \otimes_k R$ is a localization of the finitely generated R-algebra $R[a_1, \dots, a_m]$. Therefore every localization of R or S by a prime ideal is a Noetherian local ring which is formally smooth over k. It is known from commutative algebra that these are then regular local rings.

If $P \subset R$ is a maximal ideal, then $HH_i(R_P, R_P) \cong HH_i(R, R) \otimes_R R_P$ for all i by Proposition 10.5. On the other hand, $\Omega^1_{R_P/k} \cong \Omega^1_{R/k} \otimes_R R_P$ by the localization property of differentials, and therefore also $\Omega^i_{R_P/k} \cong \Omega^i_{R/k} \otimes_R R_P$ for all $i > 0$. WE conclude that the theorem holds for R if and only if it holds for R_P for all $P \subset R$ maximal by Lemma 2.20. Thus we may replace R by R_P and assume R is local. Then taking the preimage $Q \subset S$ of P in S, we may replace S by S_Q. By the lemma above we are now in the situation of Proposition 8.22, and the theorem follows from that proposition together with Lemma 10.12. $\square$

11. The homotopy category and its exact triangles

To start our work towards the construction of derived categories, we first present some auxiliary constructions for complexes in an abelian category $\mathcal{A}$ that are important in their own right. We denote the category of complexes in $\mathcal{A}$ by $C(\mathcal{A})$.

Construction 11.1. Given a morphism of complexes $f : A^\bullet \to B^\bullet$ in $C(\mathcal{A})$, the *cone* $C(f)$ of f is the complex with terms $C(f)^i = A^{i+1} \oplus B^i$ and differentials $d_f : C(f)^i \to C(f)^{i+1}$ given by the 2×2 matrix of morphisms

$$d_f = \begin{bmatrix} -d_A & 0 \\ f & d_B \end{bmatrix}$$

where d_A and d_B are the differentials of $A^\bullet$ and $B^\bullet$, respectively. (Thus for $\mathcal{A}$ the category of abelian groups, $a \in A^{i+1}, b \in B^i$ we have $d_f((a, b)) = (-d_A(a), f(a) + d_B(b))$.) This is indeed a complex because

$$\begin{bmatrix} -d_A & 0 \\ f & d_B \end{bmatrix}^2 = \begin{bmatrix} d_A \circ d_A & 0 \\ -f \circ d_A + d_B \circ f & d_B \circ d_B \end{bmatrix}$$

which is 0 because $A^\bullet, B^\bullet$ are complexes and f is a morphism of complexes.

Given a commutative diagram

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f} & B^\bullet \\ \downarrow & & \downarrow \\ A'^\bullet & \xrightarrow{g} & B'^\bullet \end{array}$$

of morphisms in $C(\mathcal{A})$, there is an obvious induced morphism $C(f) \to C(g)$; this is the *functoriality of the cone construction.*

Quite generally for a complex $A^\bullet$ and $n \in \mathbf{Z}$ the shifted complex $A^\bullet[n]$ is defined by

$$A[n]^i := A^{i+n}, \quad d_{A[n]^\bullet} = (-1)^n d_{A^\bullet}.$$

With this notation we have an exact sequence of complexes

$$0 \to B^\bullet \to C(f) \to A^\bullet[1] \to 0. \tag{21}$$

Lemma 11.2. *Let $f : A^\bullet \to B^\bullet$ be a morphism in $C(\mathcal{A})$.*

(1) *The morphisms $H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet)$ in the long exact cohomology sequence of (21) equal $H^{i+1}(f)$.*
(2) *The morphism f is a quasi-isomorphism if and only if $C(f)$ is acyclic.*

Proof. For (1), we may assume $\mathcal{A}$ is a category of modules and take $a \in Z^i(A[1]^\bullet) = Z^{i+1}(A^\bullet)$. By the proof of Proposition 3.5 the image of its class in the long exact sequence can be constructed by lifting it to $(a, 0) \in C(f)^i$ and then taking $d_f((a, 0)) = -d_A(a) + f(a) = f(a)$, which indeed represents $H^{i+1}(f)(a)$. Statement (2) follows by the long exact cohomology sequence associated with (21). □

The other standard construction is:

Construction 11.3. Given a morphism of complexes $f : A^\bullet \to B^\bullet$ in $C(\mathcal{A})$, the *cylinder* $\mathrm{Cyl}(f)$ of f is the complex with terms $\mathrm{Cyl}(f)^i = A^{i+1} \oplus B^i \oplus A^i$ and differentials $d_{\mathrm{cyl}} : \mathrm{Cyl}(f)^i \to \mathrm{Cyl}(f)^{i+1}$ given by the 3×3 matrix of morphisms

$$d_{\mathrm{cyl}} = \begin{bmatrix} -d_A & 0 & 0 \\ f & d_B & 0 \\ \mathrm{id}_A & 0 & d_A \end{bmatrix}$$

where d_A and d_B are the differentials of $A^\bullet$ and $B^\bullet$, respectively. For $\mathcal{A}$ a category of modules we have the formula

$$d_{\mathrm{cyl}}(a_{i+1}, b_i, a_i) = (-d_A(a_{i+1}), f(a_{i+1}) + d_B(b_i), a_{i+1} + d_A(a_i)). \tag{22}$$

Note that

$$\mathrm{Cyl}(f) = C(C(f)[-1] \to A) \tag{23}$$

where the morphism $C(f)[-1] \to A^\bullet$ comes from (21) after shifting by -1; it is given by $(\mathrm{id}_A, 0)$. Indeed, the terms of the two complexes are equal and equality of the differentials can be read off the matrices. From this it follows that $\mathrm{Cyl}(f)$ is a complex (which can also be checked directly) and that there is an exact sequence

$$0 \to A^\bullet \to \mathrm{Cyl}(f) \to C(f) \to 0 \tag{24}$$

in $C(\mathcal{A})$, by combining (21) and (23).

The cylinder has important chain-homotopical properties:

Proposition 11.4. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes.*

1. *The natural map of complexes $i : B^\bullet \to \mathrm{Cyl}(f)$ induced by the inclusion of $B^\bullet$ in the second component gives a chain homotopy equivalence between $B^\bullet$ and $\mathrm{Cyl}(f)$.*
2. *Another morphism of complexes $g : A^\bullet \to B^\bullet$ is homotopic to f if and only if there is a morphism of complexes $\mathrm{Cyl}(-\mathrm{id}_A) \to B^\bullet$ which composed with the natural inclusions $A^\bullet \to \mathrm{Cyl}(-\mathrm{id}_A)$ in the second and third component gives back f and g, respectively.*

Proof. For (1), define a morphism of complexes $p : \mathrm{Cyl}(f) \to B^\bullet$ by sending (a_{i+1}, b_i, a_i) to $-f(a_i) + b_i$. This is a morphism of complexes because

$$(p \circ d_{\mathrm{cyl}})(a_{i+1}, b_i, a_i) = f(a_{i+1}) + d_B(b_i) - f(a_{i+1}) - f(d_A(a_i)) = d_B(-f(a_i) + b_i).$$

By construction $p \circ i = \mathrm{id}_B$, and now we check $\mathrm{id}_{\mathrm{Cyl}(f)} - i \circ p = k \circ d_{\mathrm{cyl}} + d_{\mathrm{cyl}} \circ k$, where $k : \mathrm{Cyl}(f)^i \to \mathrm{Cyl}(f)^{i-1}$ is given by $k(a_{i+1}, b_i, a_i) = (a_i, 0, 0)$. Indeed,

$$(k \circ d_{\mathrm{cyl}})(a_{i+1}, b_i, a_i) = (a_{i+1} + d_A(a_i), 0, 0),$$

$$(d_{\mathrm{cyl}} \circ k)(a_{i+1}, b_i, a_i) = d_{\mathrm{cyl}}(a_i, 0, 0) = (-d_A(a_i), f(a_i), a_i);$$

on the other hand,

$$(a_{i+1}, b_i, a_i) - (0, -f(a_i) + b_i, 0) = (a_{i+1}, f(a_i), a_i).$$

For (2) suppose k induces a chain homotopy between f and g and consider the map $\mathrm{Cyl}(-\mathrm{id}) \to B^\bullet$ induced by the triple (k, f, g); it indeed gives back f and g after composing with the natural inclusions. We compute using formula (22) for $(a_{i+1}, \bar{a}_i, a_i) \in A^{i+1} \oplus A^i \oplus A^i$

$$((k, f, g) \circ d_{\mathrm{cyl}})((a_{i+1}, \bar{a}_i, a_i) = -(k \circ d_A)(a_{i+1}) - f(a_{i+1}) + f(d_A(\bar{a}_i)) + g(a_{i+1}) + g(d_A(a_i))$$

and

$$(d_B \circ (k, f, g))((a_{i+1}, \bar{a}_i, a_i) = (d_B \circ k)(a_{i+1}) + d_B(f(\bar{a}_i)) + d_B(g(a_i))$$

Since f and g are morphisms of complexes, equality of the two is equivalent to

$$g(a_{i+1}) - f(a_{i+1}) = (d_B \circ k + k \circ d_A)(a_{i+1})$$

which holds precisely because f and g are homotopic via k. The converse follows by reversing the argument. □

As a first application, note the following. Given an exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

in $C(\mathcal{A})$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & \mathrm{Cyl}(f) & \longrightarrow & C(f) & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle -p}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C^\bullet & \longrightarrow & 0 \end{array}$$

where the upper row is (24), $p : (a_{i+1}, b_i, a_i) \to -f(a_i)+b_i$ is the homotopy inverse of i constructed in the above proof and h is the map induced on cokernels (explicitly, $h(a_{i+1}, b_i) = -g(b_i)$).

Corollary 11.5. *The map $h : C(f) \to C^\bullet$ is a quasi-isomorphism.*

Proof. This follows from Proposition 11.4 (1) and Corollary 3.7. □

Remark 11.6. In general h is not a homotopy equivalence, even though id and i are (see Remark 11.13 below).

We now come to a crucial definition:

Definition 11.7. The *homotopy category* $K(\mathcal{A})$ is the category with the same objects as $C(\mathcal{A})$ but with morphisms

$$\mathrm{Hom}_{K(\mathcal{A})}(A, B) := \mathrm{Hom}_{C(\mathcal{A})}(A, B)/\{\phi \in \mathrm{Hom}_{C(\mathcal{A})}(A, B) : \phi \sim 0\}$$

where $\sim$ denotes homotopy equivalence of morphisms of complexes. The quotient makes sense because the ϕ homotopic to 0 form a subgroup in $\mathrm{Hom}_{C(\mathcal{A})}(A, B)$. Composition is induced from composition of morphisms in $C(\mathcal{A})$.

Remark 11.8. The category $K(\mathcal{A})$ is additive but not abelian general. Example: let $\mathcal{A}$ be the category of abelian groups, and consider the morphism $\phi : \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ as a morphism in $C(\mathcal{A})$ of complexes concentrated in degree 0. This morphism has a kernel in $C(\mathcal{A})$, namely $2\mathbf{Z}$ viewed again as a complex, but not in $K(\mathcal{A})$. Indeed, if ϕ had a kernel in $K(\mathcal{A})$, it would be represented by $2\mathbf{Z}$ because every morphism $\mathbf{Z} \to A$ with A an abelian group induces a morphism in $K(\mathcal{A})$. Now consider the morphism of complexes ψ given by

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}/2\mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Then $\phi \circ \psi \sim 0$, a homotopy being given by the identity of $\mathbf{Z}/2\mathbf{Z}$ in degree 1 and by the zero map elsewhere. Now if $2\mathbf{Z}$ were a kernel for ϕ, then ψ would factor through a morphism

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}/2\mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 2\mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

in $K(\mathcal{A})$ but that's impossible (there is no such factorization in $C(\mathcal{A})$ and no homotopy to help as the only map $\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}$ is the zero map).

Since exact sequences do not make sense in $K(\mathcal{A})$ by the above remark, we consider a substitute. A *triangle* in $K(\mathcal{A})$ is a sequence of morphisms

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

in $K(\mathcal{A})$. The basic example to have in mind is the triangle

$$A^\bullet \to B^\bullet \to C(f) \to A^\bullet[1]$$

coming from (21). An *exact (or distinguished) triangle* in $K(\mathcal{A})$ is a triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

for which there is a commutative diagram

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A^\bullet[1] \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\alpha[1]} \\ A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C(f') & \longrightarrow & A'^\bullet[1] \end{array}$$

with some $f' : A'^\bullet \to B'^\bullet$ in $K(\mathcal{A})$ such that all vertical maps are isomorphisms in $K(\mathcal{A})$. (Note that, viewed as a diagram in $C(\mathcal{A})$, the squares only commute up to homotopy!). The following statements are more or less immediate from the definition:

Lemma 11.9.

(1) *The composition of any two consecutive maps in an exact triangle is 0 in $K(\mathcal{A})$.*
(2) *If $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is an exact triangle, there is an associated long exact sequence*

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

in $\mathcal{A}$.

Proof. In (1) the triviality of the composite map $B^\bullet \to C^\bullet \to A^\bullet[1]$ follows from exact sequence (21), and that of $A^\bullet \to B^\bullet \to C^\bullet$ from exact sequence (24) and Proposition 11.4 (1). The sequence in (2) identifies with the long exact sequence associated with the exact sequence of complexes (24). □

The next lemma is a bit less straightforward.

Lemma 11.10.

(1) *A triangle*

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

in $K(\mathcal{A})$ is exact if and only if the shifted triangle

$$C^\bullet[-1] \to A^\bullet \to B^\bullet \to C^\bullet$$

is exact.

(2) *Given a commutative diagram*

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A^\bullet[1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle \alpha[1]} \\ A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C'^\bullet & \longrightarrow & A'^\bullet[1] \end{array}$$

of exact triangles in $K(\mathcal{A})$, there is a morphism $\gamma : C^\bullet \to C'^\bullet$ in $K(\mathcal{A})$ making the diagram commute.

Proof. The 'if' part of (1) follows by applying the 'only if' part twice and shifting. So suppose $A^\bullet \xrightarrow{f} B^\bullet \to C^\bullet \to A^\bullet[1]$ is exact. We may assume $C^\bullet = C(f)$ and by Proposition 11.4 (1) we may replace $B^\bullet$ by $\mathrm{Cyl}(f)$ in $K(\mathcal{A})$. But

$$C(f)[-1] \to A^\bullet \to \mathrm{Cyl}(f)^\bullet \to C(f)$$

is an exact triangle by the isomorphism (23).

For statement (2) we may again assume $C^\bullet = C(f)$ (and similarly for $C'^\bullet$), whence the statement follows by functoriality of $C(f)$. □

The two statements of the proposition are parts of the general formalism of *triangulated categories*, an axiomatic theory extracted from properties of exact triangles in $K(\mathcal{A})$. To show the power of the formalism we derive some consequences.

Corollary 11.11.

(1) *For every object $X^\bullet$ in $K(\mathcal{A})$ applying the functor $\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, \_\_)$ to the first row of the diagram in Lemma 11.10 (2) induces an exact sequence of abelian groups*

$$\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, A^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, B^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, C^\bullet).$$

Similarly, applying the contravariant functor $\mathrm{Hom}_{K(\mathcal{A})}(\_\_, X^\bullet)$ *to the first row of the diagram in Lemma 11.10 (2) induces an exact sequence*

$$\mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, X^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(B^\bullet, X^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, X^\bullet).$$

(2) *If any two of the maps* α, β, γ *in Lemma 11.10 (2) are isomorphisms in* $K(\mathcal{A})$*, then so is the third one.*

Proof. The sequence of statement (1) is a complex by Lemma 11.9 (1), so assume $f : X^\bullet \to B^\bullet$ becomes 0 in $K(\mathcal{A})$ after composing with the map $B^\bullet \to C^\bullet$. Noting that $C(X^\bullet \to 0) = X^\bullet[1]$, we have a diagram of exact triangles

$$\begin{tikzcd} X^\bullet \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, "0"] & X^\bullet[1] \arrow[r, "\mathrm{id}"] & X^\bullet[1] \arrow[d, "f[1]"] \\ B^\bullet \arrow[r] & C^\bullet \arrow[r] & A^\bullet[1] \arrow[r] & B^\bullet[1]. \end{tikzcd} \tag{25}$$

By Lemma 11.10 (2) there is a map $X^\bullet[1] \to A^\bullet[1]$ making the diagram commute, so after shifting we obtain a map $X^\bullet \to A^\bullet$ whose composition with $A^\bullet \to B^\bullet$ is f. The proof of the contravariant case is similar.

To prove (2) it is enough to consider the case where α, β are isomorphisms by Lemma 11.10 (1). We apply the contravariant form of statement (1) with $X^\bullet = C^\bullet$ to the triangles in Lemma 11.10 (2). Combined with Lemma 11.10 (1) we obtain a commutative diagram with exact rows

$$\begin{tikzcd} \mathrm{Hom}(A^\bullet, C^\bullet) & \mathrm{Hom}(B^\bullet, C^\bullet) \arrow[l] & \mathrm{Hom}(C^\bullet, C^\bullet) \arrow[l] & \mathrm{Hom}(A^\bullet[1], C^\bullet) \arrow[l] & \mathrm{Hom}(B^\bullet[1], C^\bullet) \arrow[l] \\ \mathrm{Hom}(A'^\bullet, C^\bullet) \arrow[u, "\alpha_*"] & \mathrm{Hom}(B'^\bullet, C^\bullet) \arrow[l] \arrow[u, "\beta_*"] & \mathrm{Hom}(C'^\bullet, C^\bullet) \arrow[l] \arrow[u, "\gamma_*"] & \mathrm{Hom}(A'^\bullet[1], C^\bullet) \arrow[l] \arrow[u, "\alpha_*[1]"] & \mathrm{Hom}(B'^\bullet[1], C^\bullet). \arrow[l] \arrow[u, "\beta_*[1]"] \end{tikzcd}$$

If α, β are isomorphisms, so are all vertical maps in the diagram by the five lemma, so there is $\gamma' : C'^\bullet \to C^\bullet$ in $K(\mathcal{A})$ with $\gamma' \circ \gamma = \mathrm{id}_{C'}$. This γ' makes the diagram

$$\begin{tikzcd} A'^\bullet \arrow[r] \arrow[d, "\alpha^{-1}"] & B'^\bullet \arrow[r] \arrow[d, "\beta^{-1}"] & C'^\bullet \arrow[r] \arrow[d, "\gamma'"] & A'^\bullet[1] \arrow[d, "\alpha^{-1}[1]"] \\ A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & A^\bullet[1] \end{tikzcd}$$

commute, so repeating the argument gives $\gamma'' : C^\bullet \to C'^\bullet$ in $K(\mathcal{A})$ with $\gamma'' \circ \gamma' = \mathrm{id}_{C^\bullet}$. But then composing with γ on the right gives $\gamma'' = \gamma$, so γ' is an inverse of γ in $K(\mathcal{A})$. □

We can now prove a stronger form of Lemma 11.2 (2).

Corollary 11.12. *A morphism* $f : A^\bullet \to B^\bullet$ *in* $C(\mathcal{A})$ *is a homotopy equivalence if and only if* $C(f)$ *is homotopically trivial (i.e. the identity map of* $C(f)$ *is homotopic to 0).*

Proof. Apply the second statement of the previous corollary to the commutative diagram of exact triangles

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & A^\bullet & \longrightarrow & 0 & \longrightarrow & A^\bullet[1] \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C(f) & \longrightarrow & A^\bullet[1] \end{array}$$

where the upper triangle is obtained by applying Lemma 11.10 (1) to the upper triangle in (25). □

Remark 11.13. We can now give an example showing that an exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ in $C(\mathcal{A})$ does not necessarily give rise to an exact triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $K(\mathcal{A})$. Consider the exact sequence of abelian groups $0 \to \mathbf{Z}/2\mathbf{Z} \xrightarrow{f} \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$ viewed as an exact sequence of complexes concentrated in degree 0. We have $C(f) = [\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z}]$ which is indeed quasi-isomorphic to $[0 \to \mathbf{Z}/2\mathbf{Z}]$ via the natural projection p but not homotopy equivalent. Indeed, the only possible homotopy inverse could be the natural injection $i : [0 \to \mathbf{Z}/2\mathbf{Z}] \to C(f)$ which indeed satisfies $p \circ i = \mathrm{id}$, but $i \circ p : [\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z}] \to [\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z}]$ cannot be homotopic to the identity because it is not surjective in degree 0 and no homotopy induced by a map $\mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ can remedy that.

Now were there a map $g : \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}[1]$ such that the triangle $\mathbf{Z}/2\mathbf{Z} \xrightarrow{f} \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \xrightarrow{g} \mathbf{Z}/2\mathbf{Z}[1]$ is isomorphic in $K(\mathcal{A})$ to some $A'^\bullet \xrightarrow{f'} B'^\bullet \to C(f') \to A'^\bullet[1]$, the isomorphisms identifying the sources and targets of f and f' would induce a map $C(f) \to C(f')$ in $K(\mathcal{A})$ by Lemma 11.10 (2) which would be an isomorphism by Corollary 11.11 (2). But $C(f') \cong \mathbf{Z}/2\mathbf{Z}$ by assumption whereas $C(f) \ncong \mathbf{Z}/2\mathbf{Z}$ by the above.

12. The derived category

There is another way of constructing the homotopy category, via *localization*.

Proposition 12.1. *Let $\mathcal{C}$ be a category and S a collection of morphisms in $\mathcal{C}$ containing all identity maps of objects and all compositions $s \circ t$ when $t \in \mathrm{Hom}_{\mathcal{C}}(A, B), s \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ are both in S.*

There is a category $S^{-1}\mathcal{C}$ and a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

1. *For every $s \in S$ the morphism $Q(s)$ is an isomorphism;*
2. *Every functor $F : \mathcal{C} \to \mathcal{D}$ that sends the elements of S to isomorphisms in $\mathcal{D}$ factors uniquely through Q.*

The category $S^{-1}\mathcal{C}$ is called the *localization* of $\mathcal{C}$ with respect to S. The pair $(S^{-1}\mathcal{C}, Q)$ is unique up to unique isomorphism.

Proof. One construction is as follows. Let $S^{-1}\mathcal{C}$ have the same objects as $\mathcal{C}$. For two objects A, B we define $\mathrm{Hom}_{S^{-1}\mathcal{C}}(A, B)$ as follows. Consider all possible chains of morphisms $A \cdots \leftarrow\leftarrow\rightarrow \cdots \leftarrow\rightarrow\rightarrow \cdots B$ where leftward morphisms are in S. We let $\mathrm{Hom}_{S^{-1}\mathcal{C}}(A, B)$ be the quotient of this set by the coarsest equivalence relation containing the following equivalences:

(1) $A \cdots \xrightarrow{f}\xrightarrow{g} \cdots B \sim A \cdots \xrightarrow{g\circ f} \cdots B$;
(2) $A \cdots \xleftarrow{s}\xleftarrow{t} \cdots B \sim A \cdots \xleftarrow{s\circ t} \cdots B$;
(3) $A \cdots \xleftarrow{s}\xrightarrow{s} \cdots B \sim A \cdots \xrightarrow{\mathrm{id}} \cdots B$ for $s \in S$;
(4) $A \cdots \xleftarrow{s}\xrightarrow{f} \cdots B \sim A \cdots \xrightarrow{g}\xleftarrow{t} \cdots B$ whenever $g \circ s = t \circ f$.

Morphisms are composed in the obvious way and there is a natural functor Q from $\mathcal{C}$ to this category that sends elements of S to isomorphisms by property (3). It satisfies the universal property (send a chain to a composition of $F(s)^{-1}$'s and $F(f)$'s for each leftward s and rightward f in the chain). □

Lemma 12.2. *If $\mathcal{A}$ is an abelian category, then $K(\mathcal{A})$ is the localization of $C(\mathcal{A})$ by the collection of homotopy equivalences.*

Proof. Suppose $F : C(\mathcal{A}) \to \mathcal{D}$ is a functor sending homotopy equivalences to isomorphisms. Recall from Proposition 11.4 (1) that for each complex $A^\bullet$ the natural map $i : a \mapsto (0, a, 0)$ induces a homotopy equivalence between $A^\bullet$ and $\mathrm{Cyl}(-id_A)$ with homotopy inverse $p : (a_{i+1}, \bar{a}_i, a_i) \mapsto \bar{a}_i + a_i$. Thus $F(p) = F(i)^{-1}$.

Now consider the map $j : A^\bullet \to \mathrm{Cyl}(-\mathrm{id}_A)$ given by $a \mapsto (0, 0, a)$. We have $p \circ j = \mathrm{id}_A$, so

$$F(i) = F(i) \circ F(p \circ j) = F(i) \circ F(p) \circ F(j) = F(j).$$

Now suppose $f, g : A^\bullet \to B^\bullet$ are homotopic via a map k. By Proposition 11.4 (2) the map (k, f, g) induces a morphism of complexes $\phi : \mathrm{Cyl}(-id_A) \to B^\bullet$ with $\phi \circ i = f$, $\phi \circ j = g$. But then

$$F(f) = F(\phi) \circ F(i) = F(\phi) \circ F(j) = F(g)$$

which means that F factors through the homotopy category $K(\mathcal{A})$. □

Definition 12.3. The *derived category* $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the localization of $C(\mathcal{A})$ with respect to the collection of quasi-isomorphisms of complexes.

Corollary 12.4. *One can also obtain $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ with respect to the collection of morphisms represented by quasi-isomorphisms of complexes.*

Proof. The universal functor $Q : C(\mathcal{A}) \to D(\mathcal{A})$ factors through $K(\mathcal{A})$ by Lemma 12.2 and satisfies the universal property for the collection of quasi-isomorphisms in $K(\mathcal{A})$ by definition. $\square$

The description of morphisms in the derived category furnished by the general localization construction is impractical. Here is a notion which brings it closer to the calculus of fractions for rings.

Definition 12.5. A collection S of morphisms in a category $\mathcal{C}$ is a *multiplicative system* if it satisfies the following axioms.

(1) All identity morphisms of objects of A are in S and if $t \in \mathrm{Hom}_{\mathcal{C}}(A, B), s \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ are both in S, so is $s \circ t$;
(2) Given $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and a morphism $s : A \to A'$ in S, there are morphisms $f' \in \mathrm{Hom}_{\mathcal{C}}(A', B')$ and $t : B \to B'$ in S making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ A' & \xrightarrow{f'} & B' \end{array}$$

commute. Similarly, if f' and t are given, we may complete the diagram with f and s.
(3) Given $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, there exists a morphism $s \in S$ with target A such that $f \circ s = g \circ s$ if and only if there exists $t \in S$ with source B such that $t \circ f = t \circ g$.

Construction 12.6. Given a multiplicative system S of morphisms in a category $\mathcal{C}$, we construct a category $S^{-1}\mathcal{C}$ as follows. The objects of $S^{-1}\mathcal{C}$ are to be the same as those of $\mathcal{C}$. Morphisms in $S^{-1}\mathcal{C}$ are to be equivalence classes of pairs

$$A \xleftarrow{s} A_1 \xrightarrow{f} B$$

with $s \in S$, $f \in \mathrm{Hom}_{\mathcal{C}}(A_1, B)$, subject to the following equivalence relation: two pairs $A \xleftarrow{s_1} A_1 \xrightarrow{f_1} B$ and $A \xleftarrow{s_2} A_2 \xrightarrow{f_2} B$ are equivalent if there is a third such pair $A \xleftarrow{s_3} A_3 \xrightarrow{f_3} B$ fitting in a commutative diagram

$$\begin{array}{ccccc} A & \xleftarrow{s_1} & A_1 & \xrightarrow{f_1} & B \\ {\scriptstyle \mathrm{id}}\big\uparrow & & \big\uparrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ A & \xleftarrow{s_3} & A_3 & \xrightarrow{f_3} & B \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ A & \xleftarrow{s_2} & A_2 & \xrightarrow{f_2} & B. \end{array}$$

Composition of morphisms in $S^{-1}\mathcal{C}$ is defined as follows. Given $A \stackrel{s}{\leftarrow} A_1 \stackrel{f}{\rightarrow} B$ and $B \stackrel{t}{\leftarrow} B_1 \stackrel{g}{\rightarrow} C$, we first use property (2) of multiplicative systems to find a diagram

$$\begin{tikzcd} & A' \arrow[r, "f'"] \arrow[d, "t'"'] & B_1 \arrow[r, "g"] \arrow[d, "t"] & C \\ A & A_1 \arrow[l, "s"'] \arrow[r, "f"] & B. & \end{tikzcd} \tag{26}$$

We then define the composite to be the equivalence class of $A \stackrel{s\circ t'}{\longleftarrow} A' \stackrel{g\circ f'}{\longrightarrow} C$. One checks that this composition rule indeed preserves equivalence classes.

Finally, define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ to be the identity on objects and sending each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ to the class of $A \stackrel{\mathrm{id}}{\leftarrow} A \stackrel{f}{\rightarrow} B$. This is indeed a functor because the composition of $A \stackrel{\mathrm{id}}{\leftarrow} A \stackrel{f}{\rightarrow} B$ and $B \stackrel{\mathrm{id}}{\leftarrow} B \stackrel{g}{\rightarrow} C$ is $A \stackrel{\mathrm{id}}{\leftarrow} A \stackrel{g\circ f}{\longrightarrow} B$, as can be seen by taking $f' = f$ and $t' = \mathrm{id}$ in the above diagram.

Remark 12.7. Property (3) of multiplicative systems is used during the verifications that the equivalence relation introduced above is transitive, and that composition of morphisms does not depend on choices of representatives. For instance, let us check that composition of morphisms as defined via diagram (26) does not depend on the choice of the object A' and the morphisms f', t'. Suppose we worked with a diagram

$$\begin{tikzcd} & A'' \arrow[r, "f''"] \arrow[d, "t''"'] & B_1 \arrow[r, "g"] \arrow[d, "t"] & C \\ A & A_1 \arrow[l, "s"'] \arrow[r, "f"] & B. & \end{tikzcd} \tag{27}$$

instead. Applying property (2) of multiplicative systems we get a commutative diagram

$$\begin{tikzcd} A''' \arrow[r, "u''"] \arrow[d, "u'"'] & A'' \arrow[d, "t''"] \\ A' \arrow[r, "t'"] & A_1. \end{tikzcd} \tag{28}$$

with $u' \in S$. The pair $A \stackrel{s\circ t'\circ u'}{\longleftarrow} A''' \stackrel{g\circ f''\circ u''}{\longrightarrow} B$ would induce an equivalence between $A \stackrel{s\circ t'}{\longleftarrow} A' \stackrel{g\circ f'}{\longrightarrow} C$ and $A \stackrel{s\circ t''}{\longleftarrow} A'' \stackrel{g\circ f''}{\longrightarrow} C$ if the diagram

$$\begin{tikzcd} A''' \arrow[r, "u''"] \arrow[d, "u'"'] & A'' \arrow[d, "f''"] \\ A' \arrow[r, "f'"] & B_1 \end{tikzcd} \tag{29}$$

were also commutative. This is not true in general. However, the composite maps $A''' \xrightarrow{u'} A' \xrightarrow{f'} B_1 \xrightarrow{t} B$ and $A''' \xrightarrow{u''} A'' \xrightarrow{f''} B_1 \xrightarrow{t} B$ are equal because the first is equal to $A''' \xrightarrow{u'} A' \xrightarrow{t'} A_1 \xrightarrow{f} B$ by diagram (26), which in turn equals $A''' \xrightarrow{u''} A'' \xrightarrow{t''} A_1 \xrightarrow{f} B$ by diagram (29). Finally, this composite map equals $A''' \xrightarrow{u''} A'' \xrightarrow{f''} B_1 \xrightarrow{t} B$ by diagram (27). Now property (3) of multiplicative systems implies that there is some map $u : A'''' \to A'''$ such that replacing u' and u'' by $u' \circ u$ and $u'' \circ u$ in diagram (29) it becomes commutative. Of course, diagram (28) remains commutative after this substitution.

Proposition 12.8 (Gabriel–Zisman). *Together with the functor Q the category $S^{-1}\mathcal{C}$ constructed above is the localization of $\mathcal{C}$ with respect to S.*

Proof. We check the properties in Proposition 12.1. Property (1) follows because the two-sided inverse of the class of $A \xleftarrow{\mathrm{id}} A \xrightarrow{s} B$ is represented by $B \xleftarrow{s} A \xrightarrow{\mathrm{id}} A$. Property (2) holds, because if $F : \mathcal{C} \to \mathcal{D}$ sends the morphisms in S to isomorphisms, we may factor it uniquely through $S^{-1}\mathcal{C}$ by sending the class of $A \xleftarrow{s} A_1 \xrightarrow{f} B$ to $F(f) \circ F(s)^{-1}$. That this construction respects equivalence classes follows from the definition of the equivalence relation. For it to define a functor $S^{-1}\mathcal{C} \to \mathcal{D}$ we have to check that the composition of $A \xleftarrow{s} A_1 \xrightarrow{f} B$ and $B \xleftarrow{t} B_1 \xrightarrow{g} C$ is sent to $F(g \circ f') \circ F(s \circ t')^{-1}$, with f', t' as in diagram (26). This is because applying F to the diagram implies

$$F(g) \circ F(t)^{-1} \circ F(f) \circ F(s)^{-1} = F(g) \circ F(f') \circ F(t')^{-1} \circ F(s)^{-1}$$

and F preserves composition. □

Remarks 12.9.
1. In the above constructions only 'half' of properties (2) and (3) were used. The other half can be used for defining localization via 'right fractions' represented by pairs of morphisms $A \xrightarrow{f} B_1 \xleftarrow{t} B$; the argument is similar. Since both left and right satisfy the defining universal property of localization, the resulting categories are isomorphic.
2. One can prove the stronger property that the Hom-sets in the category $S^{-1}\mathcal{C}$ arise as *filtered direct limits* of Hom-sets in $\mathcal{C}$.

Proposition 12.10. *If S is a multiplicative system of morphisms in an additive category $\mathcal{A}$, then $S^{-1}\mathcal{A}$ is also additive.*

Sketch of proof. Given two morphisms $A \xleftarrow{s} A_1 \xrightarrow{f} B$ and $A \xleftarrow{s'} A_1' \xrightarrow{f'} B$, we define their sum by introducing a 'common denominator'. Apply axiom (2) of multiplicative systems to find a commutative diagram

$$\begin{tikzcd} C \arrow[r, "r'"] \arrow[d, "r"'] & A_1 \arrow[d, "s"] \\ A_1' \arrow[r, "s'"] & A \end{tikzcd}$$

with $r \in S$. By axiom (1) $t := s' \circ r = s \circ r' \in S$ and we may represent the two morphisms above by the equivalent morphisms $A \xleftarrow{t} C \xrightarrow{f \circ r'} B$ and $A \xleftarrow{t} C \xrightarrow{f' \circ r} B$. Define their sum by the equivalence class of $A \xleftarrow{t} C \xrightarrow{f \circ r' + f' \circ r} B$. One checks that this definition is well posed and the axioms for an additive category hold (0 and $A \oplus B$ are the same objects as in A). $\square$

Now we apply the above to the derived category.

Proposition 12.11. *Let $\mathcal{A}$ be an abelian category and let $K(\mathcal{A})$ be the associated homotopy category. The collection of quasi-isomorphisms in $K(\mathcal{A})$ is a multiplicative system.*

Consequently, every morphism in $D(\mathcal{A})$ can be represented by a pair $A \xleftarrow{s} A_1 \xrightarrow{f} B$ with s a quasi-isomorphism and f a morphism in $K(\mathcal{A})$.

Proof of Proposition 12.11. We first check property (2) of multiplicative systems. Assume given morphisms $f : A^\bullet \to B^\bullet$ and $s : A^\bullet \to A'^\bullet$ in $K(\mathcal{A})$, with s a quasi-isomorphism. Using Lemma 11.10 (1) we have an exact triangle

$$C(s)[-1] \xrightarrow{g} A^\bullet \to A'^\bullet \to C(s)$$

which, using Lemma 11.10 (2), can be inserted in a commutative diagram of exact triangles

$$\begin{tikzcd} C(s)[-1] \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & A^\bullet \arrow[r, "s"] \arrow[d, "f"] & A'^\bullet \arrow[r] \arrow[d] & C(s) \arrow[d, "\mathrm{id}"] \\ C(s)[-1] \arrow[r, "f \circ g"] & B^\bullet \arrow[r] & C(f \circ g) \arrow[r] & C(s). \end{tikzcd}$$

We claim that the middle square of the diagram is the one we were looking for (in particular we may take $B' = C(f \circ g)$). For this we have to check that the map $B^\bullet \to C(f \circ g)$ in the lower triangle is a quasi-isomorphism. Since s is a quasi-isomorphism, the cone $C(s)$ is acyclic by Lemma 11.2, but then the long exact sequence associated with the lower triangle implies the claim. The proof for the other part of the square is similar.

Since $K(\mathcal{A})$ is an additive category, we can replace f by $f - g$ in property (3) and check it in the following equivalent form: there exists a morphism $s \in S$ with target

A such that $f \circ s = 0$ if and only if there exists $t \in S$ with source B such that $t \circ f = 0$. Suppose indeed that $t \circ f = 0$ for some $t : B \to C$ in S. We may insert it in the exact triangle

$$C(t)[-1] \xrightarrow{g} B^\bullet \xrightarrow{t} C^\bullet \to C(t)$$

as above. Since $t \circ f = 0$, by exactness of the sequence $\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, C(t)[-1]) \to \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, C^\bullet)$ (Corollary 11.11 (1)) we find $f' : A^\bullet \to C(t)[-1]$ with $g \circ f' = f$. Now insert f' in the exact triangle

$$C(f')[-1] \xrightarrow{s} A^\bullet \xrightarrow{f'} C(t)[-1] \to C(f').$$

Here $f' \circ s = 0$ by Lemma 11.9 (1), hence also $f \circ s = 0$. Moreover, since t is a quasi-isomorphism, the cone $C(t)$ is acyclic, hence s is also a quasi-isomorphism by the long exact sequence of the triangle above. The other implication is proven in a similar way. □

Corollary 12.12. *The derived category $D(\mathcal{A})$ is additive.*

Proof. This follows from the two previous propositions. □

For all $i \in \mathbf{Z}$ the functors $H^i : C(\mathcal{A}) \to \mathcal{A}$ given by $A^\bullet \mapsto H^i(A^\bullet), f \mapsto H^i(f)$ map quasi-isomorphisms to isomorphisms by definition, so by the universal property of localization induce functors $H^i : D(\mathcal{A}) \to \mathcal{A}$. Also, as in $K(\mathcal{A})$, define an *exact triangle in $D(\mathcal{A})$* to be a triangle isomorphic in $D(\mathcal{A})$ to a triangle of the form $A^\bullet \to B^\bullet \to C(f) \to A^\bullet[1]$. Then the statements of Lemmas 11.9 and 11.10 and their corollaries all hold for exact triangles in $D(\mathcal{A})$. But notice a new feature:

Corollary 12.13. *An exact sequence*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

in $C(\mathcal{A})$ gives rise to an exact triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

in $D(\mathcal{A})$.

Proof. This follows from Corollary 11.5. □

Remark 12.14. Given a morphism $A^\bullet \to B^\bullet$ in $D(\mathcal{A})$, we may insert it in an exact triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $D(\mathcal{A})$ as follows. First we represent the morphism by a pair $A^\bullet \xleftarrow{s} A_1^\bullet \xrightarrow{f} B^\bullet$ of morphisms in $K(\mathcal{A})$ with s a quasi-isomorphism. Setting $C^\bullet := C(f)$ we have an exact triangle $A_1^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A_1^\bullet[1]$ in $K(\mathcal{A})$, hence in $D(\mathcal{A})$, and we may construct a triangle $A^\bullet \to B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A^\bullet[1]$ isomorphic to it in $D(\mathcal{A})$ via s and the identity maps. Note that $C^\bullet$ is unique up to isomorphism in $D(\mathcal{A})$ but not up to unique isomorphism. One sometimes calls $C^\bullet$ *a cone* of the morphism $A^\bullet \to B^\bullet$ in $D(\mathcal{A})$.

Using the above remark we can verify:

Corollary 12.15. *A morphism $\phi: A^\bullet \to B^\bullet$ in $D(\mathcal{A})$ is an isomorphism in $D(\mathcal{A})$ if and only if it induces isomorphisms $H^i(\phi): H^i(A^\bullet) \to H^i(B^\bullet)$ for all i.*

Note that the corollary does *not* imply that ϕ comes from a quasi-isomorphism in $C(\mathcal{A})$!

Proof. The 'only if' part follows because the H^i are functors on $D(\mathcal{A})$. Assume now the $H^i(\phi)$ are all isomorphisms, and let $C^\bullet$ be a cone of ϕ as in the above remark. Then $H^i(C^\bullet) = 0$ for all i, and so the map $0 \to C^\bullet$ in $C(\mathcal{A})$ induces an isomorphism in $D(\mathcal{A})$. In the commutative diagram of exact triangles

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{\text{id}} & A^\bullet & \longrightarrow & 0 & \longrightarrow & A^\bullet[1] \\
{\scriptstyle \text{id}}\big\downarrow & & {\scriptstyle \phi}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \text{id}} \\
A^\bullet & \xrightarrow{\phi} & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A^\bullet[1]
\end{array}$$

three of the vertical maps are then isomorphisms, and hence so is ϕ by the derived category version of Corollary 11.11 (2). $\square$

Now that we have introduced the derived category, we can define Ext-groups in a general abelian category.

Definition 12.16. *If $\mathcal{A}$ is an abelian category and $A^\bullet$, $B^\bullet$ two objects in $C(\mathcal{A})$, we define*

$$\operatorname{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) := \operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]).$$

We shall verify in Proposition 14.12 below that in the case of one-term complexes this definition agrees with the earlier derived functor definition; in this case we simplify the notation to $\operatorname{Ext}^i_{\mathcal{A}}(A, B)$, with A, B two objects of $\mathcal{A}$. What we can do right now is to interpret elements in $\operatorname{Ext}^i_{\mathcal{A}}(A, B)$ as classes of i-fold extensions.

Construction 12.17 (Yoneda extensions)**.** Let $\mathcal{A}$ be an abelian category, and A, B two objects of $\mathcal{A}$. An *i-fold (Yoneda) extension* of A by B is an exact sequence

$$0 \to B \to E^{-i+1} \to E^{-i+2} \to \cdots \to E^0 \to A \to 0 \tag{30}$$

of objects in $\mathcal{A}$. Two extensions $0 \to B \to E^{-i+1} \to E^{-i+2} \to \cdots \to E^0 \to A \to 0$ and $0 \to B \to E'^{-i+1} \to E'^{-i+2} \to \cdots \to E'^0 \to A \to 0$ are Yoneda equivalent if there is a third extension $0 \to B \to E''^{-i+1} \to E''^{-i+2} \to \cdots \to E''^0 \to A \to 0$ fitting into a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & B & \longrightarrow & E^{-i+1} & \longrightarrow & \cdots & \longrightarrow & E^0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\big\uparrow & & \big\uparrow & & & & \big\uparrow & & \big\uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & B & \longrightarrow & E''^{-i+1} & \longrightarrow & \cdots & \longrightarrow & E''^0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & B & \longrightarrow & E'^{-i+1} & \longrightarrow & \cdots & \longrightarrow & E'^0 & \longrightarrow & A & \longrightarrow & 0.
\end{array}$$

In the case $i = 1$ this gives back the earlier definition of equivalence of extensions. It is not hard to check that this is indeed an equivalence relation but it also follows from the lemma below.

We now associate with an i-fold extension E as in (30) an element $c(E) \in \mathrm{Ext}^i_{\mathcal{A}}(A, B)$ as follows. The exact sequence (30) defines a quasi-isomorphism of complexes $s : [0 \to B \to E^{-i+1} \to E^{-i+2} \to \cdots \to E^0 \to 0] \to A$. The natural projection gives a map $f : [0 \to B \to E^{-i+1} \to E^{-i+2} \to \cdots \to E^0 \to 0] \to B[i]$ and the pair (s, f) represents a morphism $c(E) \in \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])$.

Note, in particular, that the above construction gives rise to nonzero morphisms $A \to B[i]$ in $D(\mathcal{A})$, whereas there is no such morphism in $C(\mathcal{A})$.

Proposition 12.18. *Let A and B be as above.*

(1) *Two i-fold extensions E and E' of A by B are Yoneda equivalent if and only if $c(E) = c(E')$.*
(2) *Each morphism $\phi \in \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])$ is of the form $c(E)$ for some i-fold extension E.*

To sum up, we have:

Corollary 12.19. *The assignment $E \mapsto c(E)$ induces a bijection between equivalence classes of i-fold extensions and elements in $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$.*

For the proof we use the all-important truncation functors, defined as follows.

Construction 12.20. Given a complex $A^\bullet$ in $C(\mathcal{A})$ and an integer $n \in \mathbf{Z}$, define the *(canonical) truncations* in degree n by

$$\tau_{\leq n}(A^\bullet) := [\cdots \to A^i \to A^{i+1} \to \cdots \to A^{n-1} \to Z^n(A^\bullet) \to 0 \to 0 \to \cdots]$$

and

$$\tau_{\geq n}(A^\bullet) := [\cdots \to 0 \to 0 \to A^n/B^n(A^\bullet) \to A^{n+1} \to \cdots \to A^i \to A^{i+1} \to \cdots]$$

By definition, there are natural morphisms of complexes $\tau_{\leq n}(A^\bullet) \to A^\bullet$ and $A^\bullet \to \tau_{\geq n}(A^\bullet)$ which are quasi-isomorphisms in degrees $\leq n$ and $\geq n$, respectively, and zero maps elsewhere (notice that $H^n(\tau_{\leq n}(A^\bullet)) = H^n(\tau_{\geq n}(A^\bullet) = H^n(A^\bullet).)$ Also, given a quasi-isomorphism $A^\bullet \to B^\bullet$, the induced maps $\tau_{\leq n}(A^\bullet) \to \tau_{\leq n}(B^\bullet)$ and $\tau_{\geq n}(A^\bullet) \to \tau_{\geq n}(B^\bullet)$ are quasi-isomorphisms as well. Thus $\tau_{\leq n}$ and $\tau_{\geq n}$ induce functors $D(\mathcal{A}) \to D(\mathcal{A})$.

For later use, note that for each n we have an exact sequence of complexes

$$0 \to \tau_{\leq n-1}(A^\bullet) \to \tau_{\leq n}(A^\bullet) \to [A^{n-1}/Z^{n-1}(A^\bullet) \to Z^n(A^\bullet)] \to 0$$

where the last complex is concentrated in degrees $n-1$ and n. The natural morphism of complexes

$$[A^{n-1}/Z^{n-1}(A^\bullet) \to Z^n(A^\bullet)] \to [0 \to H^n(A^\bullet)]$$

(where the second complex is placed in the same degrees) is a quasi-isomorphism. Thus using Corollary 12.13 we have an exact triangle

$$\tau_{\leq n-1}(A^\bullet) \to \tau_{\leq n}(A^\bullet) \to H^n(A^\bullet)[-n] \to \tau_{\leq n-1}(A^\bullet)[1] \tag{31}$$

in $D(\mathcal{A})$, where $H^n(A^\bullet)$ is considered as a complex concentrated in degree 0.

Similarly, there is an exact triangle

$$H^n(A^\bullet)[-n] \to \tau_{\geq n}(A^\bullet) \to \tau_{\geq n+1}(A^\bullet) \to H^n(A^\bullet)[-n+1]. \tag{32}$$

coming from the exact sequence of complexes

$$0 \to [A^n/B^n(A^\bullet) \to B^{n+1}(A^\bullet)] \to \tau_{\geq n}(A^\bullet) \to \tau_{\geq n+1}(A^\bullet) \to 0.$$

These exact triangles are very useful in inductive arguments on complexes.

Proof of Proposition 12.18. We first prove (2). Represent ϕ by a pair $A \overset{s}{\leftarrow} L^\bullet \overset{f}{\to} B[i]$, where s is a quasi-isomorphism. Replacing $L^\bullet$ by $\tau_{\geq -i-1}\tau_{\leq 0}L^\bullet$ we may assume $L^i = 0$ for $i < -i-1$ and $i > 0$. Now take the pushout of the extension $0 \to L^{-i-1} \to L^{-i} \to B^{-i+1}L \to 0$ by the composite map $L^{-i-1} \to L^{-i} \overset{f}{\to} B$, obtaining an extension $0 \to B \to L'^{-i} \to B^{-i+1}L \to 0$. Replacing $L^\bullet$ in degrees $(-i-1, -i)$ by the map $B \to L'^{-i}$ and leaving the other maps unchanged gives an i-fold extension

$$0 \to B \to L'^{-i+1} \to L^{-i+2} \to \cdots \to L^0 \overset{s}{\to} A \to 0$$

of A by B with class ϕ.

It remains to prove (1). The 'only if' part being obvious, we assume E and E' are two extensions with $c(E) = c(E')$. Represent $c(E)$ and $c(E')$ by pairs $A \overset{s}{\leftarrow} E^\bullet \overset{f}{\to} B[i]$ and $A \overset{s'}{\leftarrow} E'^\bullet \overset{f'}{\to} B[i]$ as in the construction above. By the definition of morphisms in $D(\mathcal{A})$, these two are equal if there is a pair $A \overset{t}{\leftarrow} L^\bullet \overset{g}{\to} B[i]$ equipped with morphisms $E^\bullet \leftarrow L^\bullet \to E'^\bullet$ compatible with the maps to A and $B[i]$. Truncating and modifying $L^\bullet$ as in the previous paragraph we obtain an i-fold extension of A by B equipped with maps to E and E' inducing a Yoneda equivalence. $\square$

Remark 12.21. It follows from the definition that there are natural product maps $\mathrm{Ext}^i_{\mathcal{A}}(A, B) \times \mathrm{Ext}^j_{\mathcal{A}}(B, C) \to \mathrm{Ext}^{i+j}_{\mathcal{A}}(A, C)$ induced by composing morphisms in $D(\mathcal{A})$. Via the Yoneda construction they correspond to splicing i-fold and j-fold extensions together.

13. Description in the presence of enough projectives

Now consider full subcategories of $C(\mathcal{A})$ defined as follows: $C^+(\mathcal{A})$ is the full subcategory spanned by objects $A^\bullet$ such that $A^i = 0$ for all $i \ll 0$. Similarly, $C^-(\mathcal{A})$ is spanned by objects $A^\bullet$ with $A^i = 0$ for all $i \gg 0$ and $C^b(\mathcal{A})$ is spanned by objects with $A^i \neq 0$ for all but finitely many i. Denote their respective essential images in $K(\mathcal{A})$ and $D(\mathcal{A})$ by $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ as well as $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$. Here the essential image of a functor $F : \mathcal{C} \to \mathcal{D}$ is defined as the full subcategory of $\mathcal{D}$ spanned by objects D isomorphic in $\mathcal{D}$ to some $F(C)$ with C an object of $\mathcal{C}$; we apply this notion with F the natural functors $C(\mathcal{A}) \to K(\mathcal{A})$ and $C(\mathcal{A}) \to D(\mathcal{A})$.

Lemma 13.1. *The category $D^+(\mathcal{A})$ is the full subcategory of $D(\mathcal{A})$ spanned by objects $A^\bullet$ such that $H^i(A^\bullet) = 0$ for all $i \ll 0$; similar statements hold for $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$.*

Proof. If $H^i(A^\bullet) = 0$ for $i < n$, then the natural morphism $A^\bullet \to \tau_{\geq n}(A^\bullet)$ is an isomorphism in $D(\mathcal{A})$, and by definition $\tau_{\geq n}(A^\bullet)$ is in the essential image of $C^+(\mathcal{A})$ in $D(\mathcal{A})$. The other proofs are similar. □

Recall now that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ as functors. Here F is called a *quasi-inverse* for G and vice versa.

Lemma 13.2. *The category $D^+(\mathcal{A})$ is equivalent to the localization of $K^+(\mathcal{A})$ with respect to the collection of quasi-isomorphisms in $K^+(\mathcal{A})$. Similar statements hold for $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$.*

Proof. Denote by S_+ the collection of quasi-isomorphisms in $K^+(\mathcal{A})$. The natural functor $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ maps the elements in S_+ to isomorphisms in $D(\mathcal{A})$, hence factors through a functor $S_+^{-1}K^+(\mathcal{A}) \to D^+(\mathcal{A})$. We construct a quasi-inverse as follows. For every object $A^\bullet$ in $D^+(A)$ the natural map $\phi_A : A^\bullet \to \tau_{\geq n}(A^\bullet)$ is an isomorphism in $D(\mathcal{A})$ for suitable n; we fix such an isomorphism ϕ_A for each $A^\bullet$. Given a morphism $\phi : A^\bullet \to B^\bullet$ in $D^+(A)$, the composite $\phi_B \circ \phi \circ \phi_A^{-1}$ can be represented by a pair $\tau_{\geq n}(A^\bullet) \xrightarrow{f} B_1^\bullet \xleftarrow{t} \tau_{\geq m}(B^\bullet)$ of morphisms in $K(\mathcal{A})$ by Remark 12.9 (1). Since t is a quasi-isomorphism here, the canonical map $t_1 : B_1^\bullet \to \tau_{\geq m}(B_1^\bullet)$ must be a quasi-isomorphism as well. So the pair $\tau_{\geq n}(A^\bullet) \xrightarrow{t_1 \circ f} \tau_{\geq m}(B_1)^\bullet \xleftarrow{t_1 \circ t} \tau_{\geq m}(B^\bullet)$ represents the same morphism in $D(\mathcal{A})$ but in fact represents a morphism in $S_+^{-1}K^+(\mathcal{A})$. Sending $A^\bullet$ to $\tau_{\geq n}(A^\bullet)$ and ϕ to the above morphism gives the required quasi-inverse. □

Assume now $\mathcal{A}$ has enough projectives and define $K^-(\mathcal{P})$ to be the full subcategory of $K^-(\mathcal{A})$ spanned by complexes with projective terms. Similarly, if $\mathcal{A}$ has enough injectives, define $K^+(\mathcal{I})$ to be the full subcategory of $K^+(\mathcal{A})$ spanned by complexes with injective terms.

Proposition 13.3. *If $\mathcal{A}$ has enough projectives, the composite functor*

$$K^-(\mathcal{P}) \to K^-(\mathcal{A}) \stackrel{Q}{\to} D^-(\mathcal{A})$$

induces an equivalence of categories between $K^-(\mathcal{P})$ and $D^-(\mathcal{A})$.

Similarly, if $\mathcal{A}$ has enough injectives, we have an equivalence of categories between $K^+(\mathcal{I})$ and $D^+(\mathcal{A})$.

We need a lemma.

Lemma 13.4. *Assume $\mathcal{A}$ has enough projectives (resp. injectives).*

(1) *Every complex $C^\bullet$ in $C^-(\mathcal{A})$ is quasi-isomorphic to a complex $P^\bullet$ with projective terms.*

Similarly, every complex in $C^+(\mathcal{A})$ is quasi-isomorphic to a complex $I^\bullet$ with injective terms.

(2) *An acyclic complex in $C^-(\mathcal{A})$ with projective terms is homotopically trivial, and so is an acyclic complex in $C^+(\mathcal{A})$ with injective terms.*

Proof. We postpone the proof of (1) to the end of this section. As for (2) in the projective case, notice that an acyclic complex $A^\bullet$ breaks up in short exact sequences $0 \to B^i(A^\bullet) \to A^i \to B^{i+1}(A^\bullet) \to 0$ for all i. If n is the largest index for which $A^n \neq 0$, we have $A^n = B^n(A^\bullet)$; in particular, $B^n(A^\bullet)$ is projective and the above sequence for $i = n-1$ splits as a direct sum $A^{n-1} \cong B^{n-1}(A^\bullet) \oplus B^n(A^\bullet)$. But then $B^{n-1}(A^\bullet)$ is also projective, so continuing inductively we have decompositions $A^i \cong B^i(A^\bullet) \oplus B^{i+1}(A^\bullet)$ for all i. Now $\begin{bmatrix} 0 & 0 \\ \mathrm{id}_{B^i} & 0 \end{bmatrix} : B^i(A^\bullet) \oplus B^{i+1}(A^\bullet) \to B^{i-1}(A^\bullet) \oplus B^i(A^\bullet)$ induces the required homotopy $A^i \to A^{i-1}$ between $\mathrm{id}_{A^\bullet}$ and 0. The proof in the injective case is similar. □

Corollary 13.5. *Every quasi-isomorphism in $K^-(\mathcal{P})$ (or $K^+(\mathcal{I})$) is an isomorphism.*

Proof. Let $s : P_1^\bullet \to P_2^\bullet$ be a quasi-isomorphism in $K^-(\mathcal{P})$. By Lemma 11.2 (1) the cone $C(s)$ is acyclic, hence homotopically trivial by part (2) of the above lemma. We conclude by Corollary 11.12. □

Proof of Proposition 13.3. We do the case of $K^-(\mathcal{P})$. If $S^{-1}K^-(\mathcal{P})$ denotes its localization by the collection of quasi-isomorphisms in $K^-(\mathcal{P})$, the canonical functor $K^-(\mathcal{P}) \to S^{-1}K^-(\mathcal{P})$ is an isomorphism by the previous corollary, so since the functor $K^-(\mathcal{P}) \to D^-(\mathcal{A})$ of the proposition factors through $S^{-1}K^-(\mathcal{P})$, it will be enough to construct a quasi-inverse for the induced functor $S^{-1}K^-(\mathcal{P}) \to D^-(\mathcal{A})$.

The method is the same as in the proof of Lemma 13.2. For each object $A^\bullet$ in $D^-(A)$ fix an isomorphism $\phi_A : P_A^\bullet \xrightarrow{\sim} A^\bullet$ in $D^-(\mathcal{A})$ with an object of $D^-(\mathcal{P})$; this is possible by Lemma 13.4 (1). Given a morphism $\rho : A^\bullet \to B^\bullet$ in $D^-(\mathcal{A})$, the composite $\phi_B^{-1} \circ \rho \circ \phi_A$ is a morphism $P_A^\bullet \to P_B^\bullet$ in $D^-(\mathcal{A})$. Thus we may represent $\phi_B^{-1} \circ \rho \circ \phi_A$ by a pair $P_A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} P_B^\bullet$ in $K^-(\mathcal{A})$, with s a quasi-isomorphism. Use again Lemma 13.4 (1) to find a quasi-isomorphism $t : P_C^\bullet \to C^\bullet$ where $P_C^\bullet$ is an object in $K^-(\mathcal{P})$. Then $P_A^\bullet \xleftarrow{s\circ t} P_C^\bullet \xrightarrow{f\circ t} P_B^\bullet$ still represents $\phi_B^{-1} \circ \rho \circ \phi_A$ in $D^-(\mathcal{A})$ but $s \circ t$, $f \circ t$ are now morphisms in $K^-(\mathcal{P})$, so we have in fact a morphism in $S^{-1}K^-(\mathcal{P})$; note that it does not depend on the choice of t. Now the required quasi-inverse is defined by $A^\bullet \mapsto P_A^\bullet$, $\rho \mapsto (s \circ t, f \circ t)$. □

It remains to prove Lemma 13.4 (1). It could be done via a direct construction, but we prefer to introduce a general technique that will also serve later.

Definition 13.6. A *double complex* $A^{\bullet,\bullet}$ in an abelian category $\mathcal{A}$ is a system of objects $A^{i,j}$ indexed by $\mathbf{Z} \times \mathbf{Z}$ together with morphisms $d_h^{i,j} : A^{i,j} \to A^{i+1,j}$ (*horizontal* differentials) and $d_v^{i,j} : A^{i,j} \to A^{i,j+1}$ (*vertical* differentials) satisfying

$$d_h^{i+1,j} \circ d_h^{i,j} = 0, \quad d_v^{i,j+1} \circ d_v^{i,j} = 0, \quad d_v^{i+1,j} \circ d_h^{i,j} + d_h^{i,j+1} \circ d_v^{i,j} = 0 \tag{33}$$

for all i, j.

A *morphism* of double complexes $\varphi : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is a family of morphisms $\varphi^{i,j} : A^{i,j} \to B^{i,j}$ for all pairs (i, j) compatible with the horizontal and vertical differentials.

A picture of a double complex looks like:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{i,j+2} & \xrightarrow{d_h^{i,j+2}} & A^{i+1,j+2} & \xrightarrow{d_h^{i+1,j+2}} & A^{i+2,j+2} & \longrightarrow & \cdots \\
 & & {\scriptstyle d_v^{i,j+1}}\uparrow & & {\scriptstyle d_v^{i+1,j+1}}\uparrow & & {\scriptstyle d_v^{i+2,j+1}}\uparrow & & \\
\cdots & \longrightarrow & A^{i,j+1} & \xrightarrow{d_h^{i,j+1}} & A^{i+1,j+1} & \xrightarrow{d_h^{i+1,j+1}} & A^{i+2,j+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle d_v^{i,j}}\uparrow & & {\scriptstyle d_v^{i+1,j}}\uparrow & & {\scriptstyle d_v^{i+2,j}}\uparrow & & \\
\cdots & \longrightarrow & A^{i,j} & \xrightarrow{d_h^{i,j}} & A^{i+1,j} & \xrightarrow{d_h^{i+1,j}} & A^{i+2,j} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Construction 13.7. A double complex $A^{\bullet,\bullet}$ is called *biregular* if for all $n \in \mathbf{Z}$ the set $\{(i,j) : i+j=n \text{ and } A^{i,j} \neq 0\}$ is finite. This is the case, for example, if there exists $k \in \mathbf{Z}$ such that $A^{i,j} = 0$ for $i < k, j < k$ or if $A^{i,j} = 0$ for $i > k, j > k$.

Given a biregular double complex $A^{\bullet,\bullet}$, we define the *associated simple complex* $sA^\bullet$ by setting

$$sA^n := \bigoplus_{i+j=n} A^{i,j}$$

and $d^n : sA^n \to sA^{n+1}$ given in the (i,j)-component by $d_h^{i,j} + d_v^{i,j}$. The formulas (33) ensure that this is indeed a complex.

A morphism $\varphi : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ of double complexes induces a morphism $s\varphi : sA^\bullet \to sB^\bullet$ of associated simple complexes. It is defined in degree n by the direct sum of the maps $\varphi^{i,j}$. Thus s is a functor from the category of biregular double complexes to that of simple complexes. This functor is exact because a direct sum of exact sequences is exact (exact sequences of double complexes are defined term by term, as for usual complexes).

Proposition 13.8. *Let $\varphi : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ be a morphism of biregular double complexes. If φ induces quasi-isomorphisms $A^{\bullet,j} \to B^{\bullet,j}$ for each row, then $s\varphi : sA^\bullet \to sB^\bullet$ is a quasi-isomorphism.*

Same conclusion if φ induces quasi-isomorphisms $A^{i,\bullet} \to B^{i,\bullet}$ for each column.

Proof. We do the case of columns. We have to show that the maps $H^n(s\varphi) : H^n(sA^\bullet) \to H^n(sB^\bullet)$ are isomorphisms for all n. Since the double complexes are biregular, only finitely many $A^{i,j}$ and $B^{i,j}$ contribute to $H^n(sA^\bullet)$ and $H^n(sB^\bullet)$ for fixed n. So we may assume $A^{i,j} = B^{i,j} = 0$ except for finitely many pairs i,j by setting the terms in the uninteresting range to 0. In particular, we may assume that $A^{i,\bullet} = B^{i,\bullet} = 0$ for i outside an interval $[a, b] \subset \mathbf{Z}$. Thus the morphism φ can be represented by the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A^{a,\bullet} & \longrightarrow & A^{a+1,\bullet} & \longrightarrow & \cdots & \longrightarrow & A^{b,\bullet} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & B^{a,\bullet} & \longrightarrow & B^{a+1,\bullet} & \longrightarrow & \cdots & \longrightarrow & B^{b,\bullet} & \longrightarrow & 0 \end{array} \tag{34}$$

We proceed by induction on $b - a$. If $b - a = 0$, then $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ are both concentrated in a single column and the assertion holds by assumption. Now assume we have proven the cases with $b - a < n$. Introduce the notation $A^{\geq a+1,\bullet}$ for the double complex obtained from $A^{\bullet,\bullet}$ by setting $A^{i,j} = 0$ for $i < a + 1$, and similarly for $B^{\geq a+1,\bullet}$. We have a commutative diagram of morphisms of double complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{a,\bullet} & \longrightarrow & A^{\bullet,\bullet} & \longrightarrow & A^{\geq a+1,\bullet} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^{a,\bullet} & \longrightarrow & B^{\bullet,\bullet} & \longrightarrow & B^{\geq a+1,\bullet} & \longrightarrow & 0 \end{array}$$

where on the left hand side we have the a-th columns of $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ considered as double complexes with 0's elsewhere. Applying the exact functor s we obtain a commutative diagram of morphisms of simple complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{a,\bullet} & \longrightarrow & sA^{\bullet,\bullet} & \longrightarrow & sA^{\geq a+1,\bullet} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^{a,\bullet} & \longrightarrow & sB^{\bullet,\bullet} & \longrightarrow & sB^{\geq a+1,\bullet} & \longrightarrow & 0 \end{array}$$

Here the first vertical map is a quasi-isomorphism by assumption and the last one by induction. Hence so is the middle one by Corollary 3.7. □

Corollary 13.9. *If $A^{\bullet,\bullet}$ is a biregular double complex whose rows (resp. columns) are acyclic, then $sA^{\bullet,\bullet}$ is acyclic.*

Proof. Apply the proposition with $B^{\bullet,\bullet} = 0$. □

Remark 13.10. There is a variant of the proposition which is often useful. Denote the cohomology groups of the j-th row of $A^{\bullet,\bullet}$ by $H_h^{i,j}(A^{\bullet,\bullet})$ for all i and the cohomology groups of the i-th column of $A^{\bullet,\bullet}$ by $H_v^{i,j}(A^{\bullet,\bullet})$ for all j. The identity $d_v^{i+1,j} \circ d_h^{i,j} + d_h^{i,j+1} \circ d_v^{i,j} = 0$ implies that the differentials 'in the other direction' induce complexes

$$\cdots \to H_h^{i,j-1}(A^{\bullet,\bullet}) \to H_h^{i,j}(A^{\bullet,\bullet}) \to H_h^{i,j+1}(A^{\bullet,\bullet}) \to \cdots$$

and

$$\cdots \to H_v^{i-1,j}(A^{\bullet,\bullet}) \to H_v^{i,j}(A^{\bullet,\bullet}) \to H_v^{i+1,j}(A^{\bullet,\bullet}) \to \cdots$$

called the i-th column and j-th row of cohomology, respectively. Now the variant states: if $\varphi: A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ induces quasi-isomorphisms on each row (or column) of cohomology, then $s\varphi: sA^\bullet \to sB^\bullet$ is a quasi-isomorphism. The proof is similar to the above, using canonical truncations $\tau_{\geq a+1,\bullet}(A^{\bullet,\bullet})$ of rows, instead of the 'stupid' ones used above.

Proof of Lemma 13.4 (1). We may assume $A^i = 0$ for $i > 0$. We shall construct a double complex $P^{\bullet,\bullet}$ with $P^{i,j} = 0$ for $i > 0$ or $j > 0$ whose terms are projective and is equipped with a morphism of double complexes $P^{\bullet,\bullet} \to A^\bullet$, where $A^\bullet$ is considered as a double complex with a single nonzero row. This morphism will induce a quasi-isomorphism on columns, i.e. each column of cohomology of $P^{\bullet,\bullet}$ will give a projective resolution of A^i. Proposition 13.8, together with the fact that finite direct sums of projectives are projective, will then show that the induced morphism $sP^{\bullet,\bullet} \to A^\bullet$ is a quasi-isomorphism we were looking for.

To construct $P^{\bullet,\bullet}$ we revert to homological indexing. For each i consider the exact sequence

$$0 \to B_i(A_\bullet) \to Z_i(A_\bullet) \to H_i(A_\bullet) \to 0.$$

Choose projective resolutions $P^B_{i,\bullet} \to B_i(A_\bullet)$ and $P^H_{i,\bullet} \to H_i(A_\bullet)$, respectively. By Lemma 4.4 there is a projective resolution $P^Z_{i,\bullet} \to Z_i(A_\bullet)$ fitting in an exact sequence of complexes

$$0 \to P^B_{i,\bullet} \to P^Z_{i,\bullet} \to P^H_{i,\bullet} \to 0.$$

Now repeat the procedure with the exact sequence

$$0 \to Z_i(A_\bullet) \to A_i \to B_{i-1}(A_\bullet) \to 0.$$

and the projective resolutions $P^B_{i-1,\bullet} \to B_{i-1}(A_\bullet)$ and $P^Z_{i,\bullet} \to Z_i(A_\bullet)$. It gives a projective resolution $P^Z_{i,\bullet} \to A_i$ fitting in an exact sequence of complexes

$$0 \to P^Z_{i,\bullet} \to P_{i,\bullet} \to P^B_{i-1,\bullet} \to 0.$$

Now construct a double complex $P_{\bullet,\bullet}$ out of the $P_{i,\bullet}$ with horizontal differentials induced by the composite maps

$$P_{i,j} \to P^B_{i-1,j} \to P^Z_{i-1,j} \to P_{i-1,j}$$

coming from the above diagrams and vertical maps those of the $P_{i,\bullet}$ multiplied by $(-1)^i$. With this sign rule $P_{\bullet,\bullet}$ becomes a double complex and by construction there is a morphism $P_{\bullet,\bullet} \to A_\bullet$ with the required property. The proof in the injective case is similar. $\square$

Remark 13.11. The proof shows that $P_{\bullet,\bullet}$ satisfies much more than stated in the beginning: it induces projective resolutions of each A_i, $Z_i(A_\bullet)$, $B_i(A_\bullet)$ and $H_i(A_\bullet)$ as well. Such double complexes are called *Cartan–Eilenberg resolutions.*

14. Total derived functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. It then extends uniquely to an additive functor $C(\mathcal{A}) \to C(\mathcal{B})$ and also to a functor $K(\mathcal{A}) \to K(\mathcal{B})$ on the associated homotopy categories (any homotopy between maps f and g induces a homotopy between $F(f)$ and $F(g)$). Moreover, $F : K(\mathcal{A}) \to K(\mathcal{B})$ is a *triangulated functor*, i.e. it sends exact triangles to exact triangles (this property is the natural analogue of exactness for functors between homotopy or derived categories). Now consider the following question: does there exist a triangulated functor $D(\mathcal{A}) \to D(\mathcal{B})$ making the diagram

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ {\scriptstyle F}\big\downarrow & & \\ K(\mathcal{B}) & \xrightarrow{Q} & D(\mathcal{B}) \end{array}$$

commute?

When $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, the answer is yes. Indeed, notice first that if $A^\bullet$ is an acyclic complex in $C(\mathcal{A})$, then $F(A^\bullet)$ is also acyclic (because F preserves exactness of the short exact sequences $0 \to Z^i(A^\bullet) \to A^i \to Z^{i+1}(A^\bullet) \to 0$), so by Lemma 11.2 (2) F preserves quasi-isomorphisms. But then the composite functor $K(\mathcal{A}) \xrightarrow{F} K(\mathcal{B}) \xrightarrow{Q} D(\mathcal{B})$ factors through $D(\mathcal{A})$ by the universal property of localization and the resulting functor is triangulated because F and Q are.

In general such an extension to $D(\mathcal{A})$ does not exist and even in good cases one has to restrict to the full subcategories $D^+(\mathcal{A})$ or $D^-(A)$. Moreover, we shall have to weaken the requirement of commutativity for the diagram above. Here is the formal definition.

Definition 14.1. Let $F : K^-(\mathcal{A}) \to K^-(\mathcal{B})$ be a triangulated functor. A *left derived functor* for F is a triangulated functor $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ together with a morphism of functors $\varepsilon : LF \circ Q \to Q \circ F$ that is universal in the following sense: For every pair (G, η) with $G : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ a triangulated functor and $\eta : G \circ Q \to Q \circ F$ there is a unique morphism of functors $\alpha : G \to LF$ with $\eta = \varepsilon \circ \alpha \circ Q$.

Similarly, if $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ is a triangulated functor, a *right derived functor* for F is a triangulated functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ together with a morphism of functors $\varepsilon : Q \circ F \to RF \circ Q$ such that for every pair (G, η) with $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ a triangulated functor and $\eta : Q \circ F \to G \circ Q$ there is a unique morphism of functors $\alpha : RF \to G$ with $\eta = \varepsilon \circ \alpha \circ Q$.

Of course, when LF or RF exists, it is unique up to unique isomorphism. In the case where F comes from an additive functor $\mathcal{A} \to \mathcal{B}$, one sometimes calls LF and RF *total derived functors* of F.

Proposition 14.2. *If $\mathcal{A}$ has enough projectives (resp. injectives), the left (resp. right) derived functors of F exist.*

Proof. We do the case of LF. Let $R : D^-(A) \to K^-(\mathcal{P})$ be a quasi-inverse to the functor of Proposition 13.3, and set $LF := Q \circ F|_{K^-(\mathcal{P})} \circ R$. To define ε, pick $A^\bullet \in K^-(\mathcal{A})$ and take the quasi-isomorphism $\phi_A : P^\bullet \to A^\bullet$ in $K^-(\mathcal{A})$ used in the construction of R, where $P^\bullet$ has projective terms. Now ϕ_A induces a morphism $F(\phi_A) : F(P^\bullet) \to F(A^\bullet)$; applying Q we have the required morphism $\varepsilon_A : (LF \circ Q)(A^\bullet) \to (Q \circ F)(A^\bullet)$. The definition of ε on morphisms is similar, using the induced morphisms in $K^-(\mathcal{P})$ constructed in the proof of Proposition 13.3.

To construct $\alpha : G \to LF$, consider again the above $A^\bullet$ and ϕ_A. The inverse of $(G \circ Q)(\phi_A)$ in $D^-(A)$ induces an isomorphism $(G \circ Q)(A^\bullet) \xrightarrow{\sim} (G \circ Q)(P^\bullet)$ which we may compose with $(G \circ Q)(P^\bullet) \xrightarrow{\eta} (Q \circ F)(P^\bullet) = LF(Q(A^\bullet))$. This defines α on objects; the definition on morphisms is left to the reader.

We still have to check that LF is a triangulated functor. Suppose

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1] \tag{35}$$

is an exact triangle in $D^-(\mathcal{A})$. Then

$$R(A^\bullet) \xrightarrow{g} R(B^\bullet) \to R(C^\bullet) \to R(A^\bullet)[1] \tag{36}$$

is a triangle in $K^-(\mathcal{P})$ isomorphic in $D^-(A)$ to the previous one. Moreover, it is also isomorphic in $D^-(\mathcal{A})$ to the triangle

$$R(A^\bullet) \to R(B^\bullet) \to C(g) \to R(A^\bullet)[1]$$

by the version of Lemma 11.10 (2) and Corollary 11.11 (2) for derived categories. But these last two triangles have terms in $K^-(\mathcal{P})$, so they are also isomorphic in $K^-(\mathcal{P})$ by Corollary 13.5. In particular, (36) is an exact triangle in $K^-(\mathcal{A})$, and hence applying the triangulated functor F to it we obtain an exact triangle in $K^-(\mathcal{B})$. Since Q is a triangulated functor by definition, we have proven that LF maps (35) to an exact triangle. □

In the course of the above proof we have shown:

Corollary 14.3. *When $\mathcal{A}$ has enough projectives, for each object $A^\bullet \in K(\mathcal{A})$ we have $LF(A^\bullet) \cong F(P^\bullet)$, where $P^\bullet \to A^\bullet$ is a quasi-isomorphism and $P^\bullet$ has projective terms.*

Similarly, when A has enough injectives, for each object $A^\bullet \in K(\mathcal{A})$ we have $RF(A^\bullet) \cong F(Q^\bullet)$, where $A^\bullet \to Q^\bullet$ is a quasi-isomorphism and $Q^\bullet$ has injective terms.

Note that by construction if we choose two quasi-isomorphisms $P^\bullet \to A^\bullet \leftarrow P'^\bullet$ in the corollary above, $P^\bullet$ and $P'^\bullet$ will be homotopy equivalent and hence so will be $F(P^\bullet)$ and $F(P'^\bullet)$, giving isomorphic objects in $D(\mathcal{B})$. This independence of $P^\bullet$ was built in the construction.

Definition 14.4. For $i \in \mathbf{Z}$ define the *i-th left (resp. right) derived functor* of F by $L_iF := H^{-i} \circ LF$ (resp. $R^iF := H^i \circ RF$) assuming that LF or RF exists.

Since LF and RF are triangulated functors, we have:

Corollary 14.5. *Given an exact triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $D^-(\mathcal{A})$, there is a long exact sequence*

$$\cdots \to L_iF(A^\bullet) \to L_iF(B^\bullet) \to L_iF(C^\bullet) \to L_{i-1}F(A^\bullet) \to \cdots$$

Similarly, for an exact triangle in $D^+(\mathcal{A})$ there is a long exact sequence

$$\cdots \to R^iF(A^\bullet) \to R^iF(B^\bullet) \to R^iF(C^\bullet) \to R^{i+1}F(A^\bullet) \to \cdots$$

Remark 14.6. There is a natural functor $E : \mathcal{A} \to D^b(\mathcal{A})$ sending an object A to the object in $D^b(\mathcal{A})$ represented by the complex with A in degree 0 and 0 elsewhere. When $\mathcal{A}$ has enough projectives or injectives and F is an additive functor $\mathcal{A} \to \mathcal{B}$, the composite functors $L_iF \circ E$ and $R^iF \circ E$ are exactly the derived functors introduced in Section 4. This follows from Corollary 14.3.

In the remainder of the section assume $\mathcal{A}$ is the category of modules over a fixed commutative ring R; in particular it has enough injectives and projectives. Recall that the tensor product of two complexes in $C^-(\mathcal{A})$ was defined in Definition 8.4. Using the language of double complexes it can be restated as follows: $(A \otimes_R B)^\bullet$ is the simple complex associated with the double complex $C^{\bullet,\bullet}$ with $C^{i,j} = A^i \otimes_R B^j$, horizontal differentials given by $d^i_A \otimes \mathrm{id}_B$ and vertical differentials by $(-1)^i \mathrm{id}_A \otimes d^j_B$.

Proposition 14.7. *For a fixed complex $B^\bullet$ in $K^-(\mathcal{A})$ the functor $K^-(\mathcal{A}) \to K^-(\mathcal{A})$ given by $A^\bullet \mapsto A^\bullet \otimes B^\bullet$ has a left derived functor $D^-(\mathcal{A}) \to D^-(\mathcal{A})$ denoted by $A^\bullet \mapsto A^\bullet \otimes^{\mathbf{L}} B^\bullet$.*

Moreover, the functor $B^\bullet \mapsto A^\bullet \otimes^{\mathbf{L}} B^\bullet$ respects quasi-isomorphisms, inducing a triangulated functor $D^-(\mathcal{A}) \to D^-(\mathcal{A})$.

All in all, $(A^\bullet, B^\bullet) \mapsto A^\bullet \otimes^{\mathbf{L}} B^\bullet$ induces a triangulated *bifunctor* $D^-(\mathcal{A}) \times D^-(\mathcal{A}) \to D^-(\mathcal{A})$ (i.e. a triangulated functor in both variables).

Proof. Only the second statement needs a proof. By Corollary 14.3 $A^\bullet \otimes^{\mathbf{L}} B^\bullet$ is computed by $P^\bullet \otimes B^\bullet$, where $P^\bullet \to A^\bullet$ is a quasi-isomorphism and $P^\bullet$ has projective terms. If $B^\bullet \to B'^\bullet$ is a quasi-isomorphism of complexes in $C^-(\mathcal{A})$, then so is $P^i \otimes_R B^\bullet \to P^i \otimes_R B'^\bullet$ for each $i \in \mathbf{Z}$ because P^i is flat over R. Now Proposition 13.8 implies that $P^\bullet \otimes B^\bullet \to P^\bullet \otimes B'^\bullet$ is a quasi-isomorphism. That the resulting functor is triangulated will follow from the remark below, together with the last part of the proof of Proposition 14.2. □

One defines $\mathrm{Tor}_i(A^\bullet, B^\bullet) := H^{-i}(A^\bullet \otimes^{\mathbf{L}} B^\bullet)$. In the case when $A^\bullet$ and $B^\bullet$ are one-term complexes, this is the same Tor as before.

Remark 14.8. Notice that if $P_A^\bullet \to A^\bullet$ and $P_B^\bullet \to B^\bullet$ are quasi-isomorphisms with complexes having projective terms, we have isomorphisms in $D^-(\mathcal{A})$

$$A^\bullet \otimes^{\mathbf{L}} B^\bullet \cong P_A^\bullet \otimes B^\bullet \cong P_A^\bullet \otimes P_B^\bullet$$

by the proposition above. Moreover, by the same argument in the above proof, tensoring with $P_B^\bullet$ on the right also respects quasi-isomorphisms, hence we also have

$$A^\bullet \otimes^{\mathbf{L}} B^\bullet \cong P_A^\bullet \otimes P_B^\bullet \cong A^\bullet \otimes P_B^\bullet.$$

This shows that we may compute $A^\bullet \otimes^{\mathbf{L}} B^\bullet$ and hence also the groups $\mathrm{Tor}_i(A^\bullet, B^\bullet)$ indifferently by using $P_A^\bullet$ in the first variable or $P_B^\bullet$ in the second.