ELLIPTIC CURVES
Oral exam questions, 2024/25

1. Describe the map from the group of points on an elliptic curve to its degree zero Picard group and show it is a surjective homomorphism.

2. Compute the number of nonnegative divisors of degree d on an elliptic curve over $\mathbf{F}_q$.

3. State the Weil conjectures in the case of smooth plane curves over finite fields. Show that, in the case of elliptic curves, the Riemann Hypothesis is equivalent to a bound on the number of points.

4. Describe the filtration on the group of points of an elliptic curve with good reduction over $\mathbf{Q}_p$.

5. Describe the exact sequence relating the group of points on an elliptic curve over $\mathbf{Q}$ to the Selmer and Tate–Shafarevich groups.

6. Define the height of a point on projective space over $\mathbf{Q}$ and explain how it behaves with respect to a morphism of degree d.

7. Define the Hasse–Weil L-function of an elliptic curve over $\mathbf{Q}$ (definition of factors at bad primes not needed). State the Birch–Swinnerton-Dyer conjecture.

8. Explain how a locally trivial torsor under an elliptic curve over $\mathbf{Q}$ defines an element in the Tate–Shafarevich group. (Converse not needed.)

9. State and prove the finiteness theorem of Shafarevich for elliptic curves over number fields.

10. State the criterion of Néron–Ogg–Shafarevich (proof not needed). Explain how it implies, together with the finiteness theorem of Shafarevich, the finiteness of the number of isogenies to a given elliptic curve over a number field.