ALGEBRAIC TOPOLOGY
Homework 4

1. We have seen in class that $S^1 \wedge S^1 \cong S^2$. Consider the map $S^2 \to S^2$ given by permuting the factors S^1 in $S^1 \wedge S^1$. Show that the induced map $H_2(S^2) \to H_2(S^2)$ has degree -1.

2. Assume that $f : S^i \to X$ represents the zero class in $\pi_i(X, x_0)$, where x_0 is the image of a base point $* \in S^i$. Prove that f extends to a continuous map $E^{i+1} \to X$. [*Hint:* Consider the map $S^i \times I \to E^{i+1}$ given by $(x, t) \mapsto tx + (1-t)*$.]

3. a) Verify that $\pi_i(X, X, x_0) = 0$ for all $i > 1$.

b) Deduce the converse statement to Exercise 2: if f extends to a continuous map $E^{i+1} \to X$, then f represents the zero class in $\pi_i(X, x_0)$.

4. Let X be an n-connected CW complex such that X has dimension $\leq n$ (i.e. $X = X^n$). Show that X is contractible.

[*Warning:* The assumption $X = X^n$ alone does NOT imply that $\pi_i(X) = 0$ for $i > n$!]

5. Recall that S^n can be given a CW structure with two i-cells for each $0 \leq i \leq n$, by attaching inductively two i-cells to S^{i-1} so that S^{i-1} becomes the equator of S^i. Performing this construction infinitely many times, we obtain a CW complex S^∞.

Show that S^∞ is contractible. [*Hint:* Compute homotopy groups first.]