Homology of manifolds

Def. A top. manifold of dim. n is a Hausdorff space X s.t. $\forall\, p \in X$ has an open neighbourhood homeomorphic to some open $U \subset \mathbb{R}^n$.

Ex.
1) $\forall\, U \subset \mathbb{R}^n$ open is a manifold of dim. n
2) E^n is a manifold of dim. n, S^{n-1} of dim. $n-1$
3) $\mathbb{P}^n_{\mathbb{R}}$ —||— of dim. n, $\mathbb{P}^n_{\mathbb{C}}$ of dim. $2n$
4) A torus with g holes is a manifold of dim. 2.
5) If X, Y are manifolds of dim. n & m, $X \times Y$ is a manifold of dim. $n+m$.

Rem. 1) It is known (but difficult) that a manifold of dim. n has a structure of a CW complex of dim. n. Thus in particular $H_i(X) = 0$ $i > n$ and, if X is compact, $H_i(X)$ is finitely generated $\forall i$ [because than X has only finitely many cells]

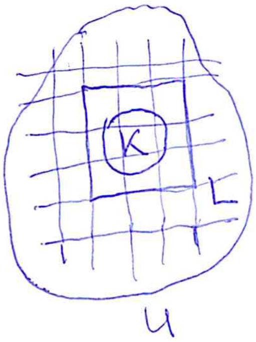
2) In fact, a manifold is a Hausdorff space that has an open covering by $U \subset X$ s.t. $U \simeq \mathbb{R}^n$.
[Indeed, if $I = (0,1)$ than $I \simeq \mathbb{R}$ by the tg function $\Rightarrow I^n \simeq \mathbb{R}^n$, but $\forall x \in \mathbb{R}^n$ has a basis of open neighbourhoods $\simeq I^n$.]

We now prove directly:

Theorem. If X is a top. manifold of dim. n, then $H_i(X) \cong 0$ $i > n$.

Lemma. If $U \subset \mathbb{R}^n$ open, then $H_i(U) = 0$ $i \geq n$.

Pf. Suppose $i > n$, $\alpha \in H_i^{\varepsilon}(U)$. Compactness argument: α comes from some $H_i(K)$ for a compact $K \subset U$.
Choose now a cubical grid in $\mathbb{R}^n$ s.t. the union L of finitely many closed cubes satisfies $K \subset L^{\circ} \subset U$.



Since the grid gives a CW decomposition of $\mathbb{R}^n$ and L is a union of some cells $H_i(\mathbb{R}^n, L) = 0$ $i > n$ by a HW exercise [no i-cells $i > n$]

But $H_i(\mathbb{R}^n) = 0$ $i > 0$ $\Rightarrow$ the rel. homology sequence gives

$$H_{i+1}(\mathbb{R}^n, L) \xrightarrow{\sim} H_i(L)$$
$$H_{i+1}(\mathbb{R}^n, U) \xrightarrow{\sim} H_i(U) \qquad i > 0.$$

Now in the comm. diagram

$$\begin{tikzcd} H_{i+1}(\mathbb{R}^n, L) \arrow[r] \arrow[d, "\simeq"] & H_{i+1}(\mathbb{R}^n, U) \arrow[d, "\simeq"] \\ H_i(L) \arrow[r] & H_i(U) \end{tikzcd}$$

$H_i(L) = 0$ $i \geq n$

and α comes from $H_i(L)$ [since it comes from $H_i(K)$].

Cor. Let X be an n-dim. manifold, $U_1, \dots, U_r \subset X$ open s.t. $U_j \simeq \mathbb{R}^n$ $\forall j$. Then $H_i(U_1 \cup \dots \cup U_r) = 0$ $\forall i > n$.

Pf. Induction on r, the case $r = 1$ being trivial. Assume the case $r-1$ known. Mayer-Vietoris:

$$[0 =] H_i(U_1 \cup \dots \cup U_{r-1}) \oplus H_i(U_r) \to H_i(U_1 \cup \dots \cup U_r) \to$$
$$\to H_{i-1}((U_1 \cup \dots \cup U_{r-1}) \cap U_r)$$
[$= 0$ by Lemma]

Proof of Theorem. Assume $\alpha \in H_i(X)$, $i > n$. Compactness argument: $\exists K \subset X$ compact: α comes from $H_i(K)$. But K can be covered by finitely many "coordinate nbhoods" $U_1, \dots, U_r$, all $\simeq \mathbb{R}^n$ $\Rightarrow$ α comes from $H_i(U_1 \cup \dots \cup U_r) \underset{\substack{\uparrow \\ \text{Cor.}}}{=} 0$

Cor. If $K \subset X$ is a compact subset, then $H_i(X, X \setminus K) = 0$ $i > n$.

[For the proof we need a Mayer-Vietoris sequence for relative homology:

Lemma. Let X be an arbitrary top. space, $U, V \subset X$ open.

~~Let $X = U \cup V$,~~

There is a long exact sequence

$$\dots \to H_{i+1}(X, U\cup V) \to H_i(X, U\cap V) \to H_i(X,U)\oplus H_i(X,V) \to H_i(X, U\cup V) \to \dots$$

Pf. Let $\mathcal{U} = \{U, V\}$. We have a short exact sequence of complexes

$$(*) \quad 0 \to S_\cdot(X)/S_\cdot(U\cap V) \to S_\cdot(X)/S_\cdot(U) \oplus S_\cdot(X)/S_\cdot(V) \to S_\cdot(X)/S_\cdot^{\mathcal{U}}(X) \to 0$$

$$(\alpha \,,\, \beta) \mapsto \alpha - \beta .$$

Here if $U\cup V = X$, $H_i(S_\cdot^{\mathcal{U}}(X)) \xrightarrow{\sim} H_i(X)$ by the small simplices thm. In general, we have only $H_i(S_\cdot^{\mathcal{U}}(X)) \xrightarrow{\sim} H_i(U\cup V)$ and therefore $H_i(S_\cdot(X)/S_\cdot^{\mathcal{U}}(X)) \simeq H_i(X, U\cup V)$ by comparing long exact sequences. Now take the long exact sequence for $(*)$.

Proof of Cor. If K is so small that $\exists\, U \subset X$, $U \simeq \mathbb{R}^n$ s.t. $K \subset U \Rightarrow H_i(X, X\setminus K) \simeq H_i(U, U\setminus K)$ by excision and since $U \simeq \mathbb{R}^n$, $H_i(U, U\setminus K) \xrightarrow{\sim} H_{i-1}(U\setminus K)$ $(i>1)$ by the relative homology sequence. But this is 0 for $i>n$ by Lemma before Thm. In general, we may cover K by finitely many compact subsets: $K = K_1 \cup \dots \cup K_m$ $\forall K_i\ \exists U_i \simeq \mathbb{R}^n$ s.t. $K_i \subset U_i$. Now $H_i(X, X\setminus(K_1\cup\dots\cup K_m)) = 0$ $i>n$ by induction on m and relative Mayer-Vietoris.

Now recall: Since $\mathbb{R}^n \setminus \{x\}$ is homotopic to S^{n-1} for $x\in\mathbb{R}^n$, the relative homology sequence

$$\dots \to H_n(\mathbb{R}^n) \to H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\}) \to H_{n-1}(\mathbb{R}^n\setminus\{x\}) \to H_{n-1}(\mathbb{R}^n)$$

induces $H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\}) \xrightarrow{\sim} H_{n-1}(\mathbb{R}^n\setminus\{x\}) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$ $\quad n>1$.

[for $n=1$ work with $\tilde{H}_0$]

If $x\neq y$ are two points of $\mathbb{R}^n$, we may find a closed ball $E^n \subset \mathbb{R}^n$ containing both, and

$$H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\}) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n\setminus E^n) \simeq H^n(\mathbb{R}^n, \mathbb{R}^n\setminus\{y\}) \simeq \mathbb{Z}.$$

as both $\mathbb{R}^n\setminus\{x\}$, $\mathbb{R}^n\setminus E^n$ are homotopic to S^{n-1}!

Hence if we choose a generator $+1$ or -1 of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ we may choose "the same" generator for $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$ for $\forall y \in \mathbb{R}^n$.

Notice: if X is a manifold, $U \subset X$ open, $U \simeq \mathbb{R}^n$, $x \in U$
$H_n(X, X \setminus \{x\}) \simeq H_n(U, U \setminus \{x\}) \simeq \mathbb{Z}$ by excision.

Def. 1) An orientation of a manifold X is a map $X \xrightarrow{\mu} \{\pm 1\}$ such that $x \in X \rightsquigarrow \mu_x \in H_n(X, X \setminus \{x\})$ and $\forall x \in X$ has a [compact] neighbourhood K s.t. $\exists \mu_K \in H_n(X, X \setminus K)$: $\forall y \in K$ μ_y is the image of μ_K by the natural map
$$H_n(X, X \setminus K) \to H_n(X, X \setminus \{y\}).$$

2) X is orientable if $\exists$ orientation on X. [In this case one sees: $\exists$ exactly two orientations, μ and $-\mu$]

Examples

1) $\mathbb{R}^n$ is orientable by the argument above. Also, S^n is orientable; in fact, $\forall \mu_x$ is the image of $1 \in H_n(S^n)$ under $H_n(S^n) \simeq H_n(S^n, S^n \setminus \{x\})$.

2) If X is orientable, so is U for $\forall U \in X$ open. In particular, $\forall U \subset \mathbb{R}^n$ open is orientable.

3) $\mathbb{RP}^n$ is orientable $\Leftrightarrow$ n is odd [HW]
But $\mathbb{CP}^n$ is always orientable. This follows from:

Rem. (see Hatcher's book for details) $\forall$ manifold X one can construct a covering space $Y \to X$ of degree 2 s.t. Y is orientable. Moreover, X is orientable $\Leftrightarrow$ $Y \to X$ is a trivial cover.

Hence if $\pi_1(X) = 1 \Rightarrow \forall$ cover is trivial $\Rightarrow$ X is orientable.

Prop. Let X be an orientable manifold with orientation μ. For $\forall K \subset X$ compact $\exists \mu_K \in H_n(X, X \setminus K)$ s.t. $\forall x \in K$ $\mu_x \in H_n(X, X \setminus \{x\})$ is the image of μ_K.

Cor. If $X = K$ is compact, $\exists \mu_X \in H_n(X)$ s.t. $\forall x \in X$ μ_x is the image of μ_X. Later we'll see: $H_n(X) \simeq \mathbb{Z}$ with generator μ_X.

Def. μ_K is called a <u>fundamental class</u> [at K]

Rem. This already shows that $\mathbb{R}P^n$ is not orientable if $2 \mid n$ because in this case we know: $H_n(\mathbb{R}P^n) = 0$.

Lemma. If $K \subset \mathbb{R}^n$ is a compact subset, ~~s.t.~~ and $\alpha \in H_{n-1}(\mathbb{R}^n \setminus K)$ s.t. its image in $H_{n-1}(\mathbb{R}^n \setminus \{x\})$ is 0 $\forall x \in K$ $\Rightarrow \alpha = 0$.

Pf. If K is a compact ball, then, as we have seen, $H_{n-1}(\mathbb{R}^n \setminus K) \simeq H_{n-1}(S^{n-1}) \simeq H_{n-1}(\mathbb{R}^n \setminus \{x\}) \Rightarrow \checkmark$

If K is a union of m compact balls $B_1, \dots, B_m$, induction on m.

Mayer-Vietoris:

$$\underbrace{H_n(\mathbb{R}^n \setminus (\bigcup_{i=2}^m B_i) \cap B_1)}_{=0} \to H_{n-1}((\mathbb{R}^n \setminus B_1) \cap (\mathbb{R}^n \setminus \bigcup_{i=2}^m B_i)) \to H_{n-1}(\mathbb{R}^n \setminus B_1) \oplus H_{n-1}(\mathbb{R}^n \setminus \bigcup_{i=2}^m B_i)$$

[Lemma p. I] By induction $\alpha = 0$ in $H_{n-1}(\mathbb{R}^n \setminus B_1)$ & $H_{n-1}(\mathbb{R}^n \setminus \bigcup_{i=2}^m B_i) \Rightarrow$ it is 0 by injectivity.

In general, α comes from $H_{n-1}(C)$, $C \subset \mathbb{R}^n \setminus K$ compact. We may cover K by compact balls $B_1, \dots, B_m$ s.t. $(\bigcup_{i=1}^m B_i) \cap C = \emptyset$. So the image of α in $H_{n-1}(\mathbb{R}^n \setminus K)$ comes from $H_{n-1}(\mathbb{R}^n \setminus \bigcup_{i=1}^m B_i)$. But if α maps to 0 in $H_{n-1}(\mathbb{R}^n \setminus \{x\})$ for $x \in B_i \cap K$, it also maps to 0 in $H_{n-1}(\mathbb{R}^n \setminus \{y\})$ $\forall y \in B_i$ [argument above].

So α maps to 0 in $H_{n-1}(\mathbb{R}^n \setminus \bigcup_{i=1}^n B_i)$, q.e.d.

<u>Proof of Prop.</u> If K is a small compact subset as in def. of orientation, nothing to prove. Otherwise $K = \bigcup_{i=1}^{m} K_i$ where K_i is as above and moreover $\exists\, U_i \subset X$ open : $U_i \simeq \mathbb{R}^n$ and $K_i \subset U_i$. Induction on m. Suppose we have $\mu_{K'}$ for $K' = \bigcup_{i=2}^{m} K_i$.

Relative Mayer-Vietoris:

$$H_{n+1}(X, X \setminus (K_1 \cap K')) \to H_n(X, X\setminus K) \to H_n(X, X\setminus K_1) \oplus H_n(X, X \setminus K') \xrightarrow{\varrho} H_n(X, X \setminus (K_1 \cap K'))$$

$H_{n+1}(X, X \setminus (K_1 \cap K')) = 0$ [previous Cor.]

Enough to show: μ_{K_1} & $\mu_{K'}$ map to the same elt in $H_n(X, X\setminus (K_1 \cap K')) \simeq H_n(U_1, U_1 \setminus (K_1 \cap K'))$

By assumption, $\varrho(\mu_{K_1}) - \varrho(\mu_{K'})$ maps to 0 in $H_n(X, X\setminus\{x\}) \simeq H_n(U_1, U_1 \setminus \{x\})$ $\forall x \in K_1 \cap K'$ $\Rightarrow$ it is 0 by Lemma.

<u>The cap-product</u>

For Poincaré duality we need one more construction.

Set $S^\cdot(X) := \mathrm{Hom}(S_\cdot(X), \mathbb{Z})$. It computes the cohomology groups $H^i(X) := H^i(X, \mathbb{Z})$.

There is a natural evaluation map

$$\langle\ ,\ \rangle : \quad \mathrm{Hom}(S_i(X), \mathbb{Z}) \times S_i(X) \to \mathbb{Z} \quad \forall i.$$

We make it into a morphism of complexes as follows.

Given a cohomological complex $C^\cdot$, view it as a homological complex with homological numbering $C_{-i} := C^{-i}$ and homological differential $d_i = (-1)^{i+1} d^i$.

In particular, we regard $S^\cdot(X)$ as a homological complex with the above conventions. $\langle\ ,\ \rangle$ then induces

$\varepsilon : S^\cdot(X) \otimes S_\cdot(X) \to \mathbb{Z}$ [where $\mathbb{Z}$ is the complex $0 \to 0 \ldots 0 \to 0 \to \underset{\deg 0}{\mathbb{Z}} \to 0 \to \ldots$]

Indeed, the degree 0 term of $S^\cdot(X) \otimes S_\cdot(X)$ is

$$\bigoplus_i \mathrm{Hom}(S_i(X), \mathbb{Z}) \otimes S_i(X)$$

So we may set $\varepsilon_0 := \bigoplus_i \langle\, , \rangle : \bigoplus_i \mathrm{Hom}(S_i(X), \mathbb{Z}) \otimes S_i(X) \xrightarrow{\Sigma} \mathbb{Z}$;

for $n \neq 0$ we set $\varepsilon_n = 0$.

Lemma. $\varepsilon : S^\cdot(X) \otimes S_\cdot(X) \to \mathbb{Z}$ is a morphism of complexes.

Pf. Since all differentials in $(\to \dots \to 0 \to \mathbb{Z} \to 0 \to \dots)$ are 0,

we have to check $\varepsilon_0 \circ d_1 = 0$.

If $f \in \mathrm{Hom}(S_i(X), \mathbb{Z})$ and $\alpha \in S_{i+1}(X)$

$$d(f \otimes \alpha) = df \otimes \alpha + (-1)^i f \otimes d\alpha$$

$$\varepsilon_0(d(f \otimes \alpha)) = \langle df, \alpha \rangle + (-1)^i \langle f, d\alpha \rangle$$

$$= (-1)^{i+1} f(d\alpha) + (-1)^i f(d\alpha) = 0.$$

Construction. Consider the composite map

$$\gamma : S^\cdot(X) \otimes S_\cdot(X) \xrightarrow{id \otimes \tilde{\Delta}} S^\cdot(X) \otimes S_\cdot(X) \otimes S_\cdot(X) \xrightarrow{\varepsilon \otimes id} \mathbb{Z} \otimes S_\cdot(X) \simeq S_\cdot(X)$$

where $\tilde{\Delta}$ is the composite $S_\cdot(X) \xrightarrow{\Delta} S_\cdot(X \times X) \xrightarrow{EZ} S_\cdot(X) \otimes S_\cdot(X)$

(diagonal + Eilenberg-Zilber)

By a previous lemma we have maps $\forall\, i, n$

$$H^i(X) \otimes H_n(X) \to H_{n-i}(S^\cdot(X) \otimes S_\cdot(X)) \xrightarrow{\gamma} H_{n-i}(S_\cdot(X)) = H_{n-i}(X)$$

[recall the homological indexing of $S^\cdot(X)$!]

This map is the cap product. As before, it does not depend on the choice of the EZ map, and Alexander-Whitney gives an explicit formula. Notation: $(\alpha, \beta) \mapsto \alpha \cap \beta$.

We'll also use a relative version

$$H^i(X, T) \times H_n(X, T) \to H_{n-i}(X).$$

Construction the same, using the following map

$$\tilde{\Delta}_T : S_\cdot(X, T) \to S_\cdot(X, T) \otimes S_\cdot(X)$$

To construct $\tilde{\Delta}_T$, note first

$$0 \to S.(T) \to S.(X) \to S.(X,T) \to 0 \quad \text{induces}$$

$$S.(T) \otimes S.(X) \to S.(X) \otimes S.(X) \to S.(X,T) \otimes S.(X) \to 0$$

and we have a commutative diagram

$$\begin{tikzcd}
S.(T) \arrow[r] \arrow[d, "\Delta"] & S.(X) \arrow[r] \arrow[d, "\Delta"] & S.(X,T) \arrow[r] \arrow[dd, dashed] & 0 \\
S.(T \times X) \arrow[d, "EZ"] & S.(X \times X) \arrow[d, "EZ"] & & \\
S.(T) \otimes S.(X) \arrow[r] & S.(X) \otimes S.(X) \arrow[r] & S.(X,T) \otimes S.(X) \arrow[r] & 0
\end{tikzcd}$$

There is the following relation with cup-products.

Note first: $\langle\, , \,\rangle : \; S^i(X) \times S_i(X) \to \mathbb{Z}$ induces

$\langle\, , \,\rangle \quad H^i(X) \times H_i(X) \to \mathbb{Z} \quad \forall i.$ [not perfect duality!!]

This can be checked directly, or by using

$$H_{-i}(S^i(X)) \otimes H_i(S_i(X)) \to H_0(S^i(X) \otimes S_i(X)) \xrightarrow{\varepsilon_0} \mathbb{Z}.$$

<u>Prop.</u> If $\alpha \in H^{n-i}(X)$, $\beta \in H^i(X)$, $\gamma \in H_n(X)$

$$\langle \alpha, \beta \cap \gamma \rangle = \langle \alpha \cup \beta, \gamma \rangle.$$

<u>Proof.</u> Calculation going through def's — omitted.

Now we can state:

<u>Theorem</u> (Poincaré duality for compact manifolds)

Let X be a connected compact orientable manifold with fundamental class $\mu_X \in H_n(X)$. [$n = \dim X$]

The map $\alpha \mapsto \alpha \cap \mu_X$ induces isomorphisms

$$H^i(X) \xrightarrow{\sim} H_{n-i}(X) \qquad \forall i.$$

<u>Cor.</u> If moreover $H_i(X)$ is f.g. free $\forall i$, the cup-product

$$H^i(X) \times H^{n-i}(X) \to H^n(X) \simeq \mathbb{Z}$$

is a perfect duality.

Pf. of Cor.

By Thm, $H_n(X) \simeq H^0(X,\mathbb{Z}) \simeq \mathbb{Z}$
and because $H_i(X)$ is free $\forall i$, $H^i(X,\mathbb{Z}) \simeq \mathrm{Hom}(H_i(X),\mathbb{Z})$ $\forall i$
by a previous Prop. So the thm becomes
$H^i(X) \simeq \mathrm{Hom}(H^{n-i}(X),\mathbb{Z})$. To see that this is induced
by cup-product, use the formula $\langle \alpha, \beta \cap \gamma \rangle = \langle \alpha \cup \beta, \gamma \rangle$.

Rem. 1) In general one can prove: cup-product induces perfect dualities $H^i(X)/\text{torsion} \times H^{n-i}(X)/\text{torsion} \to \mathbb{Z}$.
But even better with coefficients in a field F (e.g. $F=\mathbb{Q}$ or $F=\mathbb{Z}/p\mathbb{Z}$)
Then $H^i(X,F) \times H^{n-i}(X,F) \to H^n(X,F) \simeq F$
is a perfect duality. This follows from the above by tensoring everything with F.

2) In particular, we have the symmetries
$\dim H^i(X,\mathbb{Q}) = \dim H^{n-i}(X,\mathbb{Q})$ $\forall i$
In fact, $\dim H^i(X,\mathbb{Q}) = \dim H_i(X)\otimes\mathbb{Q} = \mathrm{rk}\, H_i(X)$.
This is the earliest form of Poincaré duality.

3) If moreover $n=2m$ with m odd, cup-product gives a perfect duality $H^m(X,\mathbb{Q}) \times H^m(X,\mathbb{Q}) \to \mathbb{Q}$
and moreover $\alpha \cup \beta = -\beta \cup \alpha$ by anti-commutativity of $\cup$.
But we know from linear algebra: alternating bilinear forms exist only on even-dim. vector spaces. So in this case $2 \mid \dim H^m(X,\mathbb{Q})$.

4) With $(\mathbb{Z}/2)$-coefficients everything goes through even for non-orientable manifolds [as $1 \equiv -1 \mod 2$].

We'll give Milnor's proof of Poincaré duality, based on Mayer-Vietoris. But the thm is false for non-compact manifolds! E.g. if $U \subset \mathbb{R}^n$ open, we know $H_n(U)=0$ but $H^0(U) \simeq \mathbb{Z}$! [Connected]

To overcome this problem, we modify cohomology groups in the non-compact case.

<u>Construction.</u> If $K \subset X$ compact [X arbitrary top. space]

$S_\cdot(X) \twoheadrightarrow S_\cdot(X, X\setminus K)$, so

$\mathrm{Hom}(S_\cdot(X, X\setminus K), \mathbb{Z}) \hookrightarrow \mathrm{Hom}(S_\cdot(X), \mathbb{Z}) = S^\cdot(X)$

is a <u>subcomplex</u>. Put

$$S_c^\cdot(X) := \bigcup_{\substack{K \subset X \\ \text{compact}}} \mathrm{Hom}(S_\cdot(X, X\setminus K), \mathbb{Z}) \subset S^\cdot(X)$$

This is still a subcomplex, and we put

$H_c^i(X) := H^i(S_c^\cdot(X))$ i-th cohomology group with <u>compact support</u>

<u>Note:</u> If X is compact, we can take $K = X$ above, so $S_c^\cdot(X) = S^\cdot(X)$ and $H_c^i(X) = H^i(X)$ $\forall i$. But in the non-compact case there is a difference.

We'll use a second construction, based on direct limits.

<u>Def.</u> Let $(\Lambda, \leq)$ be a partially ordered set. It is <u>directed</u> if $\forall \alpha, \beta \in \Lambda$ $\exists \gamma$: $\alpha \leq \gamma$, $\beta \leq \gamma$.

A <u>direct system</u> of abelian groups indexed by Λ is given by
* A_α : abelian group $\forall \alpha \in \Lambda$
* $\varphi_{\alpha\beta} : A_\alpha \to A_\beta$ homomorphism $\forall \alpha \leq \beta$.

The <u>direct limit</u> (or <u>colimit</u>) of the system is

$$\varinjlim A_\alpha := \bigoplus_{\alpha \in \Lambda} A_\alpha \Big/ \langle (0,..,0,a_\alpha,0,..,0,-\varphi_{\alpha\beta}(a_\alpha),0,..,0) \rangle$$

<u>Note:</u> an element of $\varinjlim A_\alpha$ is represented by some $a_\alpha \in A_\alpha$ with $a_\alpha \in A_\alpha$ identified with $a_\beta \in A_\beta$ if $\exists \gamma \geq \alpha, \beta$: $\varphi_{\alpha\gamma}(a_\alpha) = \varphi_{\beta\gamma}(a_\beta)$. [Observe: if $(a_{\alpha_1}, \ldots, a_{\alpha_r}) \in \bigoplus A_\alpha$,

We may find $\gamma \geq \alpha_1, \dots, \alpha_r$ (use induction) and identify $(a_{\alpha_1}, \dots, a_{\alpha_r})$ with $\sum \varphi_{\alpha_i \gamma}(a_{\alpha_i})$.]

$\exists$ obvious generalization to direct limits of complexes of abelian groups.

Lemma. Assume (A_α), (B_α), (C_α) are direct systems indexed by Λ s.t. $\forall \alpha \in \Lambda$ $\exists$ short exact sequence

$$0 \to A_\alpha \xrightarrow{i_\alpha} B_\alpha \xrightarrow{s_\alpha} C_\alpha \to 0$$

and $\forall$ $\alpha \leq \beta$

$$\begin{tikzcd} A_\alpha \arrow[r, "i_\alpha"] \arrow[d, "\varphi^A_{\alpha\beta}"] & B_\alpha \arrow[r, "s_\alpha"] \arrow[d, "\varphi^B_{\alpha\beta}"] & C_\alpha \arrow[d, "\varphi^C_{\alpha\beta}"] \\ A_\beta \arrow[r, "i_\beta"'] & B_\beta \arrow[r, "s_\beta"'] & C_\beta \end{tikzcd}$$

commute. Then

$$0 \to \varinjlim A_\alpha \to \varinjlim B_\alpha \to \varinjlim C_\alpha \to 0$$

is exact.

Pf. Easy from def's. E.g. injectivity: if $a_\alpha \in A_\alpha$ is s.t. $\varphi^B_{\alpha\beta}(i_\alpha(a_\alpha)) = 0$ for $\beta \geq \alpha$, then $i_\beta(\varphi^A_{\alpha\beta}(a_\alpha)) = 0$ by commutativity and $\varphi^A_{\alpha\beta}(a_\alpha) = 0$ by injectivity of i_β. But a_α is identified with $\varphi^A_{\alpha\beta}(a_\alpha)$.

Cor. A direct limit of long exact sequences remains exact [cut up in short exact sequences]

In our case: $\Lambda = \{X \setminus K : K \subset X \text{ compact}\}$

$$X \setminus K \leq X \setminus L \overset{\text{def}}{\iff} K \subset L. \; [\iff X \setminus K \supset X \setminus L]$$

This system is obviously directed: $K, L \subset K \cup L$ compact.

Observe: The $\mathrm{Hom}(S_i(X, X \setminus K), \mathbb{Z}) \subset \mathrm{Hom}(S_i(X), \mathbb{Z})$ form a direct system indexed by Λ. Indeed,

$K \subset L \Rightarrow X \setminus K \supset X \setminus L \Rightarrow S_i(X, X \setminus L) \twoheadrightarrow S_i(X, X \setminus K)$

$\Rightarrow \mathrm{Hom}(S_i(X, X \setminus K), \mathbb{Z}) \hookrightarrow \mathrm{Hom}(S_i(X, X \setminus L), \mathbb{Z})$

These are all subgroups of $\mathrm{Hom}(S_i(X), \mathbb{Z})$ and the φ_{KL} are the inclusion maps. So

$\varinjlim \mathrm{Hom}(S_i(X, X \setminus K), \mathbb{Z}) = \bigcup_K \mathrm{Hom}(S_i(X, X \setminus K), \mathbb{Z}) = S^i_c(X) \subset S^i(X)$.

This is compatible with differentials, so in fact

$$S^{\cdot}_c(X) = \varinjlim \mathrm{Hom}(S_\cdot(X, X \setminus K), \mathbb{Z})$$

Notice: the $H^i(X, X \setminus K; \mathbb{Z})$ also form a direct system indexed by Λ. Moreover:

<u>Lemma.</u> $\varinjlim_K H^i(X, X \setminus K; \mathbb{Z}) \simeq H^i_c(X, \mathbb{Z})$ [$=: H^i_c(X)$.]

<u>Proof.</u> This follows from the previous lemma applied to $\mathrm{Ker}(d^i)$, $\mathrm{Im}(d^{i-1})$ in $\mathrm{Hom}(S_\cdot(X, X \setminus K), \mathbb{Z})$.

We use the above construction to prove:

<u>Theorem.</u> If $X = U \cup V$ with U, V open, $\exists$ Mayer-Vietoris long exact sequence

$$\dots \to H^i_c(U \cap V) \to H^i_c(U) \oplus H^i_c(V) \to H^i_c(U \cup V) \to H^{i+1}_c(U \cap V) \to \dots$$

This is a "mix" of homological and cohomological M-V!

<u>Proof.</u> $\forall K, L \subset X$ compact we have relative M-V sequences

$$\dots \to H^i(X, X \setminus (K \cap L)) \to H^i(X, X \setminus K) \oplus H^i(X, X \setminus L) \to$$
$$\to H^i(X, X \setminus (K \cup L)) \to H^{i+1}(X, X \setminus (K \cap L)) \to \dots$$

These are constructed like the homological relative M-V sequences (p. III) + argument of passing from homology to cohomology as in proof of M-V for cohomology. We shall consider these sequences only for $K \subset U$, $L \subset V$.

Fix $K \subset U$, and take $\varinjlim$ over $L \subset V$ compact:

$$\cdots \to \varinjlim_{L} H^i(X, X\setminus K\cap L) \to H^i(X, X\setminus K) \oplus \varinjlim_{L} H^i(X, X\setminus L) \to \varinjlim_{L} H^i(X, X\setminus (K\cup L)) \to \cdots$$

This is exact by previous Cor. Now take $\varinjlim$ over $K \subset U$ compact:

$$\cdots \to \varinjlim_{K,L} H^i(X, X\setminus (K\cap L)) \to \varinjlim_{K} H^i(X, X\setminus K) \oplus \varinjlim_{L} H^i(X, X\setminus L) \to \varinjlim_{K,L} H^i(X, X\setminus (K\cup L)) \to \cdots$$

Since $K \subset U$, $H^i(X, X\setminus K) \simeq H^i(U, U\setminus K)$ by excision, so

$$\varinjlim_{K} H^i(X, X\setminus K) \simeq H^i_c(U) \quad \text{as } \varinjlim \text{ is over } K \subset U \text{ compact.}$$

Likewise, $\varinjlim_{L} H^i(X, X\setminus L) \simeq H^i_c(V)$

Also, $\varinjlim_{K,L} H^i(X, X\setminus (K\cap L)) \simeq \varinjlim_{K,L} H^i(U\cap V, (U\cap V)\setminus (K\cap L)) \simeq H^i_c(U\cap V)$

and finally $\varinjlim_{K,L} H^i(X, X\setminus (K\cup L)) \simeq H^i_c(U\cup V)$. ✓

<u>Now recall:</u> If X is an n-dim. oriented manifold, we have constructed $\forall K \subset X$ a fundamental class $\mu_K \in H_n(X, X\setminus K)$. Also, if $K \subset L$, μ_L maps to μ_K by the natural map

$H_n(X, X\setminus L) \to H_n(X, X\setminus K)$ by construction.

Now recall the cap-products

$H^i(X, X\setminus K) \times H_n(X, X\setminus K) \to H_{n-i}(X)$.

So cap-product with μ_K induces

$$S_K : H^i(X, X\setminus K) \to H_{n-i}(X)$$

If $K \subset L$, the diagram

$$\begin{tikzcd} H^i(X, X\setminus K) \arrow[dr, "S_K"] \arrow[d] & \\ H^i(X, X\setminus L) \arrow[r, "S_L"'] & H_{n-i}(X) \end{tikzcd}$$

commutes as μ_L maps to μ_K and cap-products are functorial.

So we may take $\varinjlim$ over $K \subset X$ and get a map

$$\mathcal{S}_X : H^i_c(X) \to H_{n-i}(X).$$

If X is compact, $\mathcal{S}$ is just cap-product with μ_X.

<u>Theorem</u> (Poincaré duality, general form): $\mathcal{S}_X$ is an isomorphism $\forall i$.

<u>Rem.</u> Before the proof, just one more construction:
if $i: U \hookrightarrow X$ is an open immersion, it induces
$i_*: H^i_c(U) \to H^i_c(X)$ [covariant functoriality for open immersions]
These are the maps in the M-V sequence for cohomology with compact support. To construct it, we use the excision isomorphisms $H^i(U, U\setminus K) \xrightarrow{\sim} H^i(X, X\setminus K)$ for $K \subset U$. This gives an isom. $H^i_c(U) \xrightarrow{\sim} \varinjlim_{K\subset U} H^i(X, X\setminus K)$.
On the other hand, $\exists$ natural map
$$\varinjlim_{K\subset U} H^i(X, X\setminus K) \to \varinjlim_{K\subset X} H^i(X, X\setminus K) = H^i_c(X).$$
The map i_* is the composite of the two.

<u>Proof.</u> 1) Case $X = \mathbb{R}^n$. We know: $H_i(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i\neq 0 \end{cases}$

Also, if $B \subset \mathbb{R}^n$ is a compact ball,
$H_i(\mathbb{R}^n, \mathbb{R}^n\setminus B) = \begin{cases} \mathbb{Z} & i=n, \text{ generated by } \mu_B. \\ 0 & i\neq n \end{cases}$

By a previous Prop., this implies
$$H^i(\mathbb{R}^n, \mathbb{R}^n\setminus B) \simeq \mathrm{Hom}(H_i(\mathbb{R}^n, \mathbb{R}^n\setminus B), \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & i=n \\ 0 & i\neq n. \end{cases}$$
This already proves the thm for $i\neq n$, as
$$H^i_c(\mathbb{R}^n) = \varinjlim_K H^i(\mathbb{R}^n, \mathbb{R}^n\setminus K) \simeq \varinjlim_B H^i(\mathbb{R}^n, \mathbb{R}^n\setminus B)$$
as $\forall K$ $\exists B \supset K$ large enough.

Also, a generator of $\mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n\setminus B), \mathbb{Z})$ sends μ_B to 1. The cap-product with μ_B defines
$$H^n(\mathbb{R}^n, \mathbb{R}^n\setminus B) \to \mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n\setminus B), H_0(\mathbb{R}^n)) \simeq$$
$$\simeq \mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n\setminus B), \mathbb{Z})$$
This has the same generator. Now take $\varinjlim$, as above.

2) Assume $X = U \cup V$, $U, V \subset X$ open and the thm is known for $U, V, U \cap V$. Then it holds for X. Indeed, consider

$$\begin{array}{ccccccccc}
\cdots \to & H^i_c(U\cap V) & \to & H^i_c(U) \oplus H^i_c(V) & \to & H^i_c(U\cup V) & \to \cdots \\
& \downarrow \mathcal{S}_{U\cap V} & & \downarrow \mathcal{S}_U \oplus \mathcal{S}_V & & \downarrow \mathcal{S}_{U\cup V} & \\
\cdots \to & H_{n-i}(U\cap V) & \to & H_{n-i}(U) \oplus H_{n-i}(V) & \to & H_{n-i}(U\cup V) & \to \cdots
\end{array}$$

where the rows are M-V sequences. One checks: the squares commute. Since $\mathcal{S}_U, \mathcal{S}_V, \mathcal{S}_{U\cap V}$ isom $\Rightarrow$ so is $\mathcal{S}_{U\cup V}$.

3) Assume $\exists$ chain $U_1 \subset U_2 \subset \ldots$ of open subsets such that $\bigcup U_j = X$ and the thm holds for $\forall U_j$. Then it holds for X. Indeed, we have direct systems indexed by $\mathbb{Z}_{>0}$:

$$H_{n-i}(U_1) \to H_{n-i}(U_2) \to H_{n-i}(U_3) \to \cdots$$

$$H^i_c(U_1) \to H^i_c(U_2) \to H^i_c(U_3) \to \cdots$$

where in the second system the maps are as in the Rem.

As $\bigcup U_j = X$, $\varinjlim H_{n-i}(U_j) \simeq H_{n-i}(X)$

Also, $\varinjlim H^i_c(U_j) \simeq H^i_c(X)$

because $H^i(U_j, U_j \setminus K) \simeq H^i(X, X\setminus K)$ if $K \subset U_j$ and $\forall K$ $\exists U_j$: $K \subset U_j$. We know: $\mathcal{S}_{U_j}: H^i_c(U_j) \xrightarrow{\sim} H_{n-i}(U_j)$

Now take $\varinjlim$.

4) $X \subset \mathbb{R}^n$ open. We then may cover X by countably many open cubes $C_j \simeq \mathbb{R}^n$ where also $C_j \cap C_{j'} \simeq \mathbb{R}^n$ for $j \neq j'$, if $\neq \emptyset$. By 1) & 2), the thm holds for $U_j := \bigcup_{i=1}^{j} C_i$. By 3), it holds for X.

5) We cover X by open sets $U \simeq \mathbb{R}^n$. These satisfy the thm by 1). Consider the system of open sets $V \subset X$ that satisfy the thm. By 3) they satisfy the assumption in Zorn's lemma $\Rightarrow$ $\exists$ maximal element $\tilde{V}$. If $\tilde{V} \neq X$, $\exists x \in X \setminus \tilde{V}$ and an open neighbhd $U \simeq \mathbb{R}^n$ of x s.t. thm holds for U, $\tilde{V}$ and also $\tilde{V} \cap U$ by 4). Thus by 2) it holds for $\tilde{V} \cup U$ ↯

[of course for X compact this argument is not needed].

Homotopy-groups

Notation: If $(X, x_0), (Y, y_0)$ pointed spaces (i.e. spaces + base points)

$$[(X, x_0), (Y, y_0)]^0 := \text{base point preserving maps up to homotopy}$$

Fix a base point $* \in S^n$.

Def. $\pi_i(X, x_0) := [(S^n, *), (X, x_0)]^0$ i-th homotopy group of X.

For $i = 0$, this is just a set: path components of X, pointed by x_0

$i = 1$: this is the fundamental group [group structure coming below]

Let $I^n :=$ standard closed n-cube, and consider $I^n / \partial I^n$ pointed by $[\partial I^n]$

Lemma. $I^n / \partial I^n \simeq S^n$

For the proof we need the smash product $X \wedge Y$ of two pointed spaces $(X, x_0), (Y, y_0)$, $X \wedge Y := X \times Y / (X \times \{y_0\} \cup \{x_0\} \times Y)$

with base point $X \times \{y_0\} \cup \{x_0\} \times Y$. It carries the quotient top.

Given pointed maps $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ an pointed map $X \wedge Y \xrightarrow{f \wedge g} X' \wedge Y'$.

Pf of Lemma: $n = 1$ clear

$n = 2$: $(I / \partial I) \wedge (I / \partial I) \simeq I^2 / \partial I^2$

$$S^1 \wedge S^1 \simeq S^2 \qquad [(\mathbb{R}^1 \cup \{\infty\}) \wedge (\mathbb{R}^1 \cup \{\infty\}) \cong \mathbb{R}^2 \cup \{\infty\}]$$

General case similar: $(I / \partial I)^{\wedge n} \simeq I^n / \partial I^n$ and

$(S^1)^{\wedge n} \simeq S^n$ by induction on n.

Cor. $\pi_i(X, x_0) = [(I^i / \partial I^i), [\partial I^i]), (X, x_0)]^0$

$= \text{maps } (I^i, \partial I^i) \to (X, x_0))$ up to homotopies constant on ∂I^i.

Now we can define the group structure on $\pi_i(X, x_0)$:

Def. If $[f], [g] \in [(I^i / \partial I^i, [\partial I^i]), (X, x_0)]^0$,

$[f] + [g]$ is represented by

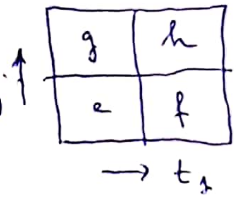
$$(f + g)(t_1, \dots, t_i) = \begin{cases} f(2t_1, t_2, \dots, t_i) & 0 \le t_1 \le \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_i) & \frac{1}{2} \le t_1 \le 1 \end{cases}$$

As in the case $i=1$, one shows this is well defined on homotopy classes and gives a group structure with unit the constant map $0: I^i / \partial I^i \to \{x_0\}$.

<u>Prop.</u> This group law is commutative for $i \geq 2$.

<u>Proof.</u> For $j = 2, \dots, i$ we have similar group laws $+_j$ using the variable t_j instead of t_1. At the same time we prove $+ = +_j \; \forall j$. Observe: if $[e], [f], [g], [h]$ are elements of $\pi_i(X, x_0)$,

$$(e+f) +_j (g+h) = (e +_j g) + (f +_j h)$$



The constant path 0 is a unit for all $+_j$.

So $f +_j g = (f+0) +_j (0+g) = (f +_j 0) + (0 +_j g) = f+g$

$\Rightarrow + = +_j$, moreover $(e+f)+(g+h) = (e+g)+(f+h)$

gives for $e = h = 0$: $\quad f+g = g+f$.

<u>Interlude</u>: function spaces.

Notation: if X, Y top. spaces, $\mathrm{Hom}(X,Y) := \{\text{cont. maps } X \to Y\}$

We put a topology on $\mathrm{Hom}(X,Y)$, the <u>compact-open</u> topology.

<u>Subbasis</u>: given $K \subset X$ compact, $U \subset Y$ open,

$$V_{K,U} := \{ f \in \mathrm{Hom}(X,Y) : f(K) \subset U \}$$

A general open set is obtained by finite $\cap$'s and arbitrary $\cup$'s.

<u>Lemma</u>. 1) Given $f \in \mathrm{Hom}(X \times Y, Z)$, the map $\tilde{f} : X \to \mathrm{Hom}(Y,Z)$ given by $x \mapsto f(x, -)$ is continuous, i.e. lies in $\mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$.

2) If moreover Y is loc. compact, $g \in \mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$, $\bar{g} : (x,y) \mapsto g(x)(y)$ is continuous, i.e. lies in $\mathrm{Hom}(X \times Y, Z)$.

<u>Cor.</u> If Y is loc. compact, $\mathrm{Hom}(X \times Y, Z) \simeq \mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$

Rem. This is a bijection of sets but one can check it is a homeomorphism in the compact-open topology.

Proof of Lemma.

1) Fix $K \subset Y$ compact, $U \subset Z$ open. We show: $\tilde{f}^{-1}(V_{K,U}) \subset X$ is open. Pick $x \in \tilde{f}^{-1}(V_{K,U})$. Then $f(\{x\} \times K) \subset U$. As f is continuous, $\forall y \in K$ $\exists U_x \subset X$, $V_y \subset Y$ open nbhds: $f(U_x \times V_y) \subset Y$.
K compact $\Rightarrow$ finitely many V_y cover K $\Rightarrow$ $\exists U_x$ that works $\forall V_y$ $\Rightarrow$ $f(U_x \times K) \subset U \Leftrightarrow U_x \subset \tilde{f}^{-1}(V_{K,U})$.

2) Enough to show: the map $Y \times \mathrm{Hom}(Y,Z) \xrightarrow{e} Z$, $(y,f) \mapsto f(y)$ is continuous. Indeed, then $\tilde{g}$ equals the composite $X \times Y \xrightarrow{(g,\mathrm{id})} \mathrm{Hom}(Y,Z) \times Y \xrightarrow{e} Y$.
If U is an open nbhd of $f(y)$, $\exists\, y \in K \subset Y$ compact: $f(K) \subset U$ as Y is loc. compact. But then $e(K \times V_{K,U}) \subset U$ by def. of $V_{K,U}$, and $y \in K$, $f \in V_{K,U}$.

Now assume (X,x), (Y,y), (Z,z) are pointed.
$\mathrm{Hom}(X,Y)^\circ :=$ pointed continuous maps $(X,x) \to (Y,y)$, endowed with the subspace top. from $\mathrm{Hom}(X,Y)$ pointed by $X \to \{y\}$.

Lemma. Given $f \in \mathrm{Hom}(X \times Y, Z)^\circ$, $\tilde{f} \in \mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$ lies in the subspace $\mathrm{Hom}(X, \mathrm{Hom}(Y,Z))^\circ \Leftrightarrow f(X \times \{y\} \cup \{x\} \times Y) = z$.

Proof. Tautology.

Cor. If Y is loc. compact, $\exists$ bijection
$$\mathrm{Hom}(X \wedge Y, Z)^\circ \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Hom}(Y,Z)^\circ)^\circ$$

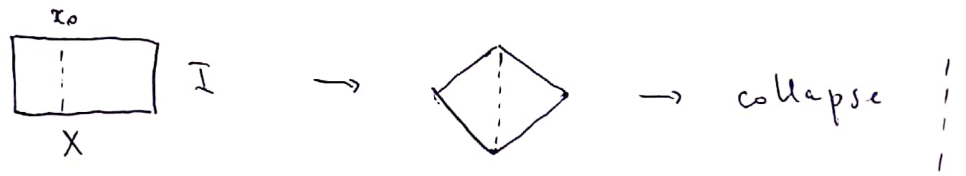
One checks: here homotopic maps correspond, so

Cor. $[X \wedge Y, Z]^\circ \xrightarrow{\sim} [X, \mathrm{Hom}(Y,Z)^\circ]^\circ$

Def. If (X, x_0) is a pointed space, define the loop space $\Omega X := \mathrm{Hom}(S^1, X)^\circ$.
It is pointed by $S^1 \to \{x_0\}$.

Similarly, set $SX := S^1 \wedge X$ the <u>suspension</u> of X.

<u>Rem.</u> This is sometimes called reduced suspension. To visualize it, note that SX is obtained from $X \times I$ by collapsing $X \times \{0\}$, $X \times \{1\}$, $\{x_0\} \times I$



<u>Rem.</u> $X \mapsto SX$, $X \mapsto \Omega X$ are functors of pointed top. spaces

<u>Cor.</u> There is an isomorphism $[SX, Y]^\circ \simeq [X, \Omega Y]^\circ$
[adjoint isom. in homotopy category]

<u>Cor.</u> $\exists$ canonical isomorphisms

$$\pi_{i+1}(X, x_0) \xrightarrow{\sim} \pi_i(\Omega X, x_0)$$
$\leftarrow$ constant path

<u>Proof.</u> $\pi_i(\Omega X, x_0) = [(S^i, *), (\Omega X, x_0)]^\circ \simeq [((S(S^i), *), (X, x_0)]^\circ$ and $S(S^i) \simeq S^{i+1}$ as we have seen. $\quad = \pi_{i+1}(X, x_0)$.

<u>Rem.</u> <u>Dependence on base point</u>: assume $\pi_0(X) = 1$, $x_0, x_1 \in X$. We have loop spaces $\Omega_{x_0}X$, $\Omega_{x_1}X$ based at x_0, x_1, resp. Given a path γ from x_0 to x_1, $f \mapsto \gamma \cdot f \cdot \gamma^{-1}$ gives a map $\Omega_{x_1}X \to \Omega_{x_0}X$ $\rightsquigarrow$ applying $\pi_i(\ )$ we get

$$\pi_i(\Omega_{x_1}X, x_1) \longrightarrow \pi_i(\Omega_{x_0}X, x_0)$$
$$\begin{matrix} \cong & & \cong \\ \pi_{i+1}(X, x_1) & \xrightarrow{\gamma_*} & \pi_{i+1}(X, x_0) \end{matrix}$$

This is a bijection [as γ_*^{-1} is an inverse].
One checks: γ_* depends only on the homotopy class of γ. Whence a <u>group action</u> of $\pi_1(X, x_0)$ on $\pi_i(X, x_0)$ $\forall i \geq 1$. If this action is trivial (e.g. $\pi_1(X) = 1$), $\pi_i(X, x_0)$ does <u>not</u> depend on x_0.

Def. Let $T \subset X$ be a subspace s.t. $x_0 \in T$,

The i-th *relative homotopy group* of X relative to T is:

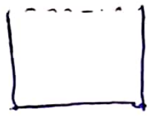
$\pi_i(X, T, x_0) := [(E^i, S^{i-1}, *), (X, T, x_0)]$

(homotopy classes of maps of triples, up to homotopies
$H: E^i \times I \to X$ s.t. $\forall t \in I$ $H_t(S^{i-1}) \subset T$, $H_t(*) = x_0$)

As for the usual π_i, $\exists$ second description with $(I^i, \partial I^i)$

Set $J^{i-1} := \partial I^{i-1} \times I \cup I^{i-1} \times \{0\} \subset \partial I^i$

(union of all faces in ∂I^i except $I^{i-1} \times \{1\}$



J^1

Then $I^i / J^{i-1} \simeq E^i$, $\partial I^i / J^{i-1} \simeq S^{i-1}$. Thus

$$\pi_i(X, T, x_0) \simeq [(I^i/J^{i-1}, \partial I^i / J^{i-1}, *), (X, T, x_0)]$$
$$\simeq [(I^i, \partial I^i, J^{i-1}), (X, T, x_0)]$$

Rem. 1) This description gives $\pi_i(X, \{x_0\}, x_0) \simeq \pi_i(X, x_0)$

2) If $T \neq \{x_0\}$, $\pi_1(X, T, x_0)$ = homotopy classes of paths from x_0 to $t_0 \in T$ $\Rightarrow$ no group structure!

3) But for $i \geq 2$ $\exists$ group law def'd like for usual $\pi_i(X, x_0)$'s. It is commutative for $i \geq 3$.

Lemma. $\exists$ canonical isomorphisms $\pi_{i+1}(X, T, x_0) \simeq \pi_i(\Omega X, \Omega T, x_0)$.

Proof. Notice: $E^i \wedge S^1 \simeq E^{i+1}$

[Indeed, $E^i \simeq I^i / J^{i-1}$ $S^1 = I / \{0,1\}$ and
$J^{i-1} \times I \cup I^i \times \{0,1\} = J^i \ (\subset \partial I^i)$]

So $\pi_i(\Omega X, \Omega T, x_0) \simeq [(E^i, S^{i-1}, x_0), (\Omega X, \Omega T, x_0)$

$\simeq [(SE^i, S(S^{i-1}), x_0), (X, T, x_0)]$

$= [(E^{i+1}, S^i, x_0), (X, T, x_0)] = \pi_{i+1}(X, T, x_0)$

[need relative version of adjoint isom]

Prop. $\exists$ long exact sequence

$$\cdots \to \pi_i(T, x_0) \to \pi_i(X, x_0) \to \pi_i(X, T, x_0) \to$$
$$\to \pi_{i-1}(T, x_0) \to \cdots \to \pi_0(T, x_0) \to \pi_0(X, x_0)$$

[for $i \leq 1$ only of pointed sets]

Proof. To define the maps, we use 2^{nd} description:

$\pi_i(X,T,x_0) \xrightarrow{\partial} \pi_{i-1}(T,x_0)$ defined by

$f: (I^i, \partial I^i, J^{i-1}) \to (X,T,x_0) \rightsquigarrow f\big|_{I^{i-1}\times\{1\}}$ and

$\pi_i(X,x_0) = \pi_i(X,\{x_0\},x_0) \xrightarrow{j_*} \pi_i(X,T,x_0)$ by

$\{x_0\} \hookrightarrow T$. So using Lemma enough to check exactness of

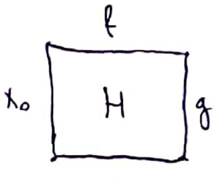
$$\pi_1(T,x_0) \xrightarrow{i_*} \pi_1(X,x_0) \xrightarrow{j_*} \pi_1(X,T,x_0) \xrightarrow{\partial} \pi_0(T,x_0) \xrightarrow{i_*} \pi_0(X,x_0)$$

We first check this is a complex. $i_* \circ \partial = 0$ and $\partial \circ j_* = 0$ easy.

$\underline{j_* \circ i_* = 0}$: A class in $\pi_1(T,x_0)$ is represented by $f: I \to T$, $f(0)=f(1)=x_0$.
Consider $H: I\times I \to T$, $(s,t) \mapsto f(s+t(1-s))$. Then
$H(0,t) = f(t)$, $H(1,t) = f(1) = x_0 \Rightarrow H$ is a homotopy of
f to the constant path $I \to \{x_0\}$.

[Note: This does not mean $[f]=0$ in $\pi_1(T,x_0)$ because for general s
$H(s,0) = f(s) \neq x_0$!]

<u>Exactness</u> at $\pi_1(X,x_0)$: Suppose $f: I \to X$, $f(0)=f(1)=x_0$ is homotopic
to $I \xrightarrow{c} \{x_0\}$ via a homotopy $H: I\times I \to X$ s.t.
$H(0,t) = f(t)$, $H(1,t) = x_0$ $\forall t$ and $H(s,0) = x_0$, $H(s,1) =: g(s) \in T$



Since $H(0,1) = f(0,1) = x_0$, $H(1,1) = x_0 \Rightarrow g: I \to X$ is a loop <u>in T</u>
And $i_*([g]) = [f]$ because $I\times I$ contractible $\Rightarrow f\cdot g^{-1}$ is
homotopic to $c: I \to \{x_0\}$.

<u>Exactness</u> at $\pi_1(X,T,x_0)$: suppose $f: I \to X$ with $f(1) = x_0$, $f(0) \in T$
satisfies $\partial([f]) = 0$. Then $\exists\ g: I \to T$: $g(1) = f(1)$, $g(0) = x_0$.
Here $g\cdot f$ is a loop around x_0, and is homotopic to
f via

$$H(s,t) := \begin{cases} g(1-t+s(1+t)) & 0 \le s \le \frac{t}{1+t} \\ f(s(1+t)-t) & \frac{t}{1+t} \le s \le 1 \end{cases}$$

Indeed, $H(s,0) = f(s)$, $H(s,1) = (g\cdot f)(s)$
$H(1,t) = f(1) = x_0$, $H(0,t) = g(1-t) \in T$.

A nontrivial result (without proof):

<u>Homotopy</u> <u>excision thm</u> (Blakers - Massey) Let $X = U_1 \cup U_2$, U_i open. $p, q \geq 1$, $u \in U_1 \cap U_2$.

If $\pi_i(U_1, U_1 \cap U_2, u) = 0$ $\quad 0 < i < p$ and $\quad$ [no cond. for $p=1, q=1$]

$\pi_i(U_2, U_1 \cap U_2, u) = 0$ $\quad 0 < i < q$,

then the natural map $\pi_i(U_2, U_1 \cap U_2, u) \to \pi_i(X, U_1, u)$ is an isomorphism for $1 \leq i < p+q-2$, and a surjection for $i = p+q-2$.

We apply the thm with $X = S^n$, $U_i = E^n_{\pm}$ [closed upper and lower hemispheres], $U_1 \cap U_2 = S^{n-1}$ [equator]. Note that this is <u>not</u> an open covering, but $E^n_+ \sim_{hom} S^n \setminus \{\text{south pole}\}$, $E^n_- \sim_{hom} S^n \setminus \{\text{north pole}\}$, $E^n_+ \cap E^n_- \sim_{hom} S^n \setminus \{\text{two poles}\}$, and homotopy groups are unchanged by homotopy equivalences [By defn.]

<u>Theorem</u>. We have $\pi_i(S^n) = \pi_{i+1}(E^{n+1}_+, S^n) = 0 \quad 0 \leq i < n$.

<u>Proof</u>. Look at the relative homotopy sequence

$$\cdots \to \pi_{i+1}(E^n_+) \to \pi_{i+1}(E^n_+, S^{n-1}) \to \pi_i(S^{n-1}) \to \pi_i(E^n_+)$$

Here $\pi_i(E^n_+) = 0 \quad i \geq 0$ as E^n_+ is contractible. So

$$\pi_{i+1}(E^n_+, S^{n-1}) \xrightarrow{\sim} \pi_i(S^{n-1}) \qquad i \geq 0.$$

Similarly, the sequence

$$\cdots \to \pi_i(E^n_-) \to \pi_i(S^n) \to \pi_i(S^n, E^n_-) \to \pi_{i-1}(E^n_-)$$

gives $\pi_i(S^n) \xrightarrow{\sim} \pi_i(S^n, E^n_-) \quad i > 0. \quad (*)$

Now we prove $\pi_{i+1}(E^{n+1}_+, S^n) = 0 \quad 0 \leq i < n$ by induction on n.

$n = 1, 2$ OK [$n=2$ e.g. by Hurewicz] Apply excision thm, assuming thm holds up to n:

$$\left.\begin{array}{ll} \pi_i(E^{n+1}_+, S^n) = 0 & 0 \leq i < n \\ \pi_i(E^{n+1}_-, S^n) = 0 & 0 \leq i < n \end{array}\right\} \Rightarrow$$

$$\begin{array}{ccc} \pi_i(E^{n+1}_+, S^n) & \xrightarrow{\sim} & \pi_i(S^{n+1}, E^{n+1}_-) \\ \parallel & & \wr\!\wr \ (*) \\ 0 \quad i \leq n & & \pi_i(S^{n+1}) \end{array} \qquad \begin{array}{l} 1 \leq i < 2n-2 \\ \text{[so for } i < n+1 \\ \text{if } n \geq 2 \text{]} \end{array}$$

[induction] $\Rightarrow$ $\pi_i(S^{n+1}) \simeq \pi_{i+1}(E^{n+2}_+, S^{n+1}) = 0 \quad i \leq n$.

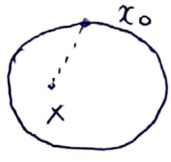
Let (X, x_0) be a pointed space. The suspension map

$\Sigma: \pi_i(X, x_0) \to \pi_{i+1}(SX, x_0)$ is defined by:

$$f: (S^i, *) \to (X, x_0) \rightsquigarrow Sf: S^1 \wedge S^i \to S^1 \wedge X$$

Lemma. Σ is a homomorphism.

Proof. We use the important observation:
If $CX := I \wedge X := I \times X / \{0\} \times X \cup I \times \{x_0\}$ is the cone of X, the embedding $X \hookrightarrow CX$, $x \mapsto (1, x)$ induces $CX/X \xrightarrow{\simeq} SX$.
Moreover, the map $(t, x) \mapsto tx + (1-t)x_0$ [x_0 = north pole] induces $CS^i \simeq E^{i+1}$, in the case $X = S^i$.



Since $\forall X$ CX is contractible [already $I \times X / \{0\} \times X$ is] the relative homotopy sequence

$$\cdots \to \pi_{i+1}(CX) \to \pi_{i+1}(CX, X) \to \pi_i(X) \to \pi_i(CX)$$

induces $\pi_{i+1}(CX, X) \xrightarrow{\simeq} \pi_i(X)$ $\forall i \geq 1$. Here
$f: S^i \to X$ corresponds to $Cf: (E^{i+1}, S^i) \to (CX, X)$ by the above.
Now Σ equals the composite of 3 homomorphisms:

$$\pi_i(X) \xrightarrow{\simeq} \pi_{i+1}(CX, X) \to \pi_{i+1}(CX/X, x_0) \xrightarrow{\simeq} \pi_{i+1}(SX),$$

because Cf is sent to Sf.

Freudenthal suspension theorem: Σ is an isomorphism
$\pi_i(S^n, *) \xrightarrow{\simeq} \pi_{i+1}(S^{n+1}, *)$ for $i \leq 2n-2$, and a surjection $i = 2n-1$.

Proof. Identify Σ with $\pi_i(S^n) \xleftarrow{\simeq} \pi_{i+1}(E^{n+1}, S^n) \xrightarrow{\alpha} \pi_{i+1}(E^{n+1}/S^n, *)$ as above. The map α factors as

$$\underbrace{\pi_{i+1}(E^{n+1}_-, S^n) \to \pi_{i+1}(S^{n+1}, E^{n+1}_+)}_{\text{isom. in this range [p. 36]}} \underbrace{\to}_{\substack{\text{homotopy equiv.}\\ \text{of pairs as}\\ E^{n+1} \text{ contractible}}} \pi_{i+1}(S^{n+1}/E^{n+1}_+, *)$$

homeomorphism $E^{n+1}_-/S^n \xrightarrow{\simeq} S^{n+1}/E^{n+1}_+$

Cor. $\pi_n(S^n) \simeq \mathbb{Z}$ $\forall n > 0$.

Proof. By thm $\pi_2(S^2) \xrightarrow{\simeq} \pi_3(S^3) \xrightarrow{\simeq} \cdots$ and $\mathbb{Z} = \pi_1(S^1) \twoheadrightarrow \pi_2(S^2)$.
So enough to prove: $\pi_2(S^2) = [S^2, S^2]^0$ is infinite.
But $\exists$ natural map $[S^2, S^2]^0 \to \mathrm{Hom}(H_2(S^2) \to H_2(S^2)) \xrightarrow{\simeq} \mathbb{Z}$
which is onto because $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$ extends to a degree n map $(S^2, \infty) \to (S^2, \infty)$ [to check this, restrict to unit circle]

Homotopy of CW complexes

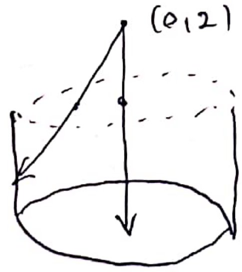
Def. Let $i : T \hookrightarrow X$ be a subspace. i is a cofibration if
$\forall\, f : X \to Y,\ h : T \times I \to Y$ with $h|_{T\times\{0\}} = f|_T$
$\exists\, H : X \times I \to Y$ s.t. $H|_{X\times\{0\}} = f,\ H|_{T\times I} = h$.
[Homotopy extension property.]

Lemma. If $T \subset X$ is closed, i is a cofibration $\iff$ the inclusion $X \times \{0\} \cup T \times I \subset X \times I$ has a retraction.

Pf. $\Leftarrow$: define $g : X\times\{0\} \cup T \times I \to Y$ by $g|_{X\times\{0\}} = f$, $g|_{T\times I} = h$. T closed $\Rightarrow$ g cont. If
$r : X \times I \to X \times\{0\} \cup T \times I$ is a retraction $\Rightarrow$
$g \circ r : X \times I \to Y$ extends h.

$\Rightarrow$: Set $Y = X\times\{0\} \cup T \times I$, $f : X \hookrightarrow X\times\{0\}$, $h = id_{T\times I}$.
The cofibration property gives a retraction.

Ex. $S^{n-1} \hookrightarrow E^n$ is a cofibration. Proof:



Projection from $(0,2)$ gives a retraction
$E^n \times I \to E^n \times \{0\} \cup S^{n-1} \times I$.

Lemma. Let $T \hookrightarrow X$ be a cofibration, $g : T \to Y$,

$$\begin{tikzcd} T \arrow[r, hook, "i"] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r, hook] & Z \end{tikzcd}$$

$Z = Y \cup X / (t \sim g(t))$

Then $Y \hookrightarrow Z$ is a cofibration.

Proof. The above diagram is a pushout $\Rightarrow$ so is the diagram

$$\begin{tikzcd} T \times I \arrow[r, "i\times id"] \arrow[d, "g\times id"'] & X \times I \arrow[d] \arrow[dd, bend left] \\ Y \times I \arrow[r] \arrow[dr, bend right] & Z \times I \arrow[d, dashed] \\ & W \end{tikzcd}$$

Given $f : Z \to W$, $Y \times I \to W$ restricting to f on $Y \times \{0\}$, the composite $T \times I \to Y \times I \to W$ extends to $X \times I \to W$ as i is a cofibration. But then $\exists$ extension $Z \times I \to W$.

Cor. If X is a CW complex, $X^n \hookrightarrow X$ is a cofibration.

Pf.
$$\begin{tikzcd} \bigcup S^{n} \arrow[r] \arrow[d] & \bigcup E^{n+1} \arrow[d] \\ X^n \arrow[r] & X^{n+1} \end{tikzcd}$$
Since $S^n \to E^{n+1}$ cofibration $\Rightarrow X^n \to X^{n+1}$ cofibration by Lemma.

Composition of cofibrations is a cofibration $\Rightarrow X^n \hookrightarrow X^m$ cofibration $\forall m > n$. As $X = \bigcup X^m$, we conclude using induction.

Rem. More generally, $T \hookrightarrow X$ is a <u>subcomplex</u> if $T \subset X$ closed, and a union of cells. (X, T) is called a <u>CW pair</u>.
In this case $T \hookrightarrow X$ is a cofibration (same proof).

Def. A pair of path-connected spaces (X, T) is <u>n-connected</u> if $\pi_i(X, T, x_0) = 0 \quad \forall x_0 \in T, \quad 0 \le i \le n$

Rem. A path-connected space is n-connected if $\pi_i(X, x_0) = 0 \quad 0 \le i \le n$
$n = 1 \Leftrightarrow$ simply connected.

Lemma. A map $f: (E^i, S^{i-1}, *) \to (X, T, x_0)$ represents $0 \in \pi_i(X, T, x_0) \Leftrightarrow f$ is homotopic (rel. S^{i-1}) to a map $E^i \to T$.
i.e. H constant on S^{i-1}.

Pf. $\Rightarrow$: Suppose $G: (E^i \times I, S^{i-1} \times I, x_0 \times I) \to (X, T, x_0)$ is a homotopy of f to the constant map $E^i \to x_0$.
Define $H: E^i \times I \to X$ by

$$H(x,t) := \begin{cases} G\left(\frac{2x}{2-t}, t\right) & 0 \le \|x\| \le \frac{2-t}{2} \\ G\left(\frac{x}{\|x\|}, 2 - 2\|x\|\right) & \frac{2-t}{2} \le \|x\| \le 1 \end{cases}$$

Then $H(x,0) = G(x,0) = f$, $H(x,1) \subset T \quad \forall x \in E^i$
and $\|x\| = 1 \Rightarrow H(x,t) = G(x,0) = H(x,0)$, i.e. H constant on S^{i-1}.

$\Leftarrow$: If $f \sim_{\text{hom}} g$ rel S^{i-1}, $g(E^i) \subset T$, with homotopy h,
set $H: (E^i \times I, S^{i-1} \times I), * \times I) \to (X, T, x_0)$ by

$$H(x,t) = \begin{cases} h(x, 2t) & 0 \le t \le \frac{1}{2} \\ g((2-2t)x + (2t-1)*) & \frac{1}{2} \le t \le 1 \end{cases}$$

Then $H(x,0) = f$, $H(x,1) = g(*) = x_0$.

Cor. 1) (X,T) is n-connected $\Leftrightarrow$ $\forall f: (E^i, S^{i-1}) \to (X,T)$ is homotopic (rel. S^{i-1}) to a map $E^i \to T$.

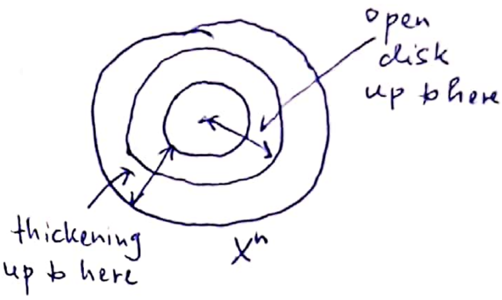
2) Given $X \supset T_1 \supset T_2$, if (X,T_1) and (T_1,T_2) n-connected $\Rightarrow$ (X,T_2) is n-connected [follows from 1)]

Example. (E^n, S^{n-1}) is n-connected [computation of π_i of spheres above]

Prop. If X is a CW complex, then (X, X^n) is n-connected.

Proof. Let $U_1 :=$ union of open disks, one in each open $(n+1)$-cell of X

$U_2 :=$ thickening of X^n inside X^{n+1}.



Then $X^{n+1} = U_1 \cup U_2$ is an open covering, and the pair $(U_1, U_1 \cap U_2)$ is homotopic to $\bigcup_{\alpha \in \Lambda} (E^{n+1}_\alpha, S^n_\alpha)$ where Λ is indexing $(n+1)$-cells.

As recalled in the above Example, $\pi_i(U_1, U_1 \cap U_2) = 0$ $\underline{0 \le i \le n+1}$.

The homotopy excision thm (applied with $p = n+2, q = 1$) gives $\pi_i(X^{n+1}, U_2) = \pi_i(X^{n+1}, X^n) = 0$ as $U_2 \underset{hom}{\sim} X^n$.

If $m > n$, Cor. 2) above + induction gives $\pi_i(X^m, X^n)$ $0 \le i \le n$

By a compactness argument [$\forall f: E^i \to X$ $f(E^i) \subset X^m$ $m \gg 0$]

$\pi_i(X, X^n) = 0$ $0 \le i \le n$.

Def. A map $X \to Y$ of CW complexes is <u>cellular</u> if $f(X^i) \subset Y^i$ $\forall i$.

Thm (Cellular Approximation Theorem) Every cont. map of CW complexes is homotopic to a cellular map.

Proof. By induction on n we construct homotopies $H^n: X \times I \to Y$ s.t.

a) $H^0(x,0) = f(x)$, $H^{n-1}(x,1) = H^n(x,0)$ $n \ge 1$

b) $H^n(x^i, 1) \subset Y^i$ $\forall x^i \in X^i$ $0 \le i \le n$

c) H^n is constant on X^{n-1}.

If $X = X^n$ for $n \gg 0$, this completes the proof.

Otherwise:

$$H(x,t) := \begin{cases} H^i(x, 2^{i+1}(t-1-2^{-i})) & 1-2^{-i} \le t \le 1-2^{-i-1} \\ H^i(x,1) & x \in X^i,\ t=1. \end{cases}$$

This is continuous on $\forall X^i \times I$ (by c)) $\Rightarrow$ also on $X \times I$.

We now define X^n by induction on n.

$\boxed{n=0}$ $\forall y \in f(X^0)$ choose a path from y to a fixed $y^0 \in Y^0$ in the path component of y. This defines a homotopy of $f|_{X^0}$ to a map $X^0 \to Y^0$. Since $X^0 \hookrightarrow X$ is a cofibration, we may extend the homotopy to X, giving a homotopy of f with a map $\tilde{f}$ s.t. $\tilde{f}(X^0) \subset Y^0$.

$\boxed{n>0}$ By induction we may assume $f(X^i) \subset Y^i$ $i<n$.
Let $\Phi_\alpha : (E^n_\alpha, S^{n-1}_\alpha) \to (X, X^{n-1})$ be the char. map of an n-cell.
Then $f \circ \Phi_\alpha : (E^n_\alpha, S^{n-1}_\alpha) \to (Y, Y^{n-1}) \subset (Y, Y^n)$.
Since $\pi_n(Y, Y^n) = 0$ [previous Prop.], $f \circ \Phi_\alpha$ is homotopic (rel. S^{n-1}_α) to a map $(E^n_\alpha, S^{n-1}_\alpha) \to (Y^n, Y^{n-1})$. Doing this for $\forall \alpha$, we get: $f|_{X^n}$ is homotopic to a map $X^n \to Y^n$. Since $X^n \hookrightarrow X$ is a cofibration, as above the homotopy extends to X and is constant on X^{n-1} by construction.

<u>Rem.</u> In the above argument we used that X is a CW complex, but for Y only that $\pi_n(Y, Y^n) = 0$. So the same proof gives:

<u>Prop.</u> Let X be a CW complex, $f : X \to Y$ a continuous map, $T \subset Y$ is a subspace s.t. $\pi_i(Y,T) = 0$ $\forall i$. Then f is homotopic to a map $X \to T$. Moreover, if $\tilde{X} \subset X$ is a subcomplex s.t. $f(\tilde{X}) \subset T$, we may choose homotopies constant on $\tilde{X}$.

<u>Theorem</u> (Whitehead) If $f : X \to Y$ is a cont. map of <u>CW complexes</u> inducing isomorphisms $\pi_i(X,*) \xrightarrow{\sim} \pi_i(Y,*)$ $\forall i$, then f is a homotopy equivalence.

<u>Rem.</u> This does <u>not</u> say that if X, Y are CW complexes s.t. $\pi_i(X) \simeq \pi_i(Y)$ $\forall i$, then $X \sim_{\text{hom}} Y$! [$\exists$ counterexamples with different $H^\sharp_i$.] But:

Cor. If X is a CW complex s.t. $\pi_i(X,*)=0$ $\forall i$
$\Rightarrow$ X is contractible

Proof: Apply Thm to $X \to *$.

The proof of Thm uses:

Construction. Let $f: X \to Y$ be a continuous map of top. spaces.
The mapping cylinder of f is
$M_f := X \times I \cup Y / (x,1) \sim f(x)$

So M_f is the cylinder $X \times I$ with Y attached "at the top" via f.
The map $X \to X \times \{0\} \to M_f$ is an embedding.
The map $Y \to M_f$ has a retraction $r: M_f \to Y$ sending $X \times I$ to $X \times \{1\}$, and the composite $M_f \xrightarrow{r} Y \to M_f$ is homotopic to id_Y [deform $X \times I$ to $X \times \{1\}$].
Finally, $f: X \to Y$ factors as $X \hookrightarrow M_f \xrightarrow{r} Y$.

Lemma. If f is a cellular map of CW complexes, then M_f is a CW complex, and $X \subset M_f$ a subcomplex.

Proof. $X \times I$ has a CW decomposition with cells $e^i \times \{0\}$, $e^i \times \{1\}$, $e^i \times (0,1) \simeq e^{i+1}$. We define a CW structure on M_f by attaching cells inductively. $n=0$ obvious. If M_f^{n-1} has been constructed, attach all cells $e^n \times \{0\}$, $e^{n-1} \times (0,1)$ from $X \times I$ and all n-cells of Y whose interior does not meet $f(X)$. If $E^n_Y \subset Y$ is a closed n-cell whose interior e^n_Y meets $f(X)$, $\exists$ closed n-cell E^n_X s.t. for the interior $f(e^n_X) \subset e^n_Y$ (as e^n_X is connected).
But then $f(\partial E^n_X) \subset \partial E^n_Y$ as f is cellular. So $f(E^n_X) \subset E^n_Y$ is a compact subset s.t. $f(\partial E^n_X) \subset \partial E^n_Y$. This is only possible if $f(E^n_X) = E^n_Y$. We attach one copy of $f(E^n_X) = E^n_Y$.
Finally we attach all $e^n \times \{1\}$ not used.

Proof of Whitehead's thm: As $M_f \xrightarrow{r} Y$ is a homotopy equivalence, we may replace Y by M_f and assume $X \hookrightarrow Y$ is a subcomplex [this uses the Lemma!]

The long exact sequence of relative homotopy

$$\cdots \to \pi_i(X,*) \to \pi_i(Y,*) \to \pi_i(Y,X,*) \to \pi_{i-1}(X,*) \to \cdots$$

then implies $\pi_i(Y,X,*) = 0 \ \forall i$. Then Prop. before Thm gives: id_Y is homotopic to a map $Y \to X$, with homotopies constant on X. So X is a deformation retract of Y, and we are done.

The last big thm on homotopy of CW complexes is:

<u>Theorem</u> (CW Approximation Theorem) Let T be a CW complex, $k \geq -1$. Assume given a path-connected top. space Y and a cont. map $f: T \to Y$ s.t. $\pi_i(T,*) \to \pi_i(Y,f(*))$ is an isom. $i<k$ & surjective $i=k$.

$\boxed{\forall n > k \text{ even } n=\infty}$ Then $\exists$ CW complex X containing T as a subcomplex and $F: X \to Y$ extending f s.t. $\pi_i(X,*) \to \pi_i(Y,F(*))$ is an isom. $i<n$ and surjective $i=n$.

Moreover, X is obtained by attaching cells of dim. $k \leq d \leq n$ to X.

<u>Def.</u> A continuous map of top. spaces $f: X \to Y$ is a <u>weak equivalence</u> if it induces isomorphisms $\pi_i(X,*) \xrightarrow{\sim} \pi_i(Y,f(*)) \ \forall i, *$.

<u>Cor. 1.</u> If Y is an arbitrary path-conn. space, $\exists$ CW complex X and a map $F: X \to Y$ s.t. F is a weak equivalence.

<u>Proof.</u> In the thm choose $k=-1$, T = point.

A map F as in the corollary is called a <u>CW replacement</u> of Y.

<u>Prop.</u> A CW replacement is unique up to homotopy equivalence. Moreover, $Y \mapsto$ homotopy class of CW replacement is a functor $\{\text{path-conn. spaces}\} \longmapsto \{\text{CW completes up to homotopy}\}$

<u>Proof.</u> Let $X_1 \to Y_1$, $X_2 \to Y_2$ be CW replacements, and let $Y_1 \xrightarrow{g} Y_2$ be a continuous map. We show: $\exists$ cont. map $X_1 \xrightarrow{f} X_2$ making

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

commute up to homotopy.

Setting $Y_1 = Y_2$ and $g = id$ will imply the first statement by Whitehead's theorem.

By replacing $X_2 \to Y_2$ by its mapping cylinder, we may assume $X_2 \hookrightarrow Y_2$. Since $\pi_i(X_2) \xrightarrow{\sim} \pi_i(Y_2)$, (X_2, Y_2) is γ-connected $\forall i$. But then By Prop. before Whitehead Thm the composite $X_1 \to Y_1 \to Y_2$ is homotopic to a map $X_1 \to X_2$, q.e.d.

For functoriality one needs to show uniqueness of f up to homotopy. We omit the proof (proceeds by refinement of arguments above).

<u>Proof of Thm.</u> We shall attach cells of dim. $> k$ to T. This will not change π_i for $i \leq k$, because we may represent el^ts of $\pi_i(X)$ by <u>cellular</u> maps $S^i \to X$ by cellular approximation thm $\Rightarrow$ their image lies in T. We proceed by induction on n. $n = 0$ OK.

$\boxed{n=1}$ If $\pi_0(T) \neq 0$, we can make T path-connected by attaching 1-cells connecting path-components. Then all $\pi_1(T, t_0)$ are isomorphic when t_0 varies, So enough to attach 1-cells to T to obtain X s.t. f extends to $X \to Y$ with $\pi_1(X, t_0) \twoheadrightarrow \pi_1(Y, y_0)$ [here $t_0 \in T$ fixed.] Let $\{\gamma_\alpha : \alpha \in \Lambda\}$ be generators of $\pi_1(Y, y_0)$. Given $\alpha \in \Lambda$, attach a 1-cell E^1_α to T with endpoints $= t_0$, and extend f to $T \cup e^1_\alpha$ via $\gamma_\alpha : E^1_\alpha \to Y$.

$\boxed{n>1}$ By induction on n we may assume (after replacing $X \rightsquigarrow T$): $\pi_i(T) \to \pi_i(Y)$ is an isom. $i < n-1$ and a surjection $i = n-1$. We may assume $T \hookrightarrow Y$ [mapping cylinder]. Choose a system of generators $(\Phi_\alpha, \varphi_\alpha) : (E^n, S^{n-1}, *) \to (Y, T, t_0)$ for $\pi_n(Y, T, *)$.

As above, define $X = T \cup \bigcup_\alpha E^n_\alpha$, with $\forall E^n_\alpha$ attached to T via $\varphi_\alpha : S^{n-1}_\alpha \to T$. Then the Φ_α extend $f : T \to Y$ to $F : X \to Y$. Consider now

$$\begin{tikzcd}
\pi_n(T) \arrow[r] \arrow[d, "id"] & \pi_n(X) \arrow[r] \arrow[d, "F"] & \pi_n(X,T) \arrow[r] \arrow[d, two heads, "f_T"] & \pi_{n-1}(T) \arrow[r] \arrow[d, "id"] & \pi_{n-1}(X) \arrow[r] \arrow[d, "F"] & \overset{0\ [\text{construction}]}{\cancel{\pi_{n-1}(X,T)}} \\
\pi_n(T) \arrow[r] & \pi_n(Y) \arrow[r] & \pi_n(Y,T) \arrow[r] & \pi_n(T) \arrow[r] & \pi_{n-1}(Y) \arrow[r] & \overset{0\ [\text{induction}]}{\cancel{\pi_{n-1}(Y,T)}}
\end{tikzcd}$$

f_T surjective by construction. Now a diagram chase gives $F : \pi_n(X) \to \pi_n(Y)$ surjective, and ~~then~~ $F : \pi_{n-1}(X) \to \pi_{n-1}(Y)$ injective.

Here is what CW replacements may be good for.

<u>Prop:</u> If $f: X \to Y$ is a weak equivalence, it induces isomorphisms

$H_i(X) \xrightarrow{\sim} H_i(Y) \quad \forall i.$

<u>Sketch of proof.</u> We may again assume $X \hookrightarrow Y$ using the mapping cylinder.
$\forall n \geq 0$ we define <u>Eilenberg subcomplexes</u> $S_\bullet^{(n)}(Y,X) \subset S_\bullet(Y,X)$ as follows:
Let $S_i^{(n,X)}(Y) \subset S_i(Y)$ be the subgroup generated by $\sigma: \Delta^i \to Y$ that map all faces of dim $< n$ of Δ^i to X. Then $S_\bullet^{(n,X)}(Y) \subset S_\bullet(Y)$ is a subcomplex, and so is $S_\bullet^{(n)}(Y,X) := S_\bullet^{(n,X)}(Y) / S_\bullet(X) \hookrightarrow S_\bullet(Y,X)$.

<u>Fact:</u> If (Y,X) is $(n-1)$-connected, then $S_\bullet^{(n)}(Y,X) \hookrightarrow S_\bullet(Y,X)$ is a chain homotopy equivalence. [Proven by constructing an explicit chain homotopy.

By assumption our (Y,X) is n-connected $\forall n \Rightarrow H_i(Y,X) \simeq H_i(S_\bullet^{(n)}(Y,X))$ $\forall i,n$. But by construction $S_i^{(n)}(Y,X) = 0 \quad i < n$. So $H_i(Y,X) = 0 \quad \forall i$
$\Rightarrow H_i(X) \simeq H_i(Y) \quad \forall i$, by the long exact relative homology sequence.

<u>Application:</u> <u>In principle</u> one can compute $\forall H_i(X)$ for X path-connected by taking a CW replacement $\tilde{X} \to X$ and computing $H_i(\tilde{X}) = H_i^{CW}(\tilde{X})$.

While at this topic, let us mention without proof:

<u>General Hurewicz theorem:</u> Let X be a path-connected space.
There is a functorial homomorphism $\pi_n(X, x_0) \xrightarrow{h_n} H_n(X)$.
If moreover X is $(n-1)$-connected, then h_n is an isomorphism, and $\tilde{H}_i(X) = 0 \quad i < n$.

The map is constructed like for $n = 1$: given $f: (S^n, *) \to (X, x_0)$, f induces $f_*: \underbrace{H_n(S^n)}_{\cong \mathbb{Z}} \to H_n(X_0)$. Now define $h_n(f) := f_*(1)$.

It depends only on the homotopy class of f.
Of course, one also has to prove that h_n is a homomorphism.
This was already nonobvious for $n = 1$.

The Hopf invariant

This invariant will be a map $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$.
When $n = 2k$ is even, we'll prove it has infinite image $\Rightarrow$ for $n = 2k$ $\pi_{2n-1}(S^n)$ is infinite.

Theorem (Serre) For $n = 2k$ $\pi_{2n-1}(S^n) \simeq \mathbb{Z} \oplus$ finite abelian group.
These groups and $\pi_n(S^n) \simeq \mathbb{Z}$ [n arbitrary] are the only infinite homotopy groups of spheres.
So at least we'll have seen all the infinite $\pi_i(S^n)$'s. We need.

Def. Let $f : X \to Y$ be a continuous map. The mapping cone $C(f)$ is obtained from the mapping cylinder $M(f)$ by contracting $X \times \{0\} \subset M(f)$ to a point.

Rem. Recall : $M(f) = X \times I \cup Y \,/\, \{(x,1) \sim f(x)\}$.
If we define the nonreduced cone $\tilde{C}(X)$ by $\tilde{C}(X) := X \times I \,\big|\, X \times \{0\}$
then $C(f) = \tilde{C}(X) \cup Y \,/\, \{(x,1) \sim f(x)\}$. In particular, $\tilde{C}(S^n) \simeq E^{n+1}$ $\Rightarrow$ if $f : S^n \to Y$ and Y is a CW complex,


E^{n+1} $C(f) = Y$ with an $(n+1)$-cell attached via f.

Prop. The maps $f : X \to Y$, $j : Y \to C(f)$ induce a long exact sequence

$$H^0(Y,A) \to H^0(X,A) \to H^1(C(f),A) \to \dots$$
$$\dots \to H^{i-1}(X,A) \to H^i(C(f),A) \xrightarrow{j^*} H^i(Y,A) \xrightarrow{f^*} H^i(X,A) \to \dots$$

Homotopic maps $f_0, f_1 : X \to Y$ induce isomorphic exact sequences.

Rem. In case $f : X \hookrightarrow Y$ we get by comparing with the relative cohomology sequence(*) that $H^i(C(f),A) \simeq H^i(Y,X;A)$
Similarly one can obtain $H_i(C(f)) \simeq H_i(Y,X)$.

Proof. Recall: $X \xrightarrow{\simeq} X \times \{0\} \hookrightarrow M(f) \leftrightarrows Y$ homotopy equiv.

The long exact sequence of rel. cohomology for $X \subset M(f)$ gives

$$\dots \to H^{i-1}(X,A) \to H^i(M(f),X;A) \to H^i(M(f),A) \to H^i(X,A)$$

$$\begin{array}{c} H^i(M(f),A) \\ \cong \\ H^i(Y,A) \end{array} \xrightarrow{f^*} H^i(X,A)$$

(*) Or by following the proof

We show: the natural map $p: M(f) \to C(f)$ collapsing $X \times \{0\}$ to P induces $H^i(M(f), X; A) \xleftarrow{\simeq} H^i(C(f), P; A)$ $[\simeq H^i(C(f), A)\ i>0]$.

Let $\bar{X} := X \times [0, 1/2] \subset M(f)$, $\bar{P} := p(\bar{X}) \subset C(f)$.

p induces an isomorphism of pairs $(M(f)\setminus X, \bar{X}\setminus X) \xrightarrow{\simeq} (C(f)\setminus\{P\}, \bar{P}\setminus\{P\})$

We have a commutative diagram (forgetting the coeff's A)

$$\begin{tikzcd}
H^i(M(f), X) & H^i(M(f), \bar{X}) \arrow[l, "\simeq"', "X \text{ def. ret. of } \bar{X}"] \arrow[r, "\simeq", "\text{excision}"'] & H^i(M(f)\setminus X, \bar{X}\setminus X) \\
H^i(C(f), P) \arrow[u, "p^*"] & H^i(C(f), \bar{P}) \arrow[l, "\simeq", "P \text{ def. retr. of } \bar{P}"'] \arrow[u, "p^*"'] \arrow[r, "\simeq", "\text{excision}"'] & H^i(C(f)\setminus\{P\}, \bar{P}\setminus\{P\}) \arrow[u, "\simeq", "p^*"']
\end{tikzcd}$$

whence the claim. If f_0, f_1 homotopic via $h: X \times I \to Y$

$$\begin{tikzcd}
X \arrow[r, "x \mapsto (x,0)"] \arrow[d, "f_0"] & X \times I \arrow[d, "h"] & X \arrow[l, "(x,1) \leftarrow x"'] \arrow[d, "f_1"] \\
Y \arrow[r, "id"] & Y & Y \arrow[l, "id"']
\end{tikzcd}$$

induces the isomorphism of exact sequences as the upper maps are homotopy equivalences.

<u>Construction</u>. Fix generators $x \in H^{2n-1}(S^{2n-1}, \mathbb{Z})$, $y \in H^n(S^n, \mathbb{Z})$.

Given $f: S^{2n-1} \to S^n$, the above coh. sequence gives

$$\cdots \to H^{2n-1}(S^n, \mathbb{Z}) \to H^{2n-1}(S^{2n-1}, \mathbb{Z}) \to H^{2n}(C(f), \mathbb{Z}) \to H^{2n}(S^n, \mathbb{Z})$$

$$\cdots \to H^{n-1}(S^{2n-1}, \mathbb{Z}) \to H^n(C(f), \mathbb{Z}) \to H^n(S^n, \mathbb{Z}) \to H^n(S^{2n-1}, \mathbb{Z})$$

Since $H^i(S^j, \mathbb{Z}) = 0$ $i \neq 0, j$, for $n \geq 2$ we get

$$\mathbb{Z} \simeq H^{2n-1}(S^{2n-1}, \mathbb{Z}) \xrightarrow[\simeq]{\partial} H^{2n}(C(f), \mathbb{Z})$$

$$H^n(C(f), \mathbb{Z}) \xrightarrow[\simeq]{j^*} H^n(S^n, \mathbb{Z}) \simeq \mathbb{Z}$$

So $\partial(x)$, $j^{*-1}(y)$ are ~~both~~ generators of $H^{2n}(C(f), \mathbb{Z})$, $H^n(C(f), \mathbb{Z})$ respectively $\Rightarrow \exists H(f) \in \mathbb{Z}$:

$$j^{*-1}(y) \cup j^{*-1}(y) = H(f) \cdot \partial(x).$$

This $H(f)$ is the <u>Hopf invariant</u> of (the homotopy class of) f.

(uses 2nd statement of previous Prop)

Rem. 1) If $g: S^n \to S^n$ has degree m, $H(g\circ f) = m^2 H(f)$
If $g: S^{2n-1} \to S^{2n-1}$ has degree m, $H(f\circ g) = m \cdot H(f)$.

2) If n is odd, $H(f) = 0$ by anticommutativity of cup-product.

Prop. If n is even, $\exists f: S^{2n-1} \to S^n$ such that $H(f) = \pm 2$.

Cor. $\pi_{2n-1}(S^n)$ is infinite for n even [use Remark 1)]

Rem. a) Already the case $n=2$ is nontrivial (this was the first case done by Hopf)
b) It can be shown: $H: \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism.

Proof of Prop. We have seen:
$S^n \vee S^n := (S^n \times *) \cup (* \times S^n) \subset S^n \times S^n$ is the $(2n-1)$-skeleton of $S^n \times S^n$ (for the CW structure coming from $* \cup S^n = S^n$)
So $S^n \times S^n = S^n \vee S^n$ + a $2n$-cell attached via an attaching map $g: S^{2n-1} \to S^n \vee S^n$. Let
$h: S^n \vee S^n \to S^n$ defined by $h(x,*) = h(*,x) = x \quad \forall x \in S^n$
Set $\boxed{f := h\circ g}$. This defines a class $[f] \in \pi_{2n-1}(S^n)$.
We have seen: $C(g) = S^n \vee S^n$ + $2n$-cell attached via g $= S^n \times S^n$.
By previous HW, $\{1\otimes y, y\otimes 1\}$ is a basis of $H^n(S^n \times S^n, \mathbb{Z}) \simeq \mathbb{Z}\oplus\mathbb{Z}$
and $(1\otimes y)\cup(y\otimes 1) = y\otimes y$ is a basis of $H^{2n}(S^n\times S^n) \simeq \mathbb{Z}$.
We have a commutative diagram for the induced map $\tilde{h}: C(g) \to C(f)$

$$\begin{array}{ccc} H^n(S^n\vee S^n, \mathbb{Z}) & \overset{j_g^*}{\underset{\sim}{\longleftarrow}} & H^n(S^n\times S^n, \mathbb{Z}) \, [\simeq H^n(C(g), \mathbb{Z})] \\ {\scriptstyle h^*}\uparrow & & \uparrow{\scriptstyle \tilde{h}^*} \\ H^n(S^n, \mathbb{Z}) & \overset{j_f^*}{\underset{\sim}{\longleftarrow}} & H^n(C(f), \mathbb{Z}) \end{array}$$

$\tilde{h}^* j_f^{*-1}(y) = j_g^{*-1} h^*(y) = 1\otimes j_g^{*-1}(y) + j_g^{*-1}(y)\otimes 1$ as the composition with the projections is the identity.
By definition $H(f)\,\partial_f(x) = j_f^{*-1}(y) \cup j_f^{*-1}(y) \in H^{2n}(C(f), \mathbb{Z})$
so $H(f)\tilde{h}^*\partial_f(x) = \tilde{h}^* j_f^{*-1}(y) \cup \tilde{h}^* j_f^{*-1}(y) =$
$= (1\otimes j_g^{*-1}(y) + j_g^{*-1}(y)\otimes 1) \cup (1\otimes j_g^{*-1}(y) + j_g^{*-1}(y)\otimes 1) =$
$= \boxed{2\,(j_g^{*-1}(y) \otimes j_g^{*-1}(y))}$ since $j_g^{*-1}(y) \cup j_g^{*-1}(y) = 0$
[because $y\cup y \in H^{2n}(S^n, \mathbb{Z}) = 0$]

On the other hand, we have isomorphisms

$$H^{2n}(C(f),\mathbb{Z}) \xleftarrow[\sim]{\partial_f} H^{2n-1}(S^{2n-1},\mathbb{Z}) \xrightarrow[\sim]{\partial_g} H^{2n}(C(g),\mathbb{Z})$$

One can check: $\partial_g \circ \partial_f^{-1} = \tilde{h}^*$.

Since x is a generator of $H^{2n-1}(S^{2n-1};\mathbb{Z})$

$y \otimes y$ —"— $H^{2n}(S^n \times S^n, \mathbb{Z}) \xrightarrow[\sim]{j_g^{*-1}} H^{2n}(C(g),\mathbb{Z}) \simeq \mathbb{Z}$

$$\begin{array}{c} \partial_g(x) = \boxed{\pm\, j_g^{*-1}(y) \otimes j_g^{*-1}(y)}, \\ \parallel \\ h^* \partial_f(x) \end{array}$$

Now compare the two formulas above.