Theorem. Let $T \subset S^n$ be a subset

$T \simeq S^m \quad 0 \le m \le n-1$

Then $H_0(S^n \setminus T) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & m = n-1 \\ \mathbb{Z} & \text{o.w.} \end{cases}$

$\boxed{i > 0} \quad H_i(S^n \setminus T) \simeq \begin{cases} \mathbb{Z} & i = n-m-1 \\ 0 & \text{o.w.} \end{cases}$

Special case (Jordan - Brouwer) $T \simeq S^{n-1}$

$\Rightarrow S^n \setminus T$ has 2 (path) components

[path comp = comp as $S^n \setminus T$ loc. path connected because T is compact]

Jordan's case: $n = 2$, $\mathbb{R}^2$ instead of S^2

but if $S^2 \setminus T$ has 2 comp's, so does $\mathbb{R}^2 \setminus T$

[~~loc. path connectedness~~ !] [$U \subset S^2$ connected open $\Rightarrow U \setminus \{x\}$ too (think of open ball)]

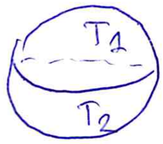
Prop. If $Y \subset S^n$, $I^m \simeq Y$ (m-cube) $0 \le m \le n \Rightarrow$

$H_i(S^n \setminus Y) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$

Prop $\Rightarrow$ Thm: Induction on m.

$m = 0$: $S^n \setminus T = \mathbb{R}^n \setminus \{pt\} \underset{\text{homotopic}}{\simeq} S^{n-1}$

use previous thm.



$m > 0$: $T = T_1 \cup T_2 \quad T_i \cong I^m$

[hemispheres] $\quad T_1 \cap T_2 \cong S^{m-1}$

Mayer-Vietoris for $S^n \setminus (T_1 \cap T_2) = (S^n \setminus T_1) \cup (S^n \setminus T_2)$:

$\ldots \to \underbrace{H_{i+1}(S^n \setminus T_1) \oplus H_{i+1}(S^n \setminus T_2)}_{0 \text{ [Prop]}} \to H_{i+1}(S^n \setminus (T_1 \cap T_2)) \to H_i(S^n \setminus T)$

$\ldots \to \underbrace{H_i(S^n \setminus T_1) \oplus H_i(S^n \setminus T_2)}_{0\ [i>0]\ \ \mathbb{Z} \oplus \mathbb{Z}\ [i=0]} \to H_i(S^n \setminus T_1 \cap T_2)$

known by induction

[$H_1(S^n \setminus (T_1 \cap T_2)) \simeq \begin{cases} \mathbb{Z} & m = n-1 \\ 0 & \text{ow} \end{cases}$

q.e.d.

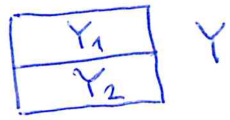
$H_0(S^n \setminus (T_1 \cap T_2)) \simeq \mathbb{Z}$ in this case]

Proof of Prop. We prove $\tilde{H}_i(S^n \setminus Y) = 0 \ \forall i$ for $Y \simeq I^m$ by induction on m.

$\underline{m=0}$: $S^n \setminus Y = S^n \setminus \{pt\} \simeq \mathbb{R}^n$ which is contractible ✓

$\underline{m>0}$: Write $Y = Y_1 \cup Y_2$ [2 closed half-cubes]

$Y_1 \cap Y_2 = I^{m-1}$



Mayer-Vietoris for $S^n \setminus (Y_1 \cap Y_2) = (S^n \setminus Y_1) \cup (S^n \setminus Y_2)$:

$$\underbrace{H_{i+1}(S^n \setminus (Y_1 \cap Y_2))}_{\substack{\| \\ 0 \ [\text{induction}]}} \to H_i(S^n \setminus Y) \to H_i(S^n \setminus Y_1) \oplus H_i(S^n \setminus Y_2)$$

So $H_i(S^n \setminus Y) \hookrightarrow H_i(S^n \setminus Y_1) \oplus H_i(S^n \setminus Y_2)$.

This also gives $\tilde{H}_0(S^n \setminus Y) \hookrightarrow \tilde{H}_0(S^n \setminus Y_1) \oplus \tilde{H}_0(S^n \setminus Y_2)$

[write $\tilde{H}_0(S^n \setminus Y) = \ker[H_0(S^n \setminus Y) \to \underbrace{H_0(P)}_{\cong \mathbb{Z}}]$ for $P \in Y_1 \cap Y_2$.]

Assume $\tilde{H}_i(S^n \setminus Y)$ contains some $\alpha \neq 0$.

Then the image of α is $\neq 0$ in $\tilde{H}_i(S^n \setminus Y_1)$ or $\tilde{H}_i(S^n \setminus Y_2)$

Set $Y^1 := Y_1$ or Y_2 (where $\alpha \neq 0$) and repeat the process by cutting Y^1 in two. Get a chain of compact subsets $Y \supset Y^1 \supset Y^2 \supset \dots$ $\bigcap_j Y^j = \{\text{point}\}$ s.t. the image of α in $\tilde{H}_i(S^n \setminus Y^j)$ ~~was~~ is $\neq 0 \ \forall j$. But $\tilde{H}_i(S^n \setminus \{\text{point}\}) = 0$!

We derive a contradiction by a "compactness argument".

α is represented by a $\mathbb{Z}$-lin. combination of maps $\Delta_i \to S^n \setminus Y$. As Δ_i is compact, $\exists\, C \subset S^n \setminus Y$ s.t. α is in the image of $\tilde{H}_i(C) \to \tilde{H}_i(S^n \setminus Y)$, $\tilde{\alpha} \mapsto \alpha$. Similarly, since $\tilde{H}_i(S^n \setminus \{pt\}) = 0$,

we find $C' \supset C$ compact, $pt \notin C'$: $\tilde{\alpha}$ maps to 0 in $\tilde{H}_i(C')$.

Since C' compact, $\exists\, j$: $C' \subset S^n \setminus Y^j$. Now

$$\begin{tikzcd}
\tilde{\alpha} \arrow[d, mapsto] & \tilde{H}_i(C) \arrow[r] \arrow[d] & \tilde{H}_i(S^n \setminus Y) \arrow[d] & \alpha \arrow[d, mapsto] & \lightning \\
0 & \tilde{H}_i(C') \arrow[r] & \tilde{H}_i(S^n \setminus Y^j) & \neq 0 &
\end{tikzcd}$$

Cor (Invariance of domain). Let $U \subset \mathbb{R}^n$ be open. Then any injective continuous map $f: U \hookrightarrow \mathbb{R}^n$ is open.

Pf. Assume $x_0 \in U$, and let $B_\varepsilon = \{ x \in \mathbb{R}^n : \|x - x_0\| \leq \varepsilon \}$ be a be a closed ball around x_0 with $B_\varepsilon \subset U$. Set $S_\varepsilon :=$ boundary of B_ε. We claim: $f(B_\varepsilon \setminus S_\varepsilon)$ is an open neighbourhood of $f(x_0)$ in $f(U)$.

$\mathbb{R}^n \setminus f(S_\varepsilon) = \mathbb{R}^n \setminus f(B_\varepsilon) \cup f(B_\varepsilon \setminus S_\varepsilon)$

Both are connected [$\mathbb{R}^n \setminus f(B_\varepsilon)$ by prop] and $\mathbb{R}^n \setminus f(S_\varepsilon)$ has 2 components by Thm $\Rightarrow f(B_\varepsilon \setminus S_\varepsilon)$ is a component $\Rightarrow$ it is open.

Cor (Invariance of dimension) If $U \subset \mathbb{R}^n$ open, $m < n$ $\Rightarrow \nexists$ injective continuous map $U \overset{g}{\hookrightarrow} \mathbb{R}^m$.

Pf. Define an injective cont. map

$$U \xrightarrow{\tilde{g}} \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$$
$$x \mapsto (g(x), 0)$$

$\operatorname{Im}(\tilde{g})$ not open $\lightning$.

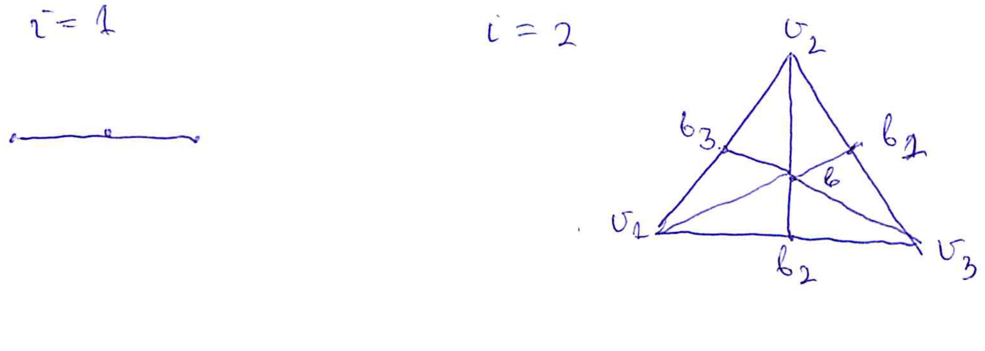
In particular, open subsets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ cannot be homeomorphic for $m \neq n$.

We now prepare for the proof of the small simplices thm.

Construction. The barycenter of the unit simplex Δ_i is $\left(\frac{1}{i+1}, \dots, \frac{1}{i+1}\right)$.

The barycentric subdivision is defined inductively: for $i=1$, divide the interval in two; for $i > 1$ join all vertices of baryc. subdiv. of faces to the barycenter [cone construction!]

$i = 1$ $\qquad$ $i = 2$



Def. Let $T \subset X$ be a subspace. Set

$S_\cdot(X,T) := \mathrm{coker}(S_\cdot(T) \hookrightarrow S_\cdot(X))$

$H_i(X,T) := H_i(S_\cdot(X,T))$ i-th relative homology

$(X,T) \rightsquigarrow S_\cdot(X,T)$ is a functor {pairs of spaces} $\to$ complexes

$(X,T) \rightsquigarrow H_i(X,T)$ are functors — " — $\to$ ab. gps

Lemma. 1) $\exists$ long exact sequence

$$\ldots \to H_{i+1}(X,T) \to H_i(T) \to H_i(X) \to H_i(X,T) \to \ldots$$

2) If $f, g : (X,T) \to (X',T')$ are maps of pairs inducing homotopic maps $X \to X'$, $T \to T'$ $\Rightarrow$ the induced maps $H_i(X,T) \to H_i(X',T')$ are the same. In particular, $H_i(X,T)$ does not change if we replace X or T by a hom. equiv. space.

Pf. Easy.

Excision theorem. Let $T_2 \subset T_1 \subset X$ be s.t. closure$(T_2) \subset$ int(T_1). Then

$$H_i(X \setminus T_2, T_1 \setminus T_2) \xrightarrow{\sim} H_i(X, T_1) \ \forall i.$$

Proof. Apply small simplices thm to $\mathcal{U} = \{T_1, X \setminus T_2\}$.

Note:
$$S_i^{\mathcal{U}}(X) = S_i(T_1) + S_i(X \setminus T_2) \subset S_i(X)$$
$$S_i(T_1 \setminus T_2) = S_i(T_1) \cap S_i(X \setminus T_2)$$

So
$$S_\cdot(X \setminus T_2) / S_\cdot(T_1 \setminus T_2) \simeq S_\cdot^{\mathcal{U}}(X) / S_\cdot(T_1)$$

Apply H_i, use small simplices thm & long exact seq.

We now compute the homology of graphs.

Def. A finite graph is a pair (X, X^0), where X = Hausdorff space, $X^0 \subset X$ finite subset, and

*) $X \setminus X^0 = e_1 \cup \ldots \cup e_n$ $e_i \subset X$ open [edges]

**) closure$(e_j) \setminus e_j = \{x,y\} \subset X^0$ or $\{x\} \subset X^0$ and $(\mathrm{cl}(e_j), e_j) \underset{\text{homeo}}{\simeq} ([0,1], (0,1))$ or $(S^1, S^1 \setminus \{pt\})$

~~Recall:~~ $\cancel{H_i(e_j) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & n \geq 1 \end{cases}}$

If $\partial e_j = \{x\}$
$H_1(\bar{e}_j) \simeq \mathbb{Z}$, $H_0(\partial e_j) \simeq \mathbb{Z}$
$\Rightarrow$ same $H_1(\bar{e}_j, \partial e_j) \simeq \mathbb{Z}$

Write $\partial e_j := \underbrace{cl(e_j)}_{=:\bar{e}_j} \setminus e_j = \{x, y\}$. The rel. hom. seq. gives

$$0 \to \underset{\substack{\wr\!\wr \\ 0}}{H_1(\bar{e}_j)} \to H_1(\bar{e}_j, \partial e_j) \to \underset{\substack{\wr\!\wr \\ \mathbb{Z}\oplus\mathbb{Z}}}{H_0(\partial e_j)} \xrightarrow{+} \underset{\substack{\wr\!\wr \\ \mathbb{Z}}}{H_0(\bar{e}_j)} \to 0$$

$\Rightarrow$ $H_1(\bar{e}_j, \partial e_j) \simeq \mathbb{Z}$ and $H_i(\bar{e}_j, \partial e_j) = 0$ $i \neq 1$
[as $H_i(\bar{e}_j) = H_i(\partial e_j) = 0$ $i > 0$]

<u>Lemma.</u> Let (X, X^0) be a finite graph. Then

$$H_i(X, X^0) = \begin{cases} \mathbb{Z}^n & i = 1 \\ 0 & \text{o.w.} \end{cases} \qquad [n = \# \{\text{edges}\}]$$

<u>Proof.</u> Let $a_j \in \bar{e}_j$ be the midpoint
$d_j \in \bar{e}_j$ the subset corresponding to $[\frac{1}{3}, \frac{2}{3}]$
$A = \{a_1, \dots, a_n\}$ $D = \bigcup d_j$. Consider

$$\begin{tikzcd}
H_i(D, D \setminus A) \arrow[r, "\sim", "\text{excision}"'] & H_i(X, X \setminus A) & H_i(X, X^0) \arrow[l, "\sim"'] \\
H_i(d_j, d_j \setminus \{a_j\}) \arrow[u] \arrow[r, "\sim", "\text{exc.}"'] & H_i(\bar{e}_j, \bar{e}_j \setminus \{a_j\}) \arrow[u] & H_i(\bar{e}_j, \partial e_j) \arrow[u] \arrow[l, "\sim"']
\end{tikzcd}$$

as ∂e_j deformation retract of $\bar{e}_j \setminus \{a_j\}$

But $H_i(D, D \setminus A) \simeq \underset{\substack{j \\ \wr\!\wr \\ \bigoplus_j H_i(d_j, \partial d_j)}}{\bigoplus} H_i(d_j, d_j \setminus \{a_j\}) \simeq \begin{cases} \bigoplus_j \mathbb{Z} & i = 1 \\ 0 & \text{o.w.} \end{cases}$

previous calculation.

<u>Cor.</u> $\exists$ exact sequence

$$0 \to H_1(X) \to \underset{\substack{\wr\!\wr \\ \mathbb{Z}^n}}{H_1(X, X^0)} \to H_0(X^0) \to H_0(X) \to 0$$

and $H_i(X) = 0$ $i > 0$

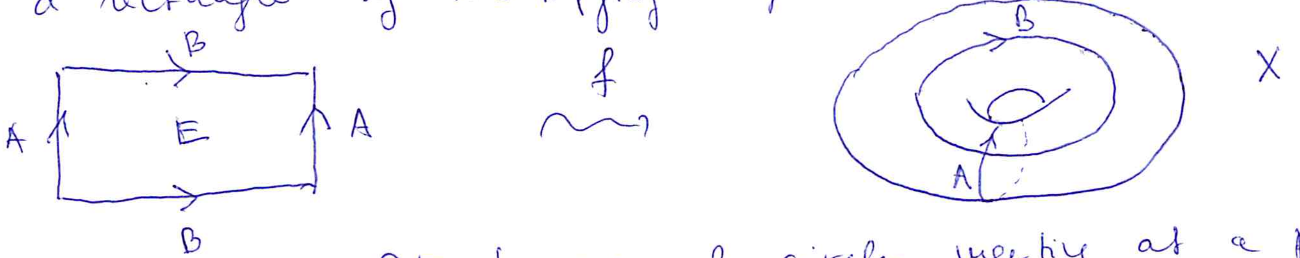
Here $H_0(X^0) \cong \mathbb{Z}^v$ $v = \#$ vertices

$H_0(X) \cong \mathbb{Z}^c$ $c = \#$ of components of X.

Prop. If X is a finite graph, set $\chi(X) := \#\text{vertices} - \#\text{edges}$
Then $H_i(X) = 0$ $i > 1$, and
$H_0(X)$, $H_1(X)$ are free abelian groups, with
$\operatorname{rk} H_0(X) - \operatorname{rk} H_1(X) = \chi(X)$. [$\Rightarrow \operatorname{rk} H_1(X) = c - \chi(X)$]

Proof. $H_1(X)$ is free (subgroup of $\mathbb{Z}^n$), so everything follows by counting ranks.

We now compute the H_i of a torus X. It is obtained from a rectangle E by identifying edges.



The boundary ∂E becomes 2 circles meeting at a point:
$X^1 := f(\partial E) = A \cup B$

Lemma 1 The map $(E, \partial E) \to (X, X^1)$ induces isom.
$H_i(E, \partial E) \xrightarrow{\sim} H_i(X, X^1)$ $\forall i$

Proof. Let $D \subset \operatorname{int}(E)$ be an open disk with center x.
Consider

$$\begin{array}{ccccc}
H_i(E, \partial E) & \xrightarrow{\sim} & H_i(E, E\setminus\{x\}) & \xleftarrow{\sim} & H_i(D, D\setminus\{x\}) \\
\downarrow & & \downarrow & & \downarrow \cong \ f|_D \text{ homeo.} \\
H_i(X, X^1) & \xrightarrow{\sim} & H_i(X, X\setminus\{f(x)\}) & \xleftarrow{\sim} & H_i(f(D), f(D)\setminus f(x)) \\
 & \uparrow & & \uparrow & \\
 & X^1 \text{ def. retract of } X\setminus f(x) & & \text{excision} &
\end{array}$$

Cor.
$$H_i(X, X^1) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & i \neq 2 \end{cases}$$

Pf. We compute $H_i(E, \partial E) \simeq H_i(E^2, S^1)$

$$\cdots \to \underset{\substack{\| \\ 0 \;\; i>1}}{H_i(E^2)} \to H_i(E^2, S^1) \to \underset{\simeq \begin{cases} 0 & i \neq 2 \\ \mathbb{Z} & i = 2 \end{cases}}{H_{i-1}(S^1)} \to \underset{\substack{\| \\ 0 \\ i > 1}}{H_{i-1}(E)}$$

and

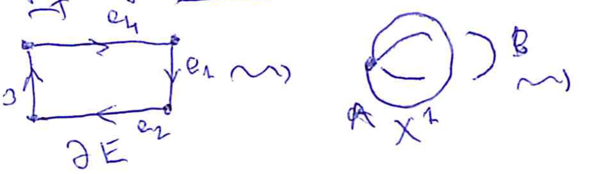
$$0 \to H_2(E^2, S^1) \to H_1(\overset{S^1}{\partial E}) \to \underset{\substack{\| \\ 0}}{H_1(E^2)} \to H_1(E^2, S^1)$$

$$\to \underset{\simeq \mathbb{Z}}{H_0(S^1)} \to \underset{\simeq \mathbb{Z}}{H_0(E^2)} \to \underset{\substack{\| \\ 0}}{H_0(E^2, S^1)} \to 0$$

Lemma 2. The map $H_1(\partial E) \to H_1(X^1)$ is the zero map.

Rem. For an arbitrary graph (X^1, X^0) $H_1(X^1, X^0) \simeq \bigoplus H_1(\bar{e}_j, \partial e_j)$ and $H_1(\bar{e}_j, \partial e_j) \simeq \ker(H_0(\partial e_j) \to H_0(\bar{e}_j))$ if $\partial e_j = \{a, b\}$
so $H_0(\partial e_j) = \mathbb{Z}a \oplus \mathbb{Z}b \Rightarrow H_1(\bar{e}_j, \partial e_j) \simeq \mathbb{Z}(a-b)$ or $\mathbb{Z}(b-a)$ (**)
Choosing $a-b$ or $b-a$ $\Leftrightarrow$ orientation of $\bar{e}_j$.

Pf of Lemma: Consider the map of graphs



$$\begin{tikzcd} 0 \arrow[r] & H_1(E^0) \arrow[r] \arrow[d] & H_1(\partial E) \arrow[r] \arrow[d] & H_1(\partial E, E^0) \arrow[r] \arrow[d] & H_0(E^0) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\ 0 \arrow[r] & H_1(X^0) \arrow[r] & H_1(X^1) \arrow[r] & H_1(X^1, X^0) \arrow[r] & H_0(X^0) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

with $H_1(E^0) \simeq 0$, $H_1(X^0) = 0$.

E^0, X^0: vertices

Counting ranks $\Rightarrow$ $\operatorname{rk} H_1(\partial E) = 1$, gen. by $e_1 + e_2 + e_3 + e_4$
$H_1(X^1) \simeq H_1(X^1, X^0) \simeq \mathbb{Z}A \oplus \mathbb{Z}B$

But $e_1 \to A$, $e_3 \to -A$, $e_2 \to B$, $e_3 \to -B$.

Prop. The homology groups of the torus are

$$H_i(X) \simeq \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 2. \end{cases}$$

(**) If $\partial \bar{e}_j = \{a\}$, $\bar{e}_j \simeq S^1$, and we are choosing the generator 1 or -1 of $H_1(S^1, \{pt\}) \simeq H_1(S^1) \simeq \mathbb{Z}$.

Proof. We compute the H_i of the finite graph X^1:
3 vertices, 4 edges $\Rightarrow$ $\chi = -1$.
X^1 connected $\Rightarrow$ $H_0(X^1) = \mathbb{Z}$, $H_1(X^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $H_i(X^1) = 0$ $i > 1$
So $H_i(X) = 0$ $i > 2$ by Lemma 1 + rel. hom. sequence.
And $\exists$ exact sequence

$$0 \to H_2(X) \to H_2(X, X^1) \xrightarrow{\partial} H_1(X^1) \to H_1(X) \to 0$$

We claim $\partial = 0$. $\exists$ comm. diagram

$$\begin{tikzcd} H_2(X, X^1) \arrow[r, "\partial"] & H_1(X_1) \\ H_2(E, \partial E) \arrow[u, leftrightarrow, "\simeq"] \arrow[r] & H_1(\partial E) \arrow[u, "0"] \end{tikzcd}$$

[Lemma 1] [Lemma 2]

Prop. follows.

Generalization (without proof): If X_g is a torus with g holes,

$$H_i(X_g) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{2g} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

[Method similar.]

CW complexes

Construction. 1) Start with a discrete set X^0 ("0-cells")
2) Define the n-skeleton X^n inductively:
Assume given an index set Λ and cont. maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$ $\forall \alpha \in \Lambda$. Set

$$X^n := X^{n-1} \cup \dot{\bigcup_{\alpha \in \Lambda}} E^n_\alpha \Big/ \sim$$ where

E^n_α = closed n-ball &
$x \sim \varphi_\alpha(x)$ $\forall x \in \partial E^n_\alpha$ $(\simeq S^{n-1})$
3) Stop at some n, or set $X = \bigcup_{n=0}^{\infty} X^n$.
For $n = \infty$ define the weak topology [or colimit top.] on X by
$U \subset X$ open $\overset{def}{\Longleftrightarrow}$ $U \cap X^n$ open $\forall n < \infty$

Def. A space constructed in this way is called a CW complex. If $\exists n: X = X^n$, X is finite dim.

Rem. 1) As a set, $X^n = X^{n-1} \,\dot\cup\, \dot\bigcup_{\alpha \in \Lambda} e^n_\alpha$ [e^n_α: open n-disk]

The e^n_α's are called n-cells, φ_α the attaching map.

There are also characteristic maps $\Phi^n_\alpha: E^n_\alpha \to X^n \subset X$ defined by $E^n_\alpha \hookrightarrow X^{n-1} \,\dot\cup\, \dot\bigcup_\alpha E^n_\alpha \to X^n$.

2) The closure of $\forall e^n_\alpha$ meets only finitely many cells. [For $cl(e^n_\alpha) = \Phi(E^n_\alpha)$ compact. If $B \subset cl(e^n_\alpha)$ contains exactly 1 point from every cell meeting $cl(e^n_\alpha)$ $\Rightarrow B \subset X$ discrete, but $B \subset cl(e^n_\alpha) \Rightarrow B$ finite.]

This explains the name: C (closure finite) + W (weak top.)

Example 1) S^n is a CW complex with 1 0-cell & 1 n-cell [$S^n \setminus \{pt\} \simeq e^n$, attaching map: $S^{n-1} \to \{pt\}$]

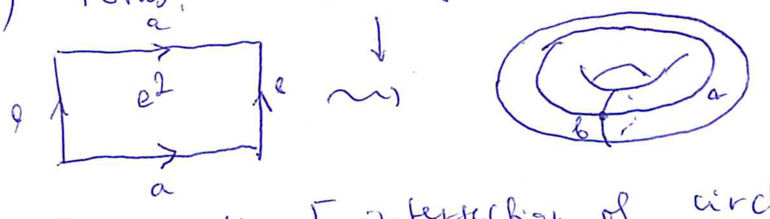
Another one has 2 m-cells $\forall\, 0 \leq m \leq n$.

[n-cells: upper & lower hemisphere. Use induction]

2) E^n is a CW complex with 1 0-cell [point] 1 $(n-1)$-cell [$\partial E^n = S^{n-1}$] and 1 n-cell [e^n]

3) 1-dim. CW complex = graph

4) Torus:



1 0-cell [intersection of circles a & b]

2 1-cells [circles $-\{pt\}$]

1 2-cell [interior of rectangle]

5) Real projective space:

$\mathbb{RP}^n = \mathbb{R}^{n+1} / \sim$ $\qquad (a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ $\overset{def}{\iff} \exists\, \lambda \neq 0: \; a_i = \lambda b_i \;\forall i$.

We have $S^n \subset \mathbb{R}^{n+1}$, and $\mathbb{RP}^n = S^n / \langle v \sim -v \rangle$

Let $E^n_+ \subset S^n$ = closed upper hemisphere. Then

$\mathbb{R}P^n = E^n_+ / \langle v \sim -v : v \in \underbrace{\partial E^n_+}_{S^{n-1}} \rangle$.

So $\mathbb{R}P^n$ = attaching 1 n-cell e^n_+ to S^{n-1}. By induction, $\mathbb{R}P^n$ is a CW complex with 1 m-cell $\forall\, 0 \le m \le n$.

6) $\mathbb{C}P^n = \mathbb{C}^{n+1}/\sim \qquad (a_0, \dots, a_n) \sim (b_0, \dots, b_n) \overset{def}{\iff} \exists \lambda \in \mathbb{C}\setminus\{0\}: a_i = \lambda b_i \ \forall i$.

We have $S^{2n+1} \subset \mathbb{C}^{n+1}$ and

$\mathbb{C}P^n = S^{2n+1} / \langle v \sim \lambda v : |\lambda| = 1 \rangle$

Define $D^{2n}_+ := \{ (w, \sqrt{1 - \|w\|^2}) \in \mathbb{C}^n \times \mathbb{C} \}$

where $\|w\|^2 = \sum |a_i|^2$ for $w = (a_1, \dots, a_n)$

Then $D^{2n}_+ \simeq E^{2n}$ and $\partial D^{2n}_+ = \{ (w, 0) : \|w\| = 1 \} \simeq S^{2n-1}$

$\forall x \in S^{2n+1} \ \exists y \in D^{2n}_+ : x \sim y$, and y is unique if $y \notin S^{2n-1}$, o.w. we restrict $\sim$ to S^{2n-1}. So:

$\mathbb{C}P^n = D^{2n}_+ / \langle v \sim \lambda v : v \in S^{2n-1} \rangle = \mathbb{C}P^{n-1} + e^{2n}_+$ attached via $S^{2n-1} \to \mathbb{C}P^{n-1}$.

By induction, $\mathbb{C}P^n$ is a CW complex with 1 $2m$-cell $\forall m$, $0 \le 2m \le 2n$.

7) $\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n$, $\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$ are ∞-dim. CW complexes.

Proposition. Let X be a CW complex.

$$H_i(X^n, X^{n-1}) = \begin{cases} \mathbb{Z}\langle \Lambda \rangle & i = n \\ 0 & i \ne n \end{cases}$$

[Here we are attaching n-cells to X^{n-1} indexed by Λ.]

Proof. Let $\Phi^n_\alpha : E^n_\alpha \to X^n$ be the char. maps $(\alpha \in \Lambda)$

$\forall \alpha \in \Lambda$ let $D^n_\alpha \subset \operatorname{int}(E^n_\alpha)$ be a closed ball

$D^n := \bigcup_{\alpha \in \Lambda} \Phi_\alpha(D^n_\alpha)$

$a_\alpha := \Phi^n_\alpha(0) \qquad A = \{ a_\alpha : \alpha \in \Lambda \}$

$$\begin{array}{ccccc} H_i(D^n, D^n\setminus A) & \overset{\text{excision}}{\xrightarrow{\simeq}} & H_i(X^n, X^n\setminus A) & \overset{\text{def. ret.}}{\xleftarrow{\simeq}} & H_i(X^n, X^{n-1}) \\ \simeq\uparrow \Phi^n_\alpha & & \uparrow \Phi^n_\alpha & & \uparrow \Phi^n_\alpha \\ \bigoplus_{\alpha\in\Lambda} H_i(D^n_\alpha, D^n_\alpha\setminus\{0\}) & \xrightarrow{\simeq} & \bigoplus_{\alpha\in\Lambda} H_i(E^n_\alpha, E^n_\alpha\setminus\{0\}) & \xleftarrow{\simeq} & \bigoplus_{\alpha\in\Lambda} H_i(E^n_\alpha, S^{n-1}_\alpha) \end{array}$$

But $H_i(E^n, S^{n-1}) = \begin{cases} \mathbb{Z} & i=n \\ 0 & i\neq n \end{cases}$ as rel. hom. seq. gives

$$\cdots \to \underset{\substack{\| \\ 0}}{H_n(E^n)} \to H_n(E^n, S^{n-1}) \to \underset{\substack{\cong \\ \mathbb{Z}}}{H_{n-1}(S^{n-1})} \to \underset{\substack{\| \\ 0}}{H_{n-1}(E^n)}$$

for $n>1$ and the end was computed before.

<u>Cor.</u> We have isomorphisms $H_i(X^{n-1}) \xrightarrow{\simeq} H_i(X^n)$ $i\neq n, n-1$ and an exact sequence

$$(*) \quad 0 \to H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) \to$$
$$\to H_{n-1}(X^n) \to 0.$$

<u>Cor.</u> 1) If X is a fin. dim. CW complex with finitely many n-cells $\Rightarrow H_n(X)$ is fin. gen.

2) If $X = X^n$, $H_i(X) = 0$ $i>n$

3) If X is a fin. dim. CW complex with no n-cells $\Rightarrow H_n(X) = 0$.

<u>Proof.</u> 2) Induction on n. $n=0$ ✓ If known for $n-1$, $\boxed{i>n-1}$ $H_i(X^{n-1}) = 0$ by induction and ~~$H_n(X^n, X^{n-1}) = 0$ by assumption. Apply $(*)$~~. $H_i(X^{n-1}) \simeq H_i(X^n)$ $i>n$.

1) Apply $(*)$ and use $H_n(X^{n-1}) = 0$ by 2).

3) Similar to 1). [Use next prop.]

The following prop. allows one to remove the finiteness assumption from 1), 3).

Prop. If $i<n$, X a CW complex, $i_n : X^n \hookrightarrow X$ induces $H_i(X^n) \xrightarrow{\sim} H_i(X)$.

Pf. We know: $H_i(X^{i+1}) \simeq H_i(X^{i+2}) \simeq \dots$

If $\dim X < \infty$, $\exists\, n>i : X^n = X$ and we are done.

O.w. use compactness argument: every $\alpha \in H_i(X)$ comes from some $H_i(C)$, $C \subset X$ compact. But C meets finitely many cells $\Rightarrow$ α comes from $H_i(X^n)$ for some $n>i$ as $C \subset X^n$ $n \gg 0$. Similarly, if $\beta \in H_i(X^n)$, $\beta \mapsto 0$ in $H_i(X)$ $\Rightarrow$ $\exists\, \underset{X^n}{\overset{}{C}} \subset \underset{X}{C'}$ compact: β comes from $H_i(C)$

$$\begin{array}{ccc} C & \subset & C' \\ \cap & & \cap \\ X^n & & X \end{array}$$

and maps to 0 in $H_i(C')$. But $C' \subset X^m$ for some $m>n$ $\Rightarrow$ β maps to 0 in $H_i(X^m) \simeq H_i(X^n) \Rightarrow \beta = 0$.

Construction. Set $C_i^{CW}(X) := H_i(X^i, X^{i-1})$

$X =$ CW complex.

$$d_i^{CW} : H_i(X^i, X^{i-1}) \xrightarrow{\partial_i} H_{i-1}(X^{i-1}) \xrightarrow{j_{i-1}} H_{i-1}(X^{i-1}, X^{i-2})$$

Lemma. $(C_\bullet^{CW}(X), d_\bullet^{CW})$ is a complex of free ab. gp.s.

Pf. Already the composite

$$H_{i-1}(X^{i-1}) \to H_{i-1}(X^{i-1}, X^{i-2}) \xrightarrow{\partial_{i-1}} H_{i-2}(X^{i-2})$$

is 0.

Consider the comm. exact diagram

$$\begin{tikzcd}
 & {[H_i(X^{i-1})=]\,0} \arrow[dr] & & H_i(X^{i+1}) \simeq H_i(X) & & \\
 & & H_i(X^i) \arrow[ur] \arrow[dr, "j_i"] & & & \\
\cdots \arrow[r] & H_{i+1}(X^{i+1}, X^i) \arrow[ur, "\partial_{i+1}"] \arrow[rr, "d_{i+1}^{CW}"] & & H_i(X^i, X^{i-1}) \arrow[rr, "d_i^{CW}"] \arrow[dr, "\partial_i"] & & H_{i-1}(X^{i-1}, X^{i-2}) \\
 & & & & H_{i-1}(X^{i-1}) \arrow[ur, "j_{i-1}"] & \\
 & & & {[H_{i-1}(X^{i-2})=]\,0} \arrow[ur] & &
\end{tikzcd}$$

We have a map

$$\begin{array}{ccc} H_i(X^i) & \to & H_i(X^{i+1}) = H_i(X) \\ \| & & \\ \ker(\partial_i) & & \end{array}$$

But $\ker(\partial_i) = \ker(d_i^{CW})$

and $H_{i+1}(X^{i+1}, X^i) \xrightarrow{\partial_{i+1}} H_i(X^i) \to H_i(X^{i+1})$ is the zero map.

So $\exists$ well-defined map

$$H_i^{CW}(X) := \ker(d_i^{CW}) / \mathrm{im}(d_{i+1}^{CW}) \to H_i(X).$$

<u>Theorem.</u> $H_i^{CW}(X) \to H_i(X)$ is an isomorphism.

<u>Proof.</u> Surjectivity: already $H_i(X^i) \twoheadrightarrow H_i(X^{i+1})$ as $H_i(X^{i+1}, X^i) = 0$

Injectivity: $\ker(H_i(X^i) \to H_i(X^{i+1})) = \mathrm{im}(\partial_{i+1})$.

<u>Example.</u> For $X = \mathbb{CP}^n$ $C_i^{CW}(X)$ is

$$\dots \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \dots$$

So $H_i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & i = 2m \quad 0 \leq m \leq n \\ 0 & \text{ow.} \end{cases}$

$H_i(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

<u>Example.</u> For $X =$ torus $C_i^{CW}(X)$ is

$$0 \to \mathbb{Z} \xrightarrow{d_2^{CW}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1^{CW}} \mathbb{Z} \to 0$$

We have $d_1^{CW} = 0$ because $H_0(X) \simeq H_0^{CW}(X) \simeq \mathbb{Z}$.

But also $d_2^{CW} = 0$ by our previous computation:

$$\begin{array}{ccc} \mathbb{Z} \simeq H_2(E, \partial E) & \longrightarrow & H_1(\partial E) \\ \downarrow \simeq & & \downarrow 0 \\ H_2(X^2, X^1) & \longrightarrow & H_1(X^1) \simeq H_1(X^1, X^0) \\ & & [2 \text{ loops}] \end{array}$$

Similarly, for a torus with g holes $C_i^{CW}(X^g)$ is

$$0 \to \mathbb{Z} \to \mathbb{Z}^{2g} \to \mathbb{Z} \to 0$$

and $d_1^{CW} = d_2^{CW} = 0 \Rightarrow$

$$H_i(X^g) \simeq \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{2g} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & \end{cases}$$

Next goal: compute $H_i(\mathbb{R}P^n)$.

We use the CW decomposition $\mathbb{R}P^n = e^0 \cup e^1 \cup \ldots \cup e^n$

$\rightsquigarrow C_\cdot^{CW}(\mathbb{R}P^n) = \mathbb{Z} \to \mathbb{Z} \to \ldots \to \mathbb{Z}$ [length $n+1$]

So we have to compute the maps $\mathbb{Z} \to \mathbb{Z}$.

$\forall$ homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ is mult. by some $m \in \mathbb{Z}$

Set $m := \deg(\varphi)$.

Lemma. Let $\rho : S^n \to S^n$ be a reflection w. r. to a hyperplane through the origin. Then $\deg(\rho_*) = -1$, where $\rho_* : H_n(S^n) \to H_n(S^n)$

Pf. Induction on n. Consider $n = 1$: $H_1(S_1) \simeq \mathbb{Z}$ with generator an oriented loop which is sent by ρ to its negative. [think of $\pi_1(S^1) \simeq H_1(S^1) \simeq \mathbb{Z}$]

$n > 1$: we used Mayer-Vietoris to compute $H_n(S^n)$:

$$\begin{array}{ccc} H_n(S^n) & \overset{\partial}{\underset{\sim}{\to}} & H_{n-1}(S^{n-1}) \\ {\scriptstyle \rho_*^n} \downarrow & & \downarrow {\scriptstyle \rho_*^{n-1}} \\ H_n(S^n) & \overset{\partial}{\underset{\sim}{\to}} & H_{n-1}(S^{n-1}) \end{array}.$$

So $\deg(\rho_*^n) = \deg(\partial \circ \rho_*^{n-1} \circ \partial^{-1}) = \deg(\partial)\deg(\rho_*^{n-1})\deg(\partial)^{-1}$
$= \deg(\rho_*^{n-1}) = -1$ by induction.

Cor. Let $\alpha : S^n \to S^n$ be the antipodal map. Then $\alpha_* : H_n(S^n) \to H_n(S^n)$ has degree $(-1)^{n+1}$

Pf. α is the composite of $n+1$ reflections.
[$S^n \subset \mathbb{R}^{n+1}$, $\rho_i : (x_0, \ldots, x_n) \mapsto (x_0, \ldots, -x_i, x_{i+1}, \ldots, x_n)$]

Consider now the CW decomposition of S^n with 2 i-cells E^i_+, E^i_- $\forall\, 0 \le i \le n$. The associated $C_\cdot^{CW}$ is

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_n^{CW}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_{n-1}^{CW}} \ldots \xrightarrow{\partial_1^{CW}} \mathbb{Z} \oplus \mathbb{Z}$$ [lenght $n+1$]

We now compute the ∂_i^{CW} in this complex.

Recall: $H_i(S^n, S^{n-1}) \simeq H_i(E^i_+, S^{n-1}) \oplus H_i(E^i_-, S^{n-1}) \simeq \mathbb{Z}\oplus\mathbb{Z}$

The antipodal map $\alpha: S^n \to S^n$ restricts to $\alpha: S^{n-1} \to S^{n-1}$ and is a homeomorphism $\Rightarrow$ if x^i is a generator of $H_i(E^i_+, S^{i-1}) \underset{\cap}{\simeq} \mathbb{Z}$ then $\alpha_* x^i$ generates $H_i(E^i_-, S^{i-1})$.

$H_i(S^i, S^{i-1})$

Lemma. $\partial_i^{CW}(x^i) = \pm(\alpha_* x^{i-1} + (-1)^i x^{i-1})$.

Proof. ∂_i^{CW} is the composite

$$H_i(S^i, S^{i-1}) \xrightarrow{\partial} H_{i-1}(S^{i-1}) \xrightarrow{j} H_{i-1}(S^{i-1}, S^{i-2})$$

$$\Rightarrow \partial_i^{CW} \circ \alpha_* = j\circ\partial\circ\alpha_* = j\circ\alpha_*\circ\partial = (-1)^i j\circ\partial = (-1)^i \partial_i^{CW}.$$

by previous lemma.

[Here use commutativity of

$$\begin{array}{ccc} H_i(S^i, S^{i-1}) & \xrightarrow{\partial} & H_{i-1}(S^{i-1}) \\ \downarrow \alpha_* & & \downarrow \alpha_* \\ H_i(S^i, S^{i-1}) & \xrightarrow{\partial} & H_{i-1}(S^{i-1}) \end{array}$$]

So in particular, $0 \neq \alpha_* x^i + (-1)^{i+1} x^i \in \ker(d_i^{CW})$.

Notice: $\ker(d_i^{CW}) = \operatorname{im}(d_{i+1}^{CW})$ has rank 1 $\forall i$.

This follows by descending induction from $H_i(S^n) = \begin{cases}\mathbb{Z} & i = \\ 0\end{cases}$

In particular, $d_i^{CW}(x^i) \neq 0$ [o.w. $d_i^{CW}\alpha_* x^i = 0$ too and $d_i^{CW} = 0$ ↯]

But then $\alpha_* x^i + (-1)^{i+1} x^i$ generates $\ker(d_i^{CW}) = \operatorname{im}(d_{i+1}^{CW})$

for if $d_i^{CW}(r\cdot\alpha_* x^i + s\cdot x^i) = 0$

$$\| \qquad ((-1)^{i} r + s)\, d_i^{CW}(x^i) \Rightarrow r = (-1)^{i+1} s.$$

But $d_{i+1}^{CW}(x^{i+1})$, $d_{i+1}^{CW}(\alpha_* x^{i+1}) = (-1)^{i+1} d_{i+1}^{CW}(x^{i+1})$ also generate $\operatorname{im}(d_{i+1}^{CW}) = \ker(d_i^{CW})$, whence the lemma.

Theorem.

$$C_\cdot^{CW}(\mathbb{P}^n) = 0 \to \mathbb{Z} \xrightarrow{(-1)^n+1} \mathbb{Z} \xrightarrow{(-1)^{n-1}+1} \mathbb{Z} \to \dots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

up to a sign

So $H_n(\mathbb{P}^n) = \begin{cases} 0 & 2 \mid n \\ \mathbb{Z} & 2 \nmid n \end{cases}$

and $H_i(\mathbb{P}^n) = \begin{cases} \mathbb{Z}/2 & 0 < i < n,\ 2 \nmid i \\ 0 & 2 \mid i \text{ or } i > n. \end{cases}$

<u>Proof.</u> Consider the map $S^n \to \mathbb{P}^n$ identifying antipodes.
It respects CW decompositions $\Rightarrow$ induces
$C_\cdot^{CW}(S^n) \to C_\cdot^{CW}(\mathbb{P}^n)$
In $C_i^{CW}(S^n) = H_i(E_{+1}^n, S^{n-1}) \oplus H_i(E_{-1}^n, S^{n-1})$
both components map via the identity map to

$$C_i^{CW}(\mathbb{P}^n) = H_i(E_{+1}^n, S^{n-1})$$

In particular, $x^i, \alpha_* x^i$ map to the same generator
y^i satisfying $d_i^{CW}(y^i) = \pm(1+(-1)^{i+1})\, y^{i-1}$.

Let now X be a <u>finite</u> CW complex.

Set $n_i(X) := \#$ of i-cells in X

<u>Def.</u> $\chi(X) := \sum (-1)^i n_i(X)$ is the <u>Euler characteristic</u>

<u>Prop.</u> $\chi(X) = \sum (-1)^i \operatorname{rk} H_i(X)$

<u>Pf.</u> Use $C_\cdot^{CW}(X)$ which is a finite complex of free ab. groups

$$n_i(X) = \operatorname{rk} C_i^{CW}(X) = \underbrace{\operatorname{rk} Z_i^{CW}(X)}_{=: z_i} + \underbrace{\operatorname{rk} B_{i-1}^{CW}(X)}_{b_{i-1}}$$

$$\chi(X) = \sum (-1)^i n_i(X) = \sum (-1)^i (z_i + b_{i-1})$$

$$= \sum (-1)^i (z_i - b_i) = \sum (-1)^i \operatorname{rk} H_i(X).$$

<u>Examples</u>

1) For a finite graph this is the same result as before.
2) If $X \subset \mathbb{R}^3$ is [the boundary of] a convex polyhedron,
$\chi(X) =$ classical Euler char. But $X \simeq S^2$
$\Rightarrow \operatorname{rk} H_0 - \operatorname{rk} H_1 + \operatorname{rk} H_2 = 2.$

<u>Conclusion</u>: $\chi(X)$ does <u>not</u> depend on CW decomposition!

Cohomology groups

Def. A cohomological complex is a sequence

$$C^\cdot: \quad \dots \to C^{i-1} \xrightarrow{d_i} C^i \xrightarrow{d_{i+1}} C^{i+1} \to \dots$$

s.t. $d_{i+1} \circ d_i = 0 \ \forall i$. We set $H^i(C^\cdot) := \mathrm{Ker}(d_{i+1}^{i+1}) / \mathrm{Im}(d_i)$

If $C_\cdot$ is a homological complex of abelian groups, A an abelian group, $\mathrm{Hom}(C_\cdot, A)$ is a cohomological complex:

$d_i: C_i \to C_{i-1} \ \rightsquigarrow \ \mathrm{Hom}(C_{i-1}, A) \to \mathrm{Hom}(C_i, A)$, $\varphi \mapsto \varphi \circ d_i$.

Def. If X is a top. space, A an abelian group, $i \geq 0$,

$H^i(X, A) := H^i(\mathrm{Hom}(S_\cdot(X), A))$ i-th cohomology group of X with coeff's in A.

$X \rightsquigarrow H^i(X, A)$ is a contravariant functor

$A \rightsquigarrow H^i(X, A)$ is a covariant functor.

Recall. if $0 \to S_1 \to S_2 \to S_3 \to 0$ is an exact seq. of abelian gps $\Rightarrow$ $0 \to \mathrm{Hom}(S_3, A) \to \mathrm{Hom}(S_2, A) \to \mathrm{Hom}(S_1, A)$ exact, but the last map is not always surjective.

[Ex. $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, $A = \mathbb{Z}$.] But:

Lemma. If $0 \to S_1 \to S_2 \xrightarrow{\varphi} S_3 \to 0$ is an exact seq. of abelian groups with S_3 free $\Rightarrow$

$$0 \to \mathrm{Hom}(S_3, A) \to \mathrm{Hom}(S_2, A) \xrightarrow{\lambda} \mathrm{Hom}(S_1, A) \to 0 \quad \text{exact.}$$

Pf. Let $e_1, e_2, e_3, \dots$ be a free basis of S_3. For $\forall e_i$ choose $f_i \in S_2$: $\varphi(f_i) = e_i$. Sending $e_i \mapsto f_i$ induces a hom. $g: S_3 \to S_2$ with $\varphi \circ g = \mathrm{id}$. So $S_2 \simeq S_1 \oplus g(S_3)$. $\Rightarrow$ $\mathrm{Hom}(S_2, A) \simeq \mathrm{Hom}(S_1, A) \oplus \mathrm{Hom}(g(S_3), A) \Rightarrow \lambda$ surj.

Cor. If $C_\cdot$ is a complex of abelian groups s.t. $H_{i-1}(C_\cdot)$ is free $\Rightarrow \exists$ canonical isom.

$$H^i(\mathrm{Hom}(C_\cdot, A)) \xrightarrow{\simeq} \mathrm{Hom}(H_i(C_\cdot), A)$$

Rem. In general $\exists$ exact sequence [universal coefficient seq.]

$$0 \to \mathrm{Ext}(H_{i-1}(C_\cdot), A) \to H^i(\mathrm{Hom}(C_\cdot, A)) \to \mathrm{Hom}(H_i(C_\cdot), A) \to 0.$$

Proof. ~~$H^i(\mathrm{Hom}(C_\cdot, A))$ is the quotient~~

$\mathrm{Ker}(\mathrm{Hom}(C_i, A) \to \mathrm{Hom}(C_{i+1}, A)) / \mathrm{Im}(\mathrm{Hom}(C_{i-1}, A) \to \mathrm{Hom}(C_i, A))$

If $B_i := \mathrm{Im}(d_{i+1} : C_{i+1} \to C_i)$ and similarly $B_{i-1} = \mathrm{Im}(d_i)$

$\mathrm{Ker}(\mathrm{Hom}(C_i, A) \to \mathrm{Hom}(C_{i+1}, A)) \xrightarrow{\sim} \mathrm{Hom}(C_i / B_i, A)$.

Also, $0 \to B_{i-1} \to C_{i-1} \to C_{i-1}/B_{i-1} \to 0$

$0 \to H_{i-1}(C_\cdot) \to C_{i-1}/B_{i-1} \xrightarrow{d_{i-1}} B_{i-2} \to 0$ show:

B_{i-2} free $\overset{\text{proof of}}{\underset{\text{Lemma}}{\Longrightarrow}}$ $C_{i-1}/B_{i-1} \simeq H_{i-1}(C_\cdot) \oplus B_{i-2}$ free $\overset{\text{Lemma}}{\Longrightarrow}$

$\mathrm{Hom}(C_{i-1}, A) \twoheadrightarrow \mathrm{Hom}(B_{i-1}, A)$. So

$H^i(\mathrm{Hom}(C_\cdot, A)) \simeq \mathrm{Hom}(C_i/B_i, A)) / \mathrm{Im}(\mathrm{Hom}(B_{i-1}, A) \to \mathrm{Hom}(C_i/B_i, A))$.

Now apply $\mathrm{Hom}(\ , A)$ & Lemma to $0 \to H_i(C_\cdot) \to C_i/B_i \to B_{i-1} \to 0$.

Relative cohomology: If $T \subset X$, define $H^i(X, T; A) := H^i(\mathrm{Hom}(S_\cdot(X,T), A))$.

Applying $\mathrm{Hom}(\ , A)$ to $0 \to S_\cdot(T) \to S_\cdot(X) \to S_\cdot(X,T) \to 0$

we get an exact sequence by Lemma as all terms are free.

So we get a long exact rel. cohomology sequence:

$\ldots \to H^i(X, T; A) \to H^i(X, A) \to H^i(T, A) \to H^{i+1}(X, T; A) \to \ldots$

Now let $\mathcal{U}$ be an open covering of X, $S^{\mathcal{U}}_\cdot(X) \hookrightarrow S_\cdot(X)$ as in thm on small simplices.

Prop. The above map induces isomorphisms $H^i(X, A) \xrightarrow{\simeq} H^i(\mathrm{Hom}(S^{\mathcal{U}}_\cdot(X), A))$.

Proof. $0 \to S^{\mathcal{U}}_\cdot(X) \to S_\cdot(X) \to S_\cdot(X)/S^{\mathcal{U}}_\cdot(X) \to 0$ has free terms $\Rightarrow$ remains exact after applying $\mathrm{Hom}(\ , A)$ [Lemma]. Also, $H_i(S_\cdot(X)/S^{\mathcal{U}}_\cdot(X)) = 0$ $\forall i$ by small simplices theorem $\Rightarrow$ $H^i(\mathrm{Hom}(S_\cdot(X)/S^{\mathcal{U}}_\cdot(X), A)) \xrightarrow{\sim} \mathrm{Hom}(H_i(S_\cdot(X)/S^{\mathcal{U}}_\cdot(X)), A) = 0$ by Cor. Now apply long exact cohomology sequence.

Cor. We have Mayer-Vietoris sequences for cohomology:

$\to H^{i-1}(U_1 \cap U_2; A) \to H^i(X, A) \to H^i(U_1, A) \oplus H^i(U_2, A) \to H^i(U_1 \cap U_2, A) \to \ldots$

Rem. We also have homology groups with coeff's in A, defined by $H_i(X, A) := H_i(S_\cdot(X) \otimes A)$. Similar properties hold.

We also have excision for cohomology.

Finally, if X is a CW complex,

$C^{\cdot}_{CW}(X, A) := \mathrm{Hom}(C_{\cdot}^{CW}(X), A)$

computes $H^{\cdot}(X, A)$ [same proof]

For instance, we have

$$H^i(S^n, A) = \begin{cases} A & i = n, 0 \\ 0 & \text{o.w.} \end{cases}$$

$$H^i(\mathbb{CP}^n, A) \simeq \begin{cases} A & 2 \mid i, \ i \leq 2n \\ 0 & 2 \nmid i \text{ or } i > 2n \end{cases}$$

$$H^i(\mathbb{RP}^n, \cancel{A}\ \mathbb{Z}/2) \simeq \begin{cases} \mathbb{Z}/2 & i \leq n \\ 0 & i > 0 \end{cases}$$

Cup-products

Let A be a comm. ring. We'll define A-bilinear product maps $H^i(X, A) \times H^j(X, A) \xrightarrow{\cup} H^{i+j}(X, A)$ satisfying nice properties. In this way

$$H^*(X, A) := \bigoplus_{i=0}^{\infty} H^i(X, A)$$

will have a non-commutative ring structure. In fact it will be a graded ring i.e. a non-comm. ring H^* s.t. $H^* = \bigoplus H^i$ as abelian group [or A-module] and the product sends $H^i \times H^j$ to H^{i+j}.

We first need.

Def. Let $C_\cdot$, $D_\cdot$ be homological complexes.

We define a complex $C_\cdot \otimes D_\cdot$ by

$$(C_\cdot \otimes D_\cdot)_n := \bigoplus_{p+q=n} C_p \otimes D_q$$

$d_n : (C_\cdot \otimes D_\cdot)_n \to (C_\cdot \otimes D_\cdot)_{n-1}$ defined by

$$d_n(c \otimes d) := d_C(c) \otimes d + (-1)^p c \otimes d_D(d)$$

for $c \in C_p$, $d \in D_q$ and extending linearly.

One checks $d_{n-1} \circ d_n = 0$ [one gets $d_C(c) \otimes d_D(d)$ with signs $(-1)^p$ and $(-1)^{p-1}$ $\Rightarrow$ they cancel out]

Rem. If $C^\cdot, D^\cdot$ are cohom. complexes, $C^\cdot \otimes D^\cdot$ is defined in the same way.

Lemma. If $C_\cdot$ is contractible (i.e. $id_{C_\cdot}$ is chain homotopic to 0) $\Rightarrow$ $C_\cdot \otimes D_\cdot$ is contractible $\forall D_\cdot$.

Pf. Assume $id_{C_\cdot} = d_C k + k d_C$ with $k: C_\bullet \to C_{\bullet+1}$.

Define $\bar{k}: (C \otimes D)_\bullet \to (C \otimes D)_{\bullet+1}$ by $\bar{k}(c \otimes d) = k(c) \otimes d$.

$d\bar{k}(c \otimes d) = d(k(c) \otimes d) = d_C k(c) \otimes d + (-1)^{p+1} k(c) \otimes d_D(d)$

$\bar{k}d(c \otimes d) = k(d_C(c)) \otimes d + (-1)^p k(c) \otimes d_D(d) \leftarrow$ if $c \in C_p$

$\Rightarrow (d\bar{k} + \bar{k}d)(c \otimes d) = id_{C_\cdot}(c) \otimes d = c \otimes d$, q.e.d.

Cor. If X is a contractible top. space $\Rightarrow$ $S_\cdot(X) \otimes S_\cdot(X)$ contractible complex.

Theorem (Eilenberg-Zilber) Let X be a top. space.
The natural map
$$S_0(X \times X) \xrightarrow{EZ_0} S_0(X) \otimes S_0(X)$$
$$(P, Q) \mapsto P \otimes Q$$
extends to a morphism of complexes
$$EZ: S_\cdot(X \times X) \to S_\cdot(X) \otimes S_\cdot(X)$$
Any two extensions are chain homotopic.

Proof. We construct EZ functorially in X.

Construction of EZ_1: Notice that
$H_0(\Delta^1 \times \Delta^1) \simeq H_0(S_\cdot(\Delta^1) \otimes S_\cdot(\Delta^1)) \simeq \mathbb{Z}$
[e.g because Δ^1 is contractible + Lemma] $\Rightarrow$ the diagram

$$\begin{tikzcd}
S_1(\Delta^1 \times \Delta^1) \arrow[r] & S_0(\Delta^1 \times \Delta^1) \arrow[r] \arrow[d] & H_0(\Delta^1 \times \Delta^1) \arrow[r] \arrow[d] & 0 \\
(S_\cdot(\Delta^1) \otimes S_\cdot(\Delta^1))_1 \arrow[r] & (S_\cdot(\Delta^1) \otimes S_\cdot(\Delta^1))_0 \arrow[r] & H_0(S_\cdot(\Delta^1) \otimes S_\cdot(\Delta^1)) \arrow[r] & 0
\end{tikzcd}$$

commutes.

The diagonal map $\Delta^1 \to \Delta^1 \times \Delta^1$ gives a class $\delta^1 \in S_1(\Delta^1 \times \Delta^1)$ whose image in $S_0(\Delta^1) \otimes S_0(\Delta^1)$ maps to 0 in H_0 $\Rightarrow$ comes from an elt $\alpha \in (S_\cdot(\Delta^1) \otimes S_\cdot(\Delta^1))_1$

Set $EZ_1(\delta^1) := \alpha$ [α is not unique, we make a choice!]

Now given $\sigma : \Delta^1 \to X \times X$, it factors as $\Delta^1 \xrightarrow{\delta^1} \Delta^1 \times \Delta^1 \to X \times X$

[to see this, compose with the projections $p_1, p_2 : X \times X \to X$]

Also, set $\sigma_i := p_i \circ \sigma : \Delta^1 \to X$ $i = 1,2$ and finally

$EZ_1(\sigma) := (\sigma_1 \otimes \sigma_2)(EZ(\delta^1)) \in (S.(X) \otimes S.(X))_1$.

Construction of EZ_n, $n > 1$: assume EZ_i $(i < n)$ has been constructed, functorially. Consider

$$\begin{tikzcd}
S_n(\Delta^n \times \Delta^n) \arrow[r] & S_{n-1}(\Delta^n \times \Delta^n) \arrow[r] \arrow[d, "EZ_{n-1}"] & S_{n-2}(\Delta^n \times \Delta^n) \arrow[d, "EZ_{n-2}"] \\
(S.(\Delta^n) \otimes S.(\Delta^n))_n \arrow[r] & (S.(\Delta^n) \otimes S.(\Delta^n))_{n-1} \arrow[r] & (S.(\Delta^n) \otimes S.(\Delta^n))_{n-2}
\end{tikzcd}$$

As before, the diagonal $\Delta^n \to \Delta^n \times \Delta^n$ gives $\delta^n \in S_n(\Delta^n \times \Delta^n)$ whose image in $(S.(\Delta^n) \otimes S.(\Delta^n))_{n-2}$ is 0. But Δ^n contractible $\Rightarrow$ $S.(\Delta^n) \otimes S.(\Delta^n)$ contractible by Lemma $\Rightarrow$ the lower row is exact $\Rightarrow$ the image of δ^n in $(S.(\Delta^n) \otimes S.(\Delta^n))_{n-1}$ comes from an elt of $(S.(\Delta^n) \otimes S.(\Delta^n))_n$. Call it $EZ_n(\delta^n)$.

Finally, if $\sigma : \Delta^n \to X \times X$ is an n-simplex, set

$EZ_n(\sigma) := (\sigma_1 \otimes \sigma_2)(EZ_n(\delta^n)) \in (S.(X) \otimes S.(X))_n$.

Homotopy: Assume $\varphi, \psi : S.(X \times X) \to S.(X) \otimes S.(X)$ both extend EZ_0 to a morphism of complexes. We construct inductively $k_i : S_i(X \times X) \to (S.(X) \otimes S.(X))_{i+1}$, functorially in X, s.t. $d_{i+1} \circ k_i + k_{i-1} \circ d_i = \varphi_i - \psi_i$ $\forall i$. $(*)$

Set $k_0 = 0$ and assume k_i has been constructed for $i < n$.

$$d_n \circ (\varphi_n - \psi_n - k_{n-1} \circ d_n) = \varphi_{n-1} \circ d_n - \psi_{n-1} \circ d_n - (d_n \circ k_{n-1}) \circ d_n \overset{(*)}{=} \varphi_{n-1} \circ d_n - \psi_{n-1} \circ d_n - (\varphi_{n-1} \circ d_n - \psi_{n-1} \circ d_n - k_{n-2} \circ \underbrace{d_{n-1} \circ d_n}_{0}) = 0.$$

So $(\varphi_n - \psi_n - k_{n-1} \circ d_n)(\delta^n) \in Z_n(S.(\Delta^n) \otimes S.(\Delta^n))$ $\Rightarrow$

$\Rightarrow$ is equal to $d_{n+1}(\beta)$ for some $\beta \in (S.(\Delta^n) \otimes S.(\Delta^n))_{n+1}$

again because $S.(\Delta^n) \otimes S.(\Delta^n)$ is contractible.

Here δ^n is as before, and we set $k_n(\delta^n) := \beta$ and $k_n(\sigma) := (\sigma_1 \otimes \sigma_2)\, k_n(\delta^n)$ for a general $\sigma: \Delta^n \to X \times X$.

Cor. [Full Eilenberg-Zilber thm] $S.(X) \otimes S.(X)$ is chain homotopy equivalent to $S.(X \times X)$.

Pf. Since $S.(\Delta^n \times \Delta^n)$ is contractible, we may reverse the argument to construct functorial maps $S.(X) \otimes S.(X) \to S.(X \times X)$ extending $P \otimes Q \mapsto (P, Q)$. As the compositions $S.(X \times X) \to S.(X) \otimes S.(X) \to$ $\to S.(X \times X)$ and $S.(X) \otimes S.(X) \to S.(X \times X) \to S.(X) \otimes S.(X)$ are functorial, identity in degree 0 and all complexes are contractible $\Rightarrow$ they are chain homotopic to the identity.

To construct cup-products, we need one more easy lemma.

Lemma. If $C^\cdot, D^\cdot$ are cohomological complexes, the natural maps $C^i \otimes D^j \to (C^\cdot \otimes D^\cdot)^{i+j}$ induce maps
$$H^i(C^\cdot) \otimes H^j(D^\cdot) \to H^{i+j}(C^\cdot \otimes D^\cdot) \quad \forall i, j.$$

Proof. Recall: $d(a \otimes b) = da \otimes b + (-1)^i a \otimes db$.
So if $a \in Z^i(C^\cdot)$, $b \in Z^j(D^\cdot) \Rightarrow a \otimes b \in Z^{i+j}(C^\cdot \otimes D^\cdot)$
Also, if $a = da' \in B^i(C^\cdot)$, $b \in Z^j(D^\cdot) \Rightarrow a \otimes b = da' \otimes b = d(a' \otimes b)$ is in $B^{i+j}(C^\cdot \otimes D^\cdot)$. So we indeed have an induced map.

Construction. Let X be a top. space, A, B abelian groups.
$S.(X) \xrightarrow{\Delta} S.(X \times X) \xrightarrow{EZ} S.(X) \otimes S.(X)$ induces
$$\mathrm{Hom}(S.(X), A) \otimes \mathrm{Hom}(S.(X), B) \to \mathrm{Hom}(S.(X) \otimes S.(X), A \otimes B)$$
$$\to \mathrm{Hom}(S.(X), A \otimes B)$$
Lemma $\Big\}$
$$H^i(X, A) \otimes H^j(X, B) \to H^{i+j}(X, A \otimes B)$$
[these do not depend on the choice of EZ any more]

If $A = B$ is a ring, composing with $m: A \otimes A \to A$ gives $H^i(X, A) \otimes H^j(X, A) \to H^{i+j}(X, A)$

The construction is functorial in X.

Rem. An explicit EZ map is given as follows.
Define maps $\lambda_i, \mu_i : \Delta^i \to \Delta^n$ by
$\lambda_i(t_0, \dots, t_i) := (t_0, \dots, t_i, 0, \dots, 0)$ "front face"
$\mu_i(t_0, \dots, t_i) := (0, \dots, 0, t_0, \dots, t_i)$ "back face"
Now for $\sigma : \Delta^n \to X \times X$ define
$$AW_n(\sigma) := \sum_{i+j=n} (\sigma_1 \circ \lambda_i) \otimes (\sigma_2 \circ \mu_j) \in (S.(X) \otimes S.(X))_n$$
[Alexander - Whitney map.] Recall: $\sigma_i = p_i \circ \sigma$
$p_i : X \times X \to X \quad i = 1,2$

A calculation shows: $AW_. : S.(X \times X) \to S.(X) \otimes S.(X)$
is a functorial morphism of complexes with
$AW_0(P, Q) = P \otimes Q$. So it is an EZ map.
This gives an explicit formula for the cup-product:
if $a : S_i(X) \to A$, $b : S_j(X) \to B$ represent
classes in $H^i(X, A)$, $H^j(X, B)$, $a \cup b$ is represented by
$a \cup b : S_{i+j}(X) \to A \otimes B$, $(\sigma : \Delta^{i+j} \to X) \mapsto a(\sigma \circ \lambda_i) \otimes b(\sigma \circ \mu_j)$

Prop. If $a \in H^i(X, A)$, $b \in H^j(X, B)$,
$$a \cup b = (-1)^{ij}\, b \cup a \in H^{i+j}(X, A \otimes B)$$

Lemma. If $C.$ is a homological complex,
$\tau : C. \otimes C. \to C. \otimes C.$, $\tau(a \otimes b) = (-1)^{ij}\, b \otimes a$
$[a \in C_i, b \in C_j]$ defines a morphism of complexes.

Proof.
$$\tau(d(a \otimes b)) = \tau(da \otimes b + (-1)^i a \otimes db)$$
$$= (-1)^{(i-1)j}\, b \otimes da + (-1)^{i + i(j-1)}\, db \otimes a$$

$$d(\tau(a \otimes b)) = (-1)^{ij}\, d(b \otimes a)$$
$$= (-1)^{ij}\, db \otimes a + (-1)^{ij+j}\, b \otimes da.$$
$$(-1)^{ij} = (-1)^{i + i(j-1)}, \quad (-1)^{ij+j} = (-1)^{(i+1)j} = (-1)^{(i-1)j}$$
$\Rightarrow$ these are equal.

Proof of Prop.

Both

$EZ: S_\cdot(X\times X) \to S_\cdot(X)\otimes S_\cdot(X)$ and $T\circ EZ: S_\cdot(X\times X) \to S_\cdot(X)\otimes S_\cdot(X)$ are functorial morphisms of complexes that coincide in degree zero $\Rightarrow$ they are chain homotopic. But then the induced maps

$H^{i+j}(S_\cdot(X)\otimes S_\cdot(X), A\otimes B) \to H^{i+j}(S_\cdot(X\times X), A\otimes B)$ $\Rightarrow$

the induced maps $H^i(X,A)\otimes H^j(X,B) \to H^{i+j}(X, A\otimes B)$.

But these are $(a,b)\mapsto a\cup b$ and $(a,b)\mapsto (-1)^{ij}\, b\cup a$.

Theorem. $H^*(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}[x]/(x^{n+1})$ as a ring.

[or graded ring — after rescaling]

Here $x \leftrightarrow$ generator of $H^2(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}$.

An explicit computation is in Hatcher. We give an other proof based on Poincaré duality (to be studied later).

Proof. Recall: $H^i(\mathbb{C}P^n, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & 2\mid i,\ 0\le i\le 2n \\ 0 & \text{otherwise.} \end{cases}$

It follows from the construction that the cup-products $H^0(\mathbb{C}P^n,\mathbb{Z})\otimes H^i(\mathbb{C}P^n,\mathbb{Z}) \to H^i(\mathbb{C}P^n,\mathbb{Z})$ give the $\mathbb{Z}$-module structure $\mathbb{Z}\otimes H^i(\mathbb{C}P^n,\mathbb{Z}) \to H^i(\mathbb{C}P^n,\mathbb{Z})$.

Set $H^i(\mathbb{C}P^n) := H^i(\mathbb{C}P^n, \mathbb{Z})$. We have to show: if x is a generator of $H^2(\mathbb{C}P^n)$, then $x\cup\dots\cup x$ [i times] is a generator of $H^{2i}(\mathbb{C}P^n)$, $\forall i$.

Induction on n: $n=1$ ✓ Assume we know the claim for n. Recall: $\mathbb{C}P^n$ is the $2n$-skeleton of $\mathbb{C}P^{n+1}$, and $\mathbb{C}P^{n+1}$ is obtained by attaching a $(2n+2)$-cell.

Let $\imath: \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$ be the natural embedding.

Lemma. The induced map $\imath^*: H^i(\mathbb{C}P^{n+1}) \to H^i(\mathbb{C}P^n)$ is an isomorphism $\forall i \le 2n$.

<u>Proof of Lemma</u>. We know: $H^i(\mathbb{C}P^{n+1}, \mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & i = 2n+2 \\ 0 & i \neq 2n+2 \end{cases}$

So $\cdots \to H^i(\mathbb{C}P^{n+1}) \xrightarrow{\iota^*} H^i(\mathbb{C}P^n) \to H^{i+1}(\mathbb{C}P^{n+1}, \mathbb{C}P^n) \to H^{i+1}(\mathbb{C}P^{n+1}) \to \cdots$
induces isomorphisms $H^i(\mathbb{C}P^{n+1}) \xrightarrow{\sim} H^i(\mathbb{C}P^n) \quad \forall i \leq 2n$

<u>Proof of Thm</u> (cont.) If x is a generator of $H^2(\mathbb{C}P^{n+1})$
$\Rightarrow$ $\iota^* x$ is a generator of $H^2(\mathbb{C}P^n)$ as both correspond to the 2-cell.
By induction, $\iota^* x \cup \ldots \cup \iota^* x$ generates $H^{2i}(\mathbb{C}P^n) \simeq \mathbb{Z} \quad \forall i \leq n$
$\Rightarrow$ $x \cup \ldots \cup x$ generates $H^{2i}(\mathbb{C}P^{n+1})$ because
$\iota^* x \cup \ldots \cup \iota^* x = \iota^*(x \cup \ldots \cup x)$ [functoriality of cup-product]
$\Rightarrow$ if $x \cup \ldots \cup x \in \mathbb{Z}$ is divisible by $n > 1$ $\Rightarrow$ so is $\iota^* x \cup \ldots \cup \iota^* x$ ↯.
It remains to see that $x \cup \ldots \cup x$ [n+1] times generates $H^{2n+2}(\mathbb{C}P^n)$.
<u>Special case</u> of Poincaré duality: the cup-product maps
$H^i(\mathbb{C}P^{n+1}) \times H^{2n+2-i}(\mathbb{C}P^{n+1}) \to H^{2n+2}(\mathbb{C}P^{n+1}) \simeq \mathbb{Z}$
induce isomorphisms $H^i(\mathbb{C}P^{n+1}) \simeq \mathrm{Hom}(H^{2n+2-i}(\mathbb{C}P^{n+1}), \mathbb{Z}) \ \forall i$.
For $i = 2$ this gives: the generator x of $H^2(\mathbb{C}P^{n+1})$ corresponds to a generator of $\mathrm{Hom}(H^{2n}(\mathbb{C}P^{n+1}), \mathbb{Z})$. By induction, a generator here sends $x \cup \ldots \cup x$ [n times] to $1 \in \mathbb{Z}$.
But by construction it corresponds to

$$H^{2n}(\mathbb{C}P^{n+1}, \mathbb{Z}) \xrightarrow{\cup x} H^{2n+2}(\mathbb{C}P^{n+1}) \simeq \mathbb{Z}$$

So $(x \cup \ldots \cup x) \cup x$ is a generator of $H^{2n+2}(\mathbb{C}P^{n+1})$.

<u>Cor.</u> $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \simeq \mathbb{Z}[x]$

<u>Pf.</u> Continue the above proof to infinity.

A similar proof gives:

<u>Theorem.</u> $H^*(\mathbb{R}P^n, \mathbb{Z}/2) \xrightarrow{\sim} (\mathbb{Z}/2)[x] / (x^{n+1})$ $\quad x \leftrightarrow$ generator of $H^1(\mathbb{R}P^n, \mathbb{Z}/2)$

$$H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)[x].$$

Proof similar. Differences: $H^i(\mathbb{RP}^n, \mathbb{Z}/2) \simeq \begin{cases} \mathbb{Z}/2 & i \leq n \\ 0 & i > n \end{cases}$

[recall: 1 n-cell in each degree $i \leq n$, differentials $= 0$ mod 2!]

Relative coh. sequence:

$$\cdots \to H^i(\mathbb{RP}^{n+1}, \mathbb{Z}/2) \to H^i(\mathbb{RP}^n, \mathbb{Z}/2) \to \underbrace{H^{i+1}(\mathbb{RP}^{n+1}, \mathbb{RP}^n, \mathbb{Z}/2)}_{\cong\, \mathbb{Z}/2 \ i = n, \ 0 \text{ otherwise}} \to \cdots$$

This gives $i^*: H^i(\mathbb{RP}^{n+1}, \mathbb{Z}/2) \to H^i(\mathbb{RP}^n, \mathbb{Z}/2)$ isom. for $i < n$.

But also for $i = n$, because:

$$0 \to H^n(\mathbb{RP}^{n+1}, \mathbb{Z}/2) \xrightarrow{i^*} H^n(\mathbb{RP}^n, \mathbb{Z}/2) \to \mathbb{Z}/2 \xrightarrow{\delta} \underbrace{H^{n+1}(\mathbb{RP}^{n+1}, \mathbb{Z}/2)}_{\cong\, \mathbb{Z}/2} \to \underbrace{H^{n+1}(\mathbb{RP}^n, \mathbb{Z}/2)}_{=\,0}$$

Since δ is surjective, must be an isom.

Hence so is i^*. Rest of proof same, using Poincaré duality for $\mathbb{RP}^n$ with $\mathbb{Z}/2$-coeff's: $H^i(\mathbb{RP}^n, \mathbb{Z}/2) \times H^{n-i}(\mathbb{RP}^n, \mathbb{Z}/2) \to \mathbb{Z}/2$ non-degenerate pairing.

Rem. In these examples $H^*(X, A)$ is commutative. But $\exists$ non-comm. examples, e.g. for $X = S^1 \times \ldots \times S^1$.

Here is a geometric application.

Theorem (Borsuk-Ulam) $\nexists f: S^n \to S^{n-1}$ continuous map s.t. $f \circ \alpha = \alpha \circ f$ [α = antipodal map]

Cor. 1. If $f: S^n \to \mathbb{R}^n$ is a cont. map s.t. $f \circ \alpha = -f$
$\Rightarrow \exists x \in S^n: f(x) = 0$

Proof. If not, $g: x \mapsto \frac{f(x)}{|f(x)|}$ is a cont. map $S^n \to S^{n-1}$ s.t. $f \circ \alpha = \alpha \circ f$ $\lightning$

Cor. 2. If $f: S^n \to \mathbb{R}^n$ is continuous, $\exists x: f(x) = f(\alpha(x))$.
[In particular, f is not injective.]

Proof. If $f(x) \neq f(\alpha(x))$ $\forall x \in S^n \Rightarrow x \mapsto f(x) - f(\alpha(x))$ contradicts Cor. 1.

Proof of Theorem. Suppose f exists.

$\boxed{n > 2}$ S^n simply connected $\Rightarrow$ $S^n \to \mathbb{RP}^n$ universal cover and similarly for $S^{n-1} \to \mathbb{RP}^{n-1}$. Here P and $\alpha(P)$ get identified $\to$ $\exists$ comm. diagram

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & S^{n-1} \arrow[d] \\ \mathbb{RP}^n \arrow[r, "\bar{f}"] & \mathbb{RP}^{n-1} \end{tikzcd}$$

and $\pi_1(\mathbb{RP}^n) \simeq \pi_1(\mathbb{RP}^{n-1}) \simeq \mathbb{Z}/2$.

Lemma. The map $\pi_1(\mathbb{RP}^n) \to \pi_1(\mathbb{RP}^{n-1})$ induced by $\bar{f}$ is $\neq 0$.

Pf. Suppose it is 0. This means: $\forall$ cover $X \to \mathbb{RP}^{n-1}$ pulls back to ~~the~~ a trivial cover of $\mathbb{RP}^n$ by $\bar{f}$. In particular, $S^{n-1} \to \mathbb{RP}^{n-1}$ pulls back to $\mathbb{RP}^n \sqcup \mathbb{RP}^n \to \mathbb{RP}^n$. Since S^n is a universal cover of $\mathbb{RP}^n$, we have:

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[dr] \arrow[d, "g"'] & & S^{n-1} \arrow[d] \\ \mathbb{RP}^n & \mathbb{RP}^n \sqcup \mathbb{RP}^n \arrow[ur] \arrow[l] & \mathbb{RP}^{n-1} \\ & & \end{tikzcd}$$

But S^n connected $\Rightarrow$ f factors through one of the components $\mathbb{RP}^n$, and so does g.

But then $\forall P \in S^n$ $g(P) = g(\alpha(P)) \Rightarrow f(P) = f(\alpha(P)) = \alpha f(P)$ ↯

Proof of Thm (cont.) Lemma: $\bar{f}_*: \pi_1(\mathbb{RP}^n) \xrightarrow{\sim} \pi_1(\mathbb{RP}^{n-1})$ $[\simeq \mathbb{Z}/2]$

$\Rightarrow$ $\bar{f}_*: H_1(\mathbb{RP}^n) \xrightarrow{\sim} H_1(\mathbb{RP}^{n-1})$ by Hurewicz. But since $H_0(\mathbb{RP}^n) \simeq \mathbb{Z}$, $\mathrm{Hom}(H_1(\mathbb{RP}^n), \mathbb{Z}/2) \simeq H^1(\mathbb{RP}^n, \mathbb{Z}/2)$ by Previous corollary. Similarly for $\mathbb{RP}^{n-1}$. So $\bar{f}$ induces

$\bar{f}^*: H^1(\mathbb{RP}^{n-1}, \mathbb{Z}/2) \xrightarrow{\sim} H^1(\mathbb{RP}^n, \mathbb{Z}/2)$ $[\simeq \mathbb{Z}/2]$

If α generates $H^1(\mathbb{RP}^{n-1}, \mathbb{Z}/2)$, $\bar{f}^*(\alpha)$ generates $H^1(\mathbb{RP}^n, \mathbb{Z}/2)$

But then $\underbrace{\alpha \cup \ldots \cup \alpha}_{n \text{ times}} \in H^n(\mathbb{RP}^{n-1}, \mathbb{Z}/2) = 0$, but

$$\bar{f}^*\alpha \cup \ldots \bar{f}^*\alpha \in H^n(\mathbb{RP}^n, \mathbb{Z}/2) \simeq \mathbb{Z}/2 \text{ is a generator } ↯$$

$\boxed{n = 2}$ The Lemma also applies for $n = 2$, and shows:

$$\bar{f}_*: \underset{\mathbb{Z}/2\mathbb{Z}}{\underset{\shortparallel}{\pi_1(\mathbb{RP}^2)}} \to \underset{\mathbb{Z}}{\underset{\shortparallel}{\underset{\pi_1(S^1)}{\underset{\shortparallel}{\pi_1(\mathbb{RP}^1)}}}} \text{ is } \neq 0 \quad ↯$$

[Recall: if S^1 is the unit circle, $x \mapsto x^2$ is a double cover identifying P with $\alpha(P)$.]