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Confining Thin Elastic Sheets and Folding Paper

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Abstract

Crumpling a sheet of paper leads to the formation of complex folding patterns over several length scales. This can be understood on the basis of the interplay of a nonconvex elastic energy, which favors locally isometric deformations, and a small singular perturbation, which penalizes high curvature. Based on three-dimensional nonlinear elasticity and by using a combination of explicit constructions and general results from differential geometry, we prove that, in agreement with previous heuristic results in the physics literature, the total energy per unit thickness of such folding patterns scales at most as the thickness of the sheet to the power $5/3$. For the case of a single fold we also obtain a corresponding lower bound.

1. Introduction and main results

1.1. Overview

We consider compressed elastic sheets, and discuss the scaling of the elastic energy with respect to the thickness h. This situation, corresponding to the *crumpling* of paper (see Fig. 1a), has received the attention of several authors [2, 5, 12, 14, 15, 19, 20] in the physics literature. They focus on specific types of canonical singularities and, on the basis of heuristic arguments, propose that the elastic energy per unit thickness scales as $h^{5/3}$ for small h.

More recently, Venkataramani [27, 28] has given the first proofs supporting the validity of this prediction. He focused on a specific single-fold deformation (see Fig. 1b), which is believed to be the basic building block for the folding patterns appearing in crumpling, and obtained a rigorous proof of the scaling of the energy as $h^{5/3}$ under some simplifying geometric assumptions, including use of a linearization of the Föppl–von Kármán elastic energy.

In this paper we consider these problems in the context of nonlinear three-dimensional elasticity. The key technical ingredient is an optimal construction for a

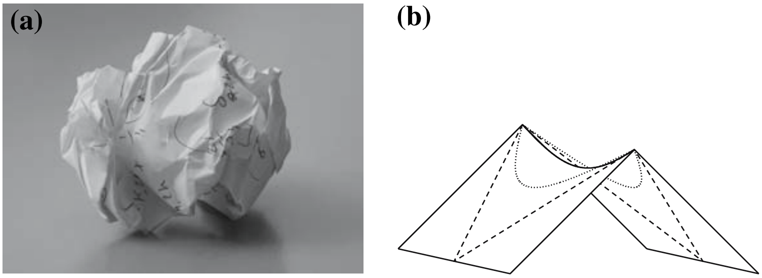

Fig. 1. A crumpled sheet of paper and a special approximation of a single fold. The *dotted lines* delimit the part of the sheet where the construction is bent; the *dashed lines* are the ones on which we impose the piecewise affine boundary conditions (see Proof of Lemma 2.1)

single fold, under boundary conditions which are suitable for using it as a building block for more complex constructions (Lemma 2.1). This leads to an approximation result for piecewise affine isometric maps (briefly, *origami maps*), which can be visualized as the deformations one obtains by folding a finite number of times a sheet of paper (Theorem 1.2). Choosing a particular origami map will then give the scaling result for the crumpling problem, that is for the problem of determining the energy needed to fold an elastic sheet into a fixed (small) ball in $\mathbb{R}^3$.

The second main result of this paper is an approximation theorem. We show that the origami maps are dense in the class of compressive deformations with respect to uniform convergence (Theorem 1.8). This result is strongly related to the Nash–Kuiper theorem on isometric C^1 immersions, which indeed constitutes the starting point of the proof [16, 17, 23] (see also [7]). Combining these results we obtain constructions approximating any generic compressive map (Theorem 1.1), hence a characterization of the Γ-limit of $h^{-\alpha}E_h$ for all $\alpha \in (0, 5/3)$. Here E_h is the elastic energy per unit thickness, see (1.3) below. Related Γ-convergence results have been obtained for $\alpha = 0$ by LEDRET and RAOULT [18], and for $\alpha \geqq 2$ by FRIESECKE and JAMES and MÜLLER [8, 9]. The regime $5/3 \leqq \alpha < 2$ remains open. We recall that the optimal energy scaling for sheets which obey compressive Dirichlet boundary conditions is linear in h, that is, corresponds to $\alpha = 1$ [1]. Therefore our result shows that uniform convergence is a much softer compression criterion. A very different situation arises for clamped membranes to which a force is applied, see [4]. For a summary of the related literature see [9].

Finally, we prove that our single-fold construction is optimal, in the sense that we provide a bound from below with the same scaling, but a different constant, under boundary conditions corresponding to a single fold (Theorem 1.3). Optimality of the global construction remains open.

This paper is organized as follows. In Section 1.2 we present the three-dimensional model and state the main results. Then (Section 1.3) we move on to a simplified two-dimensional model based on the Kirchhoff–Love ansatz, which is more suitable for heuristic discussions as well as for explicit constructions. The confinement problem is discussed in Section 1.4. In the rest of the introduction we sketch the main ideas of the proofs, to motivate the general strategy. In particular, in Section 1.5 we discuss the results on a single fold, in Section 1.6 the approximation of origami maps, and in Section 1.7 the relation between origami maps and short maps. The following sections contain the proofs. Sections 2–5 are concerned with the upper bounds: Section 2 proves the single-fold result, Section 3 discusses the low-energy approximation of origami maps, and Section 4 the approximation of generic short maps. In Section 5 we extend the previous results to three-dimensional elasticity. Finally, Section 6 gives the lower bound for a single fold in the framework of both two- and three-dimensional elasticity. A summary of the notation used can be found at the end of the paper.

1.2. Main results

The deformation $\psi : \Omega \times (0, h) \to \mathbb{R}^3$ of a thin sheet is determined by minimizing the elastic energy

$$\int_{\Omega\times(0,h)} W(\nabla\psi(x, y, z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

among all ψ which meet additional requirements, specified later. Here Ω is a bounded Lipschitz subset of $\mathbb{R}^2$ and $h > 0$ the (small) thickness of the sheet. We assume as usual that the stored-energy function $W : \mathbb{R}^{3\times3} \to [0, \infty]$ is $SO(3)$-invariant, that is

$$W(QF) = W(F), \quad \forall Q \in SO(3), \ \ F \in \mathbb{R}^{3\times3},$$

and that it vanishes only on $SO(3)$ and has quadratic growth, in the sense that

$$c\,\mathrm{dist}^2(F, SO(3)) \leqq W(F) \quad \text{for all} \quad F \in \mathbb{R}^{3\times3}, \tag{1.1}$$

and

$$W(F) \leqq C\,\mathrm{dist}^2(F, SO(3)) \quad \text{in a neighborhood of } SO(3). \tag{1.2}$$

We factor out the trivial h-dependence arising from the size of the domain and focus on the elastic energy per unit thickness

$$E_h(\psi, \Omega) := \frac{1}{h}\int_{\Omega\times(0,h)} W(\nabla\psi(x, y, z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{1.3}$$

When studying the behavior of sequences of deformations $\psi_k : \Omega \times (0, h_k) \to \mathbb{R}^3$, where $h_k \to 0$ as $k \to \infty$, it is natural to introduce some notion of convergence towards two-dimensional deformations $u : \Omega \to \mathbb{R}^3$. We say that $\psi_k \in W^{1,2}(\Omega \times (0, h_k); \mathbb{R}^3)$ *converges uniformly* to $u \in W^{1,2}(\Omega; \mathbb{R}^3)$ if

$$\lim_{k\to\infty} \operatorname*{ess\,sup}_{(x,y,z)\in\Omega\times(0,h_k)} |\psi_k(x, y, z) - u(x, y)| = 0. \tag{1.4}$$

It turns out that the qualitative behavior of thin sheets is very different under tension and under compression. The compressive case corresponds to limiting maps u which are short. Precisely, we say that $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ is short if

$$\nabla u^T \nabla u \leqq \mathrm{Id} \quad \text{almost everywhere}, \tag{1.5}$$

that is, if $\mathrm{Id} - \nabla u^T \nabla u \in \mathbb{R}^{2\times 2}$ is positive semidefinite. Condition (1.5) is equivalent to $u \circ \gamma$ being shorter than γ for any curve $\gamma : [0, 1] \to \Omega$. Typical examples of short maps are the constant map $(x, y) \to (0, 0, 0)$, the map $(x, y) \to (x, 0, 0)$, as well as the identity $(x, y) \to (x, y, 0)$. LeDret and Raoult have shown that for every short map u and for every sequence $h_k \to 0$ there are $\psi_k \in W^{1,2}(\Omega \times (0, h_k); \mathbb{R}^3)$ which converge uniformly to u and such that $E_{h_k}(\psi_k, \Omega) \to 0$. This proves that short maps are low-energy deformations for thin sheets, and corresponds to the fact that thin bodies can relax compression by forming fine-scale oscillations. We show that the actual energy is much lower. Precisely, we prove that short maps are uniform limits of sequences ψ_k whose energy $E_{h_k}(\psi_k, \Omega)$ goes to zero faster than h_k^α, for every $\alpha \in (0, 5/3)$.

Theorem 1.1. *Let Ω be an open bounded Lipschitz domain, $\alpha < 5/3$, and let W obey* (1.2). *Then for every short map $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ and every sequence $h_k \to 0$ we can find a sequence of deformations $\psi_k \in C^\infty(\Omega \times (0, h_k); \mathbb{R}^3)$ converging to u uniformly and such that*

$$\lim_{k\to\infty} \frac{1}{h_k^\alpha} E_{h_k}(\psi_k, \Omega) = 0.$$

This theorem can also be expressed in the language of Γ-convergence: together with the results by LEDRET and RAOULT [18], it implies that, for every $\alpha \in (0, 5/3)$, the Γ-limit of $h^{-\alpha} E_h$ with respect to uniform convergence (see also the discussion at the end of this Section) is the functional

$$\Gamma - \lim_{h\to 0} \frac{1}{h^\alpha} E_h(\cdot, \Omega)(u) = \begin{cases} 0 & \text{if } u \text{ is short} \\ \infty & \text{otherwise.} \end{cases} \tag{1.6}$$

In the limiting case $\alpha = 5/3$ we can still find a sequence ψ_k with $h^{-5/3} E_{h_k}[\psi_k]$ bounded, for a special class of limiting maps u, the origami maps. We say that u is an origami map if u is composed of finitely many affine isometries separated by sharp folds (see Fig. 2). More precisely an origami map is defined as a pair $(\mathcal{T}, u)$ where $\mathcal{T}$ is a family of (not too degenerate) disjoint triangles covering Ω, and u is affine on each triangle of $\mathcal{T}$, with $\nabla u \in O(2, 3)$. To each such triangulation $\mathcal{T}$ we associate its cardinality $|\mathcal{T}|$ and a degeneracy factor $\Gamma(\mathcal{T})$, defined in (1.10) to be a number which controls the largest side length, the smallest angle, and the maximal number of edges meeting at a point in the grid. To each origami map u we associate its maximum jump φ, defined as the maximum discontinuity of ∇u across the boundary of a triangle. Details are given in Section 1.6.

Theorem 1.2. *Let Ω be an open bounded Lipschitz domain, and $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ be an origami map. Then, for every sequence $h_k \to 0$ we can find a sequence of deformations $\psi_k \in C^\infty(\Omega \times (0, h_k); \mathbb{R}^3)$ converging to u uniformly and such that for sufficiently large k*

$$\frac{1}{h_k} \int_{\Omega\times(0,h_k)} \mathrm{dist}^2(\nabla \psi_k, SO(3))\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \leqq C_{\Gamma(\mathcal{T})}\, |\mathcal{T}|\, \varphi^2\, h_k^{5/3},$$

Fig. 2. Two origami maps. The first one is sometimes called Miura origami

where φ *denotes the maximum jump of* ∇u *in* Ω *and* $C_{\Gamma(\mathcal{T})}$ *is a nondecreasing function of* $\Gamma(\mathcal{T})$.

If $\varphi\, C_{\Gamma(\mathcal{T})} \leqq c_*$, *where* c_* *is a positive constant only depending on* W, *and (1.2) holds, then for large* k

$$E_{h_k}(\psi_k, \Omega) \leqq c_*^{-1} C_{\Gamma(\mathcal{T})}\, |\mathcal{T}|\, \varphi^2\, h_k^{5/3}.$$

In the following we shall not indicate explicitly the dependence of constants on W; a summary of the notation used is given at the end of the paper.

The scheme of proof of the two above theorems is the following. First we prove Theorem 1.2, based on an explicit construction (Lemma 2.1) for each fold of the origami map. In order to simplify the construction we work here first with the reduced two-dimensional functional I_h, introduced in the next subsection, and later prove that the result extends to E_h. Theorem 1.1 follows from Theorem 1.2 and the fact that origami maps with small jumps are uniformly dense in the class of short maps (see Section 1.7).

Our third main result states that the construction on a single fold presented in Lemma 2.1 and used in the proof of Theorem 1.2, is indeed optimal.

Theorem 1.3. *Let* $\psi^{\pm} : \mathbb{R}^3 \to \mathbb{R}^3$ *be two affine isometries such that* $\psi^+(x, 0, 0) = \psi^-(x, 0, 0)$, *and let* $Q = \{(x, y) : |x| + \tau|y| \leqq l\}$, *for some* $\tau, l > 0$. *Then* (i) *for* *any* $h \leqq C \min\{l\varphi\tau^3, l\varphi^{-1/2}\}$ *and any* $\psi \in W^{1,2}(Q \times (0,h); \mathbb{R}^3)$ *which obeys the boundary condition*

$$\psi = \psi^\pm \quad \text{on} \quad \{(x,y,z) : |x| + \tau|y| = l,\ \sigma < |y| < 2\sigma,\ 0 < z < h\},$$

where

$$\sigma := \frac{l^{2/3}h^{1/3}}{\varphi^{1/3}}, \quad \varphi := |\nabla\psi^+ - \nabla\psi^-|,$$

one has

$$E_h(\psi, Q) \geqq c_\tau \varphi^{7/3} l^{1/3} h^{5/3}.$$

Further, (ii) *for any* $h < l\varphi$ *there is* $\psi' \in W^{1,2}(Q \times (0,h); \mathbb{R}^3)$ *which obeys the stronger boundary condition*

$$\psi = \psi^\pm \quad \text{on} \quad \left\{(x,y,z) : |x| + \tau|y| = l,\ |y| > \frac{h}{\varphi}\min\{\tau, 1\},\ 0 < z < h\right\},$$

such that

$$E_h(\psi', Q) \leqq C_\tau \varphi^{7/3} l^{1/3} h^{5/3}.$$

Both constants depend only on τ *and* W.

This result demonstrates the validity (in the framework of three-dimensional elasticity) of the scaling law $\varphi^{7/3}l^{1/3}h^{5/3}$ for a single fold over an infinite strip obtained in [20, 19, 29, 28, 27]. The upper bound provided by Theorem 1.2 apparently gives a different scaling in φ and l because there we are working with a generic triangulation $\mathcal{T}$, this corresponds to the difference between Lemma 2.1 and Lemma 2.5 in Section 2. The proof of part (i) of Theorem 1.3 is given at the end of Section 6; that of part (ii) at the end of Section 5.

In closing, we note that in the related literature it is customary to consider, in place of the notion of convergence (1.4) for sequences $\psi_k \in W^{1,2}(\Omega \times (0, h_k); \mathbb{R}^3)$, the notion of weak convergence in $W^{1,2}(\Omega \times (0,1); \mathbb{R}^3)$ for the rescaled deformations $\tilde\psi_k(x,y,z) := \psi_k(x,y,h_k z)$ to the limit $\tilde v(x,y,z) := v(x,y)$. For the problem of interest here there is no significant difference. In particular, whenever $E_{h_k}(\psi_k, \Omega)$ is uniformly bounded, the coercivity assumption (1.1) ensures the weak compactness in $W^{1,2}(\Omega \times (0,1); \mathbb{R}^3)$ of $\tilde\psi_k$. By the uniqueness of the limit, it is not necessary to extract a subsequence, hence (1.4) and a uniform energy bound imply weak convergence to the same limit. Therefore the Γ-convergence theorem stated in (1.6) holds also with respect to weak convergence of the rescalings.

1.3. The two-dimensional elastic model

The qualitative behavior of the functional E_h for small h can be understood by considering special deformations ψ given by a simplified version of the Kirchhoff–Love ansatz. Starting from an immersion of Ω into $\mathbb{R}^3$, that is, from a map $u \in C^2(\Omega; \mathbb{R}^3)$ such that ∇u has full rank everywhere, one defines $\psi : \Omega \times (0, h) \to \mathbb{R}^3$ by

$$\psi(x, y, z) := u(x, y) + zN(x, y), \quad \text{where } N := \frac{u_x \wedge u_y}{|u_x \wedge u_y|}$$

is the normal to the surface $u(\Omega)$. The idea is that u defines a deformation of the lower face of the sheet, and fibers which are normal to the lower face are deformed into fibers which are normal to the deformed lower face.

Under hypothesis (1.2) there exists a positive parameter $c_* > 0$ only depending on W such that, if

$$\|\nabla u^{\mathrm{T}} \nabla u - \mathrm{Id}_2\|_{L^\infty(\Omega;\mathbb{R}^{2\times2})} + h\|\nabla^2 u\|_{L^\infty(\Omega;\mathbb{R}^{3\times2\times2})} \leqq c_* \tag{1.7}$$

and $\psi = u + zN$, then, for a suitable constant C independent of h,

$$\begin{aligned} E_h(\psi, \Omega) \leqq C \Bigg[\int_\Omega \left[(u_x^2 - 1)^2 + (u_y^2 - 1)^2 + (u_x \cdot u_y)^2 \right] \mathrm{d}x\, \mathrm{d}y \\ + h^2 \int_\Omega \left[(N \cdot u_{xx})^2 + (N \cdot u_{xy})^2 + (N \cdot u_{yy})^2 \right] \mathrm{d}x\, \mathrm{d}y \Bigg]. \end{aligned} \tag{1.8}$$

This (standard) statement is proven in Lemma 5.1, and shows that in the study of upper bounds for the three-dimensional elastic energy it is sufficient to focus on the functional

$$I_h(u, \Omega) := \int_\Omega \left[|(\nabla u)^{\mathrm{T}} \nabla u - \mathrm{Id}_2|^2 + h^2 |\nabla^2 u|^2 \right] \mathrm{d}x\, \mathrm{d}y.$$

The first term in the above estimate is called the *stretching energy* and takes into account the stretching and compression of the lower surface $\Omega \times \{0\}$. Geometrically this term corresponds to the L^2 distance from the identity of the first fundamental form of the parameterized surface $u(\Omega)$. The second term in (1.8), called the *bending energy*, can be seen as the L^2 norm of the second fundamental form. If u is an isometry taking Ω into a cylinder of radius R, then the stretching energy vanishes and the bending energy equals $h^2|\Omega|/R^2$. Indeed, this deformation stretches the upper surface $\Omega \times \{h\}$ more than the lower one $\Omega \times \{0\}$, hence they cannot be at the same time isometric. The difference in strain is of order h/R, leading to an energy per unit volume of order h^2/R^2. For simplicity we replace the second fundamental form with the full second gradient in the definition of I_h.

1.4. The confinement problem

The *confinement problem* consists of studying deformations which confine a sheet $\Omega \times (0, h)$ into a small ball in $\mathbb{R}^3$. Precisely, we consider the variational problem

$$m(\Omega, r, h) := \inf \left\{ I_h(u, \Omega) : u(\Omega) \subset r B^3 \right\}.$$

Here $B^3 = \{x \in \mathbb{R}^3 : |x| < 1\}$ is the unit ball of $\mathbb{R}^3$. Since any admissible deformation u for a given r is also admissible for any larger r, m is nondecreasing in r. We now discuss the different regimes in order of decreasing r, focusing on the small-h behavior.

The functional I_h vanishes if and only if u is an affine isometry. If r is larger than the radius r_{circ} of the circumscribed circle to Ω (that is, the radius of the smallest ball in $\mathbb{R}^2$ which contains Ω) then there are isometric affine maps from $\mathbb{R}^2$ to $\mathbb{R}^3$ taking Ω into $r B^3$. In particular $m = 0$ for $r \geqq r_{\text{circ}}$.

For smaller r a more complex behavior appears. For some values of $r < r_{\text{circ}}$ it can still be possible to achieve the confinement with (non-affine) deformations u which are smooth isometries. More precisely, let us consider the critical radius

$$r_{\text{critical}} := \inf \left\{ r > 0 : \exists u \in W^{2,2}(\Omega; \mathbb{R}^3),\ \nabla u \in O(2,3),\ u(\Omega) \subset r B^3 \right\},$$

where $O(2, 3)$ is the set of linear isometries from $\mathbb{R}^2$ to $\mathbb{R}^3$. Of course $r_{\text{critical}} \leqq r_{\text{circ}}$ (a simple example where $r_{\text{critical}} < r_{\text{circ}}$ is given by the unit square, for which $r_{\text{critical}} = 1/2 < r_{\text{circ}} = 1/\sqrt{2}$, see Fig. 3), and in the regime $r \in (r_{\text{critical}}, r_{\text{circ}})$ one expects to observe the energy scaling

$$m(\Omega, r, h) \sim c(\Omega, r) h^2, \quad \forall r \in (r_{\text{critical}}, r_{\text{circ}}).$$

The regime of interest here is $r \leqq r_{\text{critical}}$. One key observation is that necessarily $r_{\text{critical}} > 0$, due to the rigidity of the isometry constraint:

Rigidity of isometries: For any $W^{2,2}$ map u with gradient in $O(2, 3)$ the following holds: for any $x \in \Omega$ there is either a neighborhood of x, or a segment containing x and with endpoints in $\partial\Omega$, on which u is affine.

This was proven by POGORELOV [25, 26] under the assumption that $u \in C^1$, and $\mathcal{H}^2(N(\Omega)) = 0$. The latter assumption follows by the area formula from the condition $\nabla u \in O(2, 3)$, if u is sufficiently smooth. In the smooth case the rigidity also follows from the results by HARTMAN and NIRENBERG [10], a direct proof was given by MASSEY [21] (see also [6]). Using techniques developed by KIRCHHEIM [13], PAKZAD [24] has proven directly the above result for $W^{2,2}$ isometries, and MÜLLER and PAKZAD [22] have shown that $W^{2,2}$ isometries are C^1 and obey $\mathcal{H}^2(N(\Omega)) = 0$. Rigidity fails if only C^1-smoothness is assumed, as can be seen from the Nash–Kuiper theorem discussed in 1.7 below [16, 17, 23].

As a corollary of these results,

$$r_{\text{critical}} \geqq \sup \left\{ \rho > 0 : \rho B^2 \subset \Omega \right\} > 0.$$

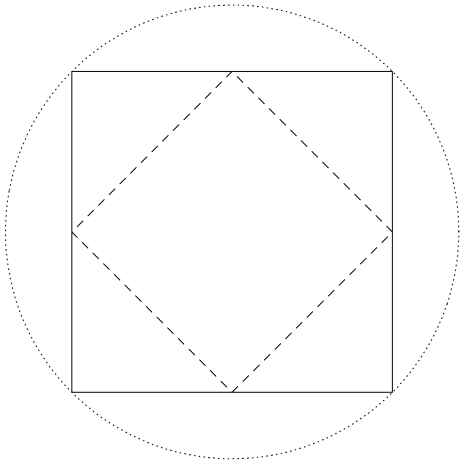

Fig. 3. The circumscribed circle to the unit square has radius $r_{\text{circ}} = 1/\sqrt{2}$, and this is the smallest confining radius with affine isometries. At the same time, a sequence of C^∞ isometries can be constructed which approximates the folding of the unit square along the *dashed lines*. The resulting square can be embedded into every ball of radius $r > r_{\text{critical}} = 1/2$

Higher-dimensional cases of this kind of statement are discussed, in a smooth setting, in [29].

We address here the confinement problem in the range $r \in (0, r_{\text{critical}})$. In this case there are no isometries with finite bending energy (since I_h is finite only on $W^{2,2}$ maps). Therefore the optimal deformations are not exactly isometric, and the optimal energy is determined by a nontrivial interplay of the two energy terms. We shall give a construction which is suitable for arbitrarily small r, and for any bounded Ω, with a scaling $h^{5/3}$. Conical constructions allow in some cases, for r close to r_{critical}, to confine with smaller energy, proportional to $h^2 \ln 1/h$. For example, a conical deformation of the unit square with vertex in the center (which can be thought of as an approximation of an isometry with a point singularity) can map the square into any sphere whose radius is larger than $1/2\sqrt{2}$. We shall not address this special regime here.

The heuristic arguments of [2, 15, 20] and the results of [27, 28] lead to the following conjecture.

Conjecture. For every bounded Lipschitz domain Ω there exists $r^*(\Omega) \leqq r_{\text{critical}}(\Omega)$ such that, if $r < r^*(\Omega)$, there exist two constants $0 < c_1(\Omega, r) \leqq c_2(\Omega, r) < \infty$ and $h(\Omega, r) > 0$ such that

$$c_1(\Omega, r)h^{5/3} \leqq m(\Omega, r, h) \leqq c_2(\Omega, r)h^{5/3},$$

for every $h < h(\Omega, r)$.

In this paper we prove the bound from above:

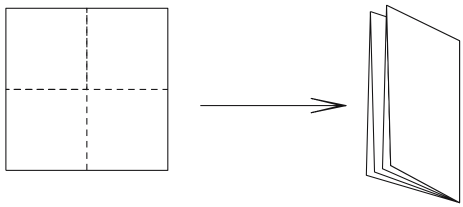

Fig. 4. The map used in Remark 1.5

Theorem 1.4. *Let* $\Omega \subset \mathbb{R}^2$ *be bounded, and let* $r > 0$. *There exist positive constants* $h^* = h^*(r)$ *and* $C^* = C^*(\operatorname{diam} \Omega, r)$ *such that for every* $h < h^*$ *there is a map* $u_h \in C^\infty(\Omega; \mathbb{R}^3)$ *with*

$$u_h(\Omega) \subset rB^3, \quad I_h(u_h, \Omega) \leqq C^* h^{5/3}.$$

The idea of the proof is to first construct a map which realizes the confinement by using sharp folds between isometric pieces, much as one would fold a sheet of paper (see Figs. 2b and 4). Sharp folds, that is, discontinuous gradients, give however, infinite elastic energy. A similar construction with small elastic energy is then obtained by inserting appropriate smooth constructions in place of the sharp folds. It is therefore clear that the crucial point is the analysis of a single fold, which is discussed in the next section.

1.5. The approximation of a single fold

Given points a, b, c, and d in $\mathbb{R}^2$, we denote by $[abcd]$, $[abc]$, and $[ab]$ the convex envelopes of the sets $\{a, b, c, d\}$, $\{a, b, c\}$, and $\{a, b\}$, respectively.

Definition (single fold, see Fig. 5). We say that a pair $([abcd], v)$ is a *single fold along* $[ac]$ if a, b, c, and d are the ordered vertices of a convex quadrilateral in $\mathbb{R}^2$, $v \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^3)$, and there exist $F_1, F_2 \in O(2, 3)$ such that

$$\nabla v = F_1 \text{ on } [acb], \quad \nabla v = F_2 \text{ on } [acd].$$

The continuity of v implies a compatibility condition between F_1 and F_2, namely, if n is a unit vector normal to $[ac]$, then $F_1 n^\perp = F_2 n^\perp$. As usual,

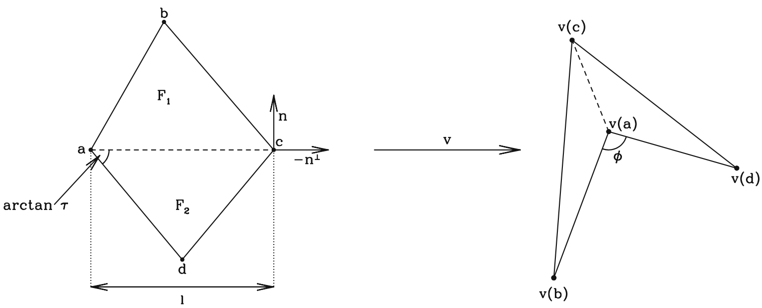

Fig. 5. A single fold deformation. See also Fig. 1b

$(x_1, x_2)^\perp := (-x_2, x_1)$. The angle φ between $F_1 n$ and $F_2 n$ will be called the *angle of the fold* $([abcd], v)$, while the length l of $[ac]$ will be called the *length of the fold*. The *slope of the fold* is the number τ defined as the smallest slope (in absolute value) between the ones defined by the angles of $[acd]$ and $[acb]$ in a and c.

An isolated single fold $([abcd], v)$ can be approximated by smooth isometries in the uniform convergence by means of a natural one-dimensional construction. To see this, notice that v has the form

$$v(p) = v(a) + \gamma\left((p-a)\cdot n\right) + (F_1 n^\perp)\left((p-a)\cdot n^\perp\right),$$

where $\gamma(t) := tF_1 n$ if $t > 0$, $\gamma(t) := tF_2 n$ if $t < 0$ and the normal n to $[ac]$ is assumed to be oriented towards b. For every $\varepsilon > 0$ we can then define $u_\varepsilon \in C^2(\mathbb{R}^2; \mathbb{R}^3)$ by

$$u_\varepsilon(p) = v(a) + \gamma_\varepsilon\left((p-a)\cdot n\right) + (F_1 n^\perp)\left((p-a)\cdot n^\perp\right),$$

where $\gamma_\varepsilon \in C^2(\mathbb{R}; \mathbb{R}^3)$ is a smooth curve parameterized by arc length, taking values in the plane spanned by $F_1 n$ and $F_2 n$, and such that $\gamma_\varepsilon(t) = \gamma(t)$ if $|t| > \varepsilon$. This curve can be chosen in such a way that it satisfies

$$\|\gamma_\varepsilon - \gamma\|_{L^\infty(\mathbb{R};\mathbb{R}^3)} \leqq \varepsilon, \quad \|\gamma_\varepsilon''\|_{L^\infty(\mathbb{R};\mathbb{R}^3)} \leqq \frac{C}{\varepsilon}$$

(see Lemma 2.2 and Fig. 10 below). Then we shall have

$$\|v - u_\varepsilon\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^3)} \leqq \varepsilon, \quad I_h(u_\varepsilon; [abcd]) \leqq C\, l\, \frac{h^2}{\varepsilon}.$$

A similar one-dimensional construction has been recently used by Horak, Lord and Peletier [11] in the study of compressed cylinders; indeed [11, Lemma 3.2] gives a bound analogous to (1.9) below. In our setting, we obtain the following non-optimal estimate:

Remark 1.5. Under the same assumptions as Theorem 1.4, and for sufficiently small h, there are maps u_h such that

$$u_h(\Omega) \subset rB^3, \quad I_h(u_h, \Omega) \leqq C^*(r, \Omega) h^{4/3}.$$

To see this, consider for simplicity $\Omega = (0,1)^2$, and let $v_r \in W^{1,\infty}(\Omega; B(0,r))$ be obtained by folding Ω over horizontal and vertical lines spaced by r, as in Figs. 2b and 4. This map is affine on each of finitely many squares, see Fig. 6a. Up to an area of order ε^2 around each corner, we can use γ_ε as a smooth replacement for v_r along the edges, see Fig. 6b. In the small squares around the corners instead we use a smooth interpolation. The total energy of the deformation u_ε so constructed is then the sum of the contribution from the edges and the one from the vertices. We can easily estimate both in the relevant case $\varepsilon > h$. On the one hand, the edges have total length $2/r$, and their energy per unit length is Ch^2/ε (by the argument above). On the other hand there are r^{-2} vertices, and on an area of order ε^{-2} around each

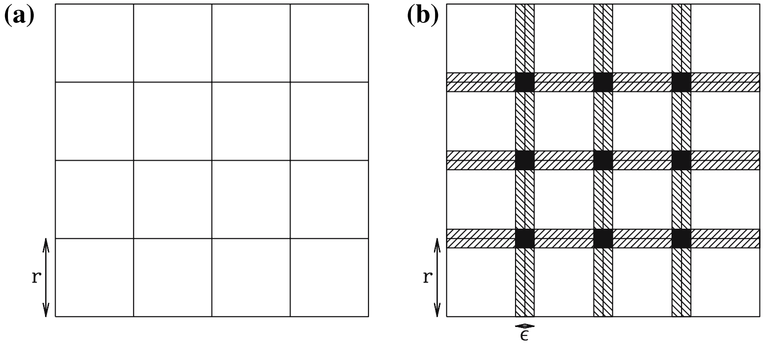

Fig. 6. Sketch of the one-dimensional construction leading to Remark 1.5. **a** The map v_r is an affine isometry on each of the squares of side r. **b** The map is then modified in an ε-neighborhood of the jump set of the gradient (that is, of the sides of the squares); the simple construction based on γ_ε can only be used outside of the smaller *black squares* of side ε

of them the energy density is of order one (because of the stretching term). Hence the total energy of this construction can be estimated as

$$I_h(u_\varepsilon, Q) \leqq C\left(\frac{h^2}{r\varepsilon} + \frac{\varepsilon^2}{r^2}\right). \tag{1.9}$$

Optimizing the value of ε we obtain $\varepsilon = cr^{1/3}h^{2/3}$ (which is admissible provided $h \leqq cr$) and correspondingly

$$I_h(u_{\varepsilon(h,r)}, Q) \leqq C\frac{h^{4/3}}{r^{4/3}}.$$

This argument clarifies that the simple approximation of the single fold by cylinders does not lead to the conjectured optimal scaling law. We have therefore to resort to a more complex, genuinely two-dimensional construction. The construction, presented in Section 2 (Lemma 2.1), is inspired by the pattern observed on a real sheet of paper, see Fig. 8, and corresponds to the illustration of Fig. 1b. The basic idea is that the fold will be sharper around the corners—that is, close to the points a and c—and smoother in the center. This permits to reduce the bending energy in the central part of the fold, and at the same time reduce the area of the "bad regions" around the corners: at a heuristic level, it can be thought of as using a different value of ε in the two terms entering (1.9). The cost of this operation resides in the cost of the transition layer between the corners and the (central part of the) edges, which generates in particular a small tension on the folding line $[ac]$. The result is that for each single fold $([abcd], v)$ with length $l > 0$ a family of deformations u_h can be obtained so that u_h agrees with v up to the gradient on the boundary of $[abcd]$, is smooth away from the two points a and c, and

$$I_h(u_h, [abcd]_h) \leqq C_\tau \varphi^2 l^{1/3} h^{5/3},$$

where $\varphi \in (0, \pi)$ is the angle of the fold and C_τ is a constant (explicitly computable, see Section 2) depending only on the slope of the fold τ. In this estimate we denote by $[abcd]_h$ the domain obtained by taking away from $[abcd]$ the balls $B(a,h)$ and $B(c,h)$. On these balls u_h shall be extended by means of a smooth interpolation. A similar estimate would not be possible in the entire domain. Precisely, $I_h(u,[abcd]) = \infty$ for all u which coincide with v up to the gradient on $\partial[abcd]$. To see this, notice that $I_h(u,[abcd])$ is larger than the squared $W^{1,2}$ norm of ∇u (times a small factor h^2), hence it can be finite only if the boundary values for ∇u are in $H^{1/2}(\partial[abcd],\mathbb{R}^3)$. But functions with jump discontinuities, such as ∇v on $\partial[abcd]$, are not in $H^{1/2}$.

Theorem 1.4 then follows by using Lemma 2.1 (in place of the one-dimensional construction of Remark 1.5) to modify the map v_r discussed above around the discontinuities of its gradient, as sketched in Fig. 7. Details are given in Section 3 below.

The structure of single folds had been previously studied heuristically in the physics literature, see for example [15, 20]. The first mathematical justification was given by Venkataramani [27, 28], although in a simplified setting. More precisely, he considers a single fold with an infinite stripe as reference configuration, and uses for the energy functional a theory of the Föppl–von Kármán type, where the contribution of the tangential displacements to rotations is linearized around the reference state (in particular, this leads to two different linearizations on the two sides of the fold). The plate model used here (that is, I_h) includes instead a complete, nonlinear treatment of rotations. This generates a few additional terms in the upper bound that we need to control, but at the same time leads us to a simpler argument for the lower bound, which is based on a transparent balancing of stretching and bending energies, see Section 6. Furthermore, the upper bound we obtain with the fully geometrically nonlinear elastic model I_h allows to treat directly the three-dimensional model E_h. The adaptation of the argument for the lower bound for I_h to the case of E_h is however nontrivial, and shall in particular require to apply the Friesecke–James–Müller rigidity theorem [8].

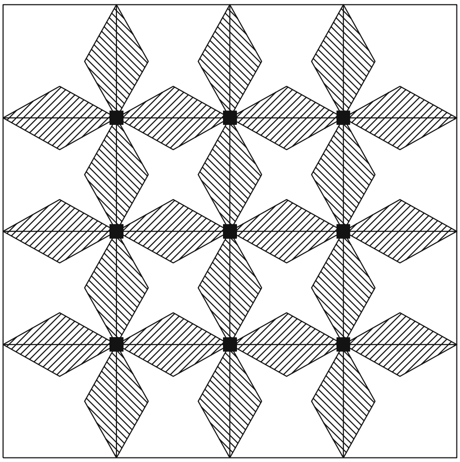

Fig. 7. Sketch of the construction leading from Lemma 2.1 to Theorem 1.4. The Lemma is applied separately on each shaded region. On the *black squares* a generic smooth interpolation is used

Fig. 8. A real sheet of paper undergoing a single fold. The ideal folding line is drawn to put in evidence the stretching

The fact that our treatment of single folds allows as reference configurations generic convex quadrilatera $[abcd]$, instead of infinite stripes only as in [28, 27], is crucial in order to patch together the constructions for different adjacent single folds, and thus approximate generic origami maps. This procedure is explained in detail in the next subsection.

1.6. Origami maps and their approximation in energy

It is clear that we can use Lemma 2.1 to approximate much more general deformations than the v_r defined in Fig. 4. The class of limit maps that we can directly reach in this way consists of what we call origami maps.

Definition (origami maps, see Fig. 2). An *origami map* on a bounded open set $\Omega \subset \mathbb{R}^2$ is a pair $(\mathcal{T}, v)$ where

(i) $\mathcal{T} = \{T_j\}_j$ is a finite family of (closed, nondegenerate) triangles with disjoint interiors such that Ω is contained in $\Omega_{\mathcal{T}} := \bigcup_{T_j \in \mathcal{T}} T_j$; each pair (T_i, T_j) is either disjoint, or shares a corner, or shares a side.
(ii) $v \in W^{1,\infty}(\Omega_{\mathcal{T}}; \mathbb{R}^3)$ is affine on each T_j, and $\nabla v \in O(2, 3)$ almost everywhere.

This definition implies that, if $T_1, T_2 \in \mathcal{T}$ have a side $[ac]$ in common, then there are points $b \in T_1$ and $d \in T_2$ such that $([abcd], v)$ is a single fold along $[ac]$. We shall in particular choose as b and d the barycenters of T_1 and T_2, so that the domains of the single folds so constructed are disjoint (up to a null set).

To every triangulation $\mathcal{T}$ as the ones used in the definition of an origami map we associate the maximum side length $l_{\max}$ of the triangles in $\mathcal{T}$, their minimum slope $\tau_{\min}$, and the maximum number ν of triangles of $\mathcal{T}$ that meet at a point. The *degeneracy factor* of $\mathcal{T}$ is the number

$$\Gamma(\mathcal{T}) := \max\left\{l_{\max}, \nu, \frac{1}{\tau_{\min}}\right\}. \tag{1.10}$$

We shall denote by $|\mathcal{T}|$ the cardinality of $\mathcal{T}$, that is, the number of triangles contained in $\mathcal{T}$.

By repeatedly using Lemma 2.1 and by patching together the different constructions for each single fold in $(\mathcal{T}, v)$ we can extend our approximation result with optimal scaling energy to the class of origami maps.

Theorem 1.6. *Let $(\mathcal{T}, v)$ be an origami map on the bounded domain Ω and let φ be the maximum jump of ∇v. Then for every h small enough [depending only on Ω and $\mathcal{T}$, see* (3.1) *below], there is a deformation $u_h \in C^2(\Omega; \mathbb{R}^3)$ such that*

$$\|v - u_h\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqq C_{\Gamma(\mathcal{T})}\, \varphi\, h^{1/3}, \tag{1.11}$$

$$I_h(u_h, \Omega) \leqq C_{\Gamma(\mathcal{T})}\, |\mathcal{T}|\, \varphi^2\, h^{5/3}, \tag{1.12}$$

and

$$\|\mathrm{dist}(\nabla u_h, O(2,3))\|_{L^\infty(\Omega)} + h\|\nabla^2 u_h\|_{L^\infty(\Omega;\mathbb{R}^{3\times2\times2})} \leqq C_{\Gamma(\mathcal{T})}\, \varphi, \tag{1.13}$$

where $C_{\Gamma(\mathcal{T})}$ is a nondecreasing function of $\Gamma(\mathcal{T})$.

The parameter φ in the above statement corresponds, up to an inessential, universal factor, to the largest angle between those of the various folds composing $(\mathcal{T}, v)$.

In the study of buckling of cylinders, origami maps with periodic boundary conditions are of interest (the so-called Yoshimura pattern, see for example [11]). Our argument applies with no change to that situation. Precisely:

Remark 1.7. If $(\mathcal{T}, v)$ are defined on a rectangle, with periodic boundary conditions on one or both pairs of opposite sides, then the sequence u_h inherits the same periodicity.

1.7. The closure of origami maps in the uniform convergence

To bridge between Theorems 1.2 and 1.1 we shall show that the class of short maps is the closure in the uniform convergence of origami maps. That the closure is no larger follows from the fact that short maps are those for which the gradient takes values in the convex hull of $O(2,3)$, whereas the gradient of origami maps takes values in $O(2,3)$. We therefore only need to perform a construction. This is subtle, since origami maps have a quite rigid structure.

As a first example, consider the deformation of Fig. 2a (the so-called Miura origami). This is a periodic origami map, and after scaling gives rise to origami maps uniformly converging to a biaxial compression. Similar patterns have recently been observed in the buckling of compressed thin films on compliant substrates [3]. This construction is however too rigid for our purposes, and does not permit to deal with non-affine limits. We shall therefore propose in Section 4 a different pattern, which allows us to approximate uniformly not only affine biaxial compressions, but all short maps.

We shall perform an explicit construction of the origami map only in the case of C^1 isometries, that is, for $v \in C^1$ with $\nabla v \in O(2,3)$ everywhere. The relevant feature is that, if $\tilde{v}$ is obtained from v by triangulation and scaling, then $\tilde{v}$ is locally uniformly close to an affine short map, which in turn is close to a scaled isometry.

This permits to control the qualitative geometry of the construction, see Section 4 below.

That this special case is sufficient follows from the celebrated theorem by Nash and Kuiper about isometric immersions of Riemannian manifolds in Euclidean spaces, which in our setting states that for every short map $v \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ and every $\delta > 0$ there is a deformation $u \in C^1(\Omega;\mathbb{R}^3)$ that is an isometry (that is, $\nabla u \in O(2,3)$ everywhere) and such that $\|u - v\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqq \delta$ [23, 16, 17] (see also [7]). The crucial point here is the C^1 regularity of u. Indeed, as discussed above the same cannot hold with C^2 or $W^{2,2}$ regularity.

Theorem 1.8. *On bounded Lipschitz domains, the uniform closure of the set of origami maps is the set of short maps.*

More precisely, let u be a short map on a bounded Lipschitz set $\Omega \subset \mathbb{R}^2$. Then for every $\delta > 0$ we can find an origami map $(\mathcal{T}_\delta, v_\delta)$ on Ω such that

$$\|u - v_\delta\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqq \delta, \quad |\varphi_\delta| \leqq \delta,$$

where φ_δ is the maximum jump of ∇v_δ in Ω. Furthermore $\Gamma(\mathcal{T}_\delta) \leqq C$ for some universal constant C.

2. The approximation of a single fold: upper bound

We present below the construction of a single fold with optimal energy, as discussed in Section 1.5 above.

Lemma 2.1. *Let $([abcd], v)$ be a single fold, with length l, angle φ, and slope τ. Then for every $\sigma < l/8$ there exists a deformation u_σ such that*

$$u_\sigma \in W^{1,\infty}([abcd];\mathbb{R}^3) \cap C^2([abcd] \setminus \{a,c\};\mathbb{R}^3),$$
$$(u_\sigma, \nabla u_\sigma) = (v, \nabla v) \text{ on } \partial[abcd],$$

which obeys

$$\|u_\sigma - v\|_{L^\infty([abcd];\mathbb{R}^3)} \leqq C_\tau\, \varphi\, l^{2/3}\, \sigma^{1/3}, \tag{2.1}$$

$$\|\nabla u_\sigma - \nabla v\|_{L^\infty([abcd];\mathbb{R}^{3\times 2})} \leqq C_\tau\, \varphi, \tag{2.2}$$

$$\|\nabla^2 u_\sigma\|_{L^\infty([abcd]_\sigma;\mathbb{R}^{3\times 2\times 2})} \leqq C_\tau\, \frac{\varphi}{\sigma}, \tag{2.3}$$

and with elastic energy bounded by

$$I_h(u_\sigma, [abcd]_\sigma) \leqq C\left[\varphi^4 \tau_*^5 l^{1/3} \sigma^{5/3} + \varphi^8 \tau_*^7 \sigma^2 + h^2\left(\frac{\varphi^2 l^{1/3}}{\tau_* \sigma^{1/3}} + \varphi^4 \tau_*\right)\right], \tag{2.4}$$

where $\tau_* = \min\{\tau, 1\}$ and we have denoted by $[abcd]_\sigma$ the domain obtained by taking away from $[abcd]$ the balls $B(a,\sigma)$ and $B(c,\sigma)$, and C_τ is a nonincreasing function of τ.

If $h = \sigma$, then (2.4) *reduces to*

$$I_h(u_h, [abcd]_h) \leqq C_\tau \varphi^2 l^{1/3} h^{5/3}. \tag{2.5}$$

Proof of Lemma 2.1, first part. We first observe that, if $[abcd] \subset [ab'cd']$, then any construction u used for $([abcd], v)$ can be extended (setting $u = v$ outside) to a construction u' for $([ab'cd'], v)$, with exactly the same estimates. This shows that the estimates can only improve with increasing τ, and in particular that it suffices to consider the case $\tau \leqq 1$.

After a change of coordinates, we can assume that $0 < \varphi < \pi$ and

$$v(x, y) = \begin{cases} xe_1 + ye_2, & y \geqq 0 \\ xe_1 + y\cos\varphi\, e_2 + y\sin\varphi\, e_3, & y \leqq 0. \end{cases}$$

Let

$$Q := \Big\{(x, y) \in \mathbb{R}^2 : 0 \leqq x \leqq l,\ |y| \leqq \tau \max\{x, l - x\}\Big\}.$$

Clearly $[ac] \subset Q \subset [abcd]$, hence it will suffice to focus on the rhombus Q. In order to prove the lemma we shall construct a deformation

$$u \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^3) \cap C^2((0, l) \times \mathbb{R}; \mathbb{R}^3), \tag{2.6}$$

$$u = v \text{ on } \mathbb{R}^2 \setminus Q \tag{2.7}$$

which obeys the estimates (2.1), (2.2), (2.3) and (2.4); (2.5) is then an immediate consequence. The estimates involving $[abcd]_\sigma$ shall be proved in consequence of analogous estimates on the domain

$$Q_\sigma := \{(x, y) \in Q : \min\{x, l - x\} > \sigma'\},$$

which makes some computations simpler. Here $\sigma' = \sigma/\sqrt{2}$, and in the following we write σ for σ', since a constant factor does not change the estimates. We divide the proof into various steps.

I. Structure of the deformation u. We shall construct u by scaling a universal fold profile γ to a position-dependent fold width f, and inserting a suitable correction β.

Given a function $f \in C^\infty([0, l]; \mathbb{R})$ such that

$$0 \leqq f(x) \leqq \tau \min\{x, l - x\},$$

we define the subdomain $\Omega \subset Q$ (see Fig. 9) by

$$\Omega := \{(x, y) \in (0, l) \times \mathbb{R} : |y| < f(x)\}.$$

Given $\gamma \in C^2([-1, 1]; \mathbb{R}^3)$ and $\beta \in C^2(\Omega; \mathbb{R}^3)$, we define $u : \mathbb{R}^2 \to \mathbb{R}^3$ as

$$u(x, y) := \begin{cases} \beta(x, y)e_1 + f(x)\gamma(y/f(x)), & \text{if } (x, y) \in \Omega, \\ v(x, y), & \text{else.} \end{cases}$$

We shall choose γ parameterized by arc length and orthogonal to the e_1 direction, that is,

$$(\gamma')^2 = 1, \quad \gamma \cdot e_1 = 0. \tag{2.8}$$

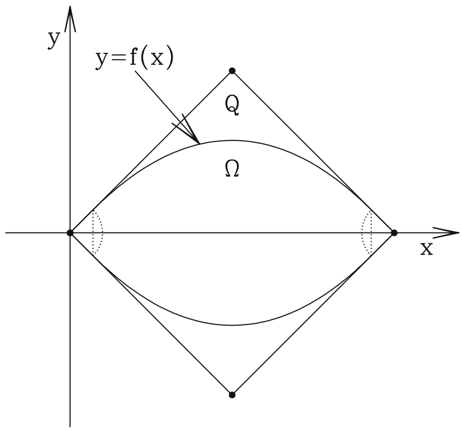

Fig. 9. Subdivision of the domain in the construction of the proof of Lemma 2.1

This curve constitutes a suitable one-dimensional interpolation between the two values taken by the gradient of v on the two sides of the fold: thus we require

$$\gamma(t) = t\, e_2 \quad \text{in a neighborhood of } t = 1$$
$$\gamma(t) = t\, (\cos\varphi e_2 + \sin\varphi e_3) \quad \text{in a neighborhood of } t = -1.$$

Analogously we require that, for every fixed $x \in [0, l]$,

$$\beta(x, y) = x \quad \text{in a neighborhood of } |y| = f(x). \tag{2.9}$$

This ensures a smooth matching at the boundary of Ω, up to the two points $(0, 0)$ and $(l, 0)$, including condition (2.6).

The profile f, the curve γ, and the function β will be chosen in order to achieve the claimed scalings. In particular γ is constructed in Lemma 2.2, f in Lemma 2.4, and β is defined in Equation (2.18).

In order to compute the energy we have to express ∇u and $\nabla^2 u$ in terms of f, γ and β. To this end it is useful to introduce the following functions related to γ:

$$\eta(t) := \gamma(t) - t\gamma'(t),$$
$$\zeta(t) := \gamma'(t) \cdot \eta(t) = \gamma'(t) \cdot \gamma(t) - t.$$

For later reference we notice that $\eta = 0 = \zeta$ around ± 1. For every $(x, y) \in \Omega$ the gradient of u is given by

$$u_x(x, y) = \beta_x(x, y)e_1 + f'(x)\,\eta\left(\frac{y}{f(x)}\right), \tag{2.10}$$

$$u_y(x, y) = \beta_y(x, y)e_1 + \gamma'\left(\frac{y}{f(x)}\right), \tag{2.11}$$

while the second derivatives are

$$u_{xx}(x,y) = \beta_{xx}(x,y)e_1 + f''(x)\,\eta\left(\frac{y}{f(x)}\right) + \frac{y^2 f'(x)^2}{f(x)^3}\gamma''\left(\frac{y}{f(x)}\right), \quad (2.12)$$

$$u_{xy}(x,y) = \beta_{xy}(x,y)e_1 - \frac{yf'(x)}{f(x)^2}\gamma''\left(\frac{y}{f(x)}\right), \quad (2.13)$$

$$u_{yy}(x,y) = \beta_{yy}(x,y)e_1 + \frac{1}{f(x)}\gamma''\left(\frac{y}{f(x)}\right). \quad (2.14)$$

II. Computation of the stretching energy and the choice of β. The stretching energy, which involves only first derivatives of u, is well defined on $\mathbb{R}^2$, and it vanishes on $\mathbb{R}^2 \setminus \Omega$. It remains to compute

$$\int_\Omega (u_x^2 - 1)^2 + (u_x \cdot u_y)^2 + (u_y^2 - 1)^2 \,\mathrm{d}x\,\mathrm{d}y.$$

For every $(x,y) \in \Omega$, according to (2.10) and (2.11), and taking into account (2.8), we find

$$(u_x^2 - 1)^2 = \left((f'\eta)^2 + \beta_x^2 - 1\right)^2 \leqq 2\left[(f')^4\eta^4 + (\beta_x^2 - 1)^2\right], \quad (2.15)$$

$$(u_y^2 - 1)^2 = \left((\gamma')^2 + \beta_y^2 - 1\right)^2 = \beta_y^4, \quad (2.16)$$

$$(u_x \cdot u_y)^2 = (f'\zeta + \beta_x\beta_y)^2 = \left((f'\zeta + \beta_y) + \beta_y(\beta_x - 1)\right)^2, \quad (2.17)$$

where u, β and their derivatives are computed at (x,y), f is computed at x, and η and ζ at $y/f(x)$.

We now choose β so that β_x is close to 1, and $f'\zeta + \beta_y = 0$. Precisely, we set

$$\beta(x,y) := x - f(x)f'(x)\omega\left(\frac{y}{f(x)}\right), \quad (x,y) \in \Omega \quad (2.18)$$

where

$$\omega(t) := \int_{-1}^{t} \zeta(s)\,\mathrm{d}s.$$

Since $\zeta = 0$ around ± 1, and $\int_{-1}^{1} \zeta(t)\,\mathrm{d}t = 0$, we see that $\omega = 0$ around ± 1, and hence (2.9) holds. Furthermore $f'\zeta + \beta_y = 0$ so that (2.17) reduces to

$$(u_x \cdot u_y)^2 = \beta_y^2(\beta_x - 1)^2, \quad \text{on} \quad \Omega. \quad (2.19)$$

Now that β has been chosen we introduce

$$\xi(t) := t\omega'(t) - \omega(t) = t\zeta(t) - \omega(t),$$

and then we compute

$$\beta_x = 1 + (f')^2\xi - f\,f''\,\omega, \quad \beta_y = -f'\zeta, \quad (2.20)$$

$$\beta_{xx} = -\frac{y^2(f')^3}{f^3}\zeta' + 3f'\,f''\xi - f\,f'''\omega, \quad (2.21)$$

$$\beta_{yy} = -\frac{f'}{f}\zeta', \quad \beta_{xy} = \frac{y(f')^2}{f^2}\zeta' - f''\zeta, \quad (2.22)$$

(again, ζ, ξ, and ω are evaluated at y/f). We are now in the position to express the stretching energy in terms of f and γ only. By (2.16) and (2.20) we have

$$\begin{aligned}\int_\Omega (u_y^2 - 1)^2 \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \beta_y^4 \,\mathrm{d}x\,\mathrm{d}y &= \int_0^l \int_{-f(x)}^{f(x)} f'(x)^4 \zeta(y/f)^4 \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^l f'(x)^4 f(x)\,\mathrm{d}x \int_{-1}^1 \zeta(t)^4 \,\mathrm{d}t. \qquad (2.23)\end{aligned}$$

By (2.20)

$$(\beta_x - 1)^2 \leqq 2\left[(f')^4 \xi(y/f)^2 + (f\,f'')^2 \omega(y/f)^2\right]$$

and thus by (2.19)

$$\begin{aligned}\int_\Omega (u_x \cdot u_y)^2 \,\mathrm{d}x\,\mathrm{d}y &\leqq \int_\Omega \beta_y^2 (\beta_x - 1)^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leqq 2\int_0^l f'(x)^6 f(x)\,\mathrm{d}x \int_{-1}^1 \zeta(t)^2 \xi(t)^2 \,\mathrm{d}t \qquad (2.24)\\ &\quad + 2\int_0^l f'(x)^2 f''(x)^2 f(x)^3 \,\mathrm{d}x \int_{-1}^1 \zeta(t)^2 \omega(t)^2 \,\mathrm{d}t.\end{aligned}$$

Finally, as $(\beta_x + 1)^2 \leqq C(1 + (\beta_x - 1)^2)$, we have

$$(\beta_x^2 - 1)^2 \leqq C\left[(f')^4 \xi^2 + (f')^8 \xi^4 + (f\,f'')^2 \omega^2 + (f\,f'')^4 \omega^4\right].$$

Thus from (2.15) it follows

$$\begin{aligned}\frac{1}{C}\int_\Omega (u_x^2 - 1)^2 \,\mathrm{d}x\,\mathrm{d}y &\leqq \int_0^l f'(x)^4 f(x)\,\mathrm{d}x \int_{-1}^1 \eta(t)^4 \,\mathrm{d}t \qquad (2.25)\\ &\quad + \int_0^l f'(x)^4 f(x)\,\mathrm{d}x \int_{-1}^1 \xi(t)^2 \,\mathrm{d}t \\ &\quad + \int_0^l f'(x)^8 f(x)\,\mathrm{d}x \int_{-1}^1 \xi(t)^4 \,\mathrm{d}t \\ &\quad + \int_0^l f''(x)^2 f(x)^3 \,\mathrm{d}x \int_{-1}^1 \omega(t)^2 \,\mathrm{d}t \\ &\quad + \int_0^l f''(x)^4 f(x)^5 \,\mathrm{d}x \int_{-1}^1 \omega(t)^4 \,\mathrm{d}t.\end{aligned}$$

In conclusion (2.23), (2.24) and (2.25) produce an estimate for the stretching energy in terms of a sum of products of integrals depending only on f or on γ.

Before concluding the proof of Lemma 2.1 we introduce in the next three lemmas the curves γ and f.

Lemma 2.2. *For all $\varphi \in (-\pi, \pi)$ there is a curve $\gamma \in C^2([-1, 1]; \mathbb{R}^3)$ such that*

$$\begin{aligned} \gamma(t) \cdot e_1 = 0 \quad \textit{and} \quad \gamma'(t)^2 = 1 \qquad & \forall t \in [-1, 1] \\ \gamma(t) = t\, e_2 \qquad & \textit{in a neighborhood of } t = 1 \\ \gamma(t) = t\,(\cos\varphi e_2 + \sin\varphi e_3) \quad & \textit{in a neighborhood of } t = -1 \end{aligned}$$

which satisfies the estimates

$$|\gamma(t) - te_2| \leqq C\varphi \quad \forall t \in [-1, 1], \tag{2.26}$$

$$|\gamma'(t) - e_2| \leqq C\varphi \quad \forall t \in [-1, 1], \tag{2.27}$$

$$|\gamma''(t)| \leqq C\varphi \quad \forall t \in [-1, 1]. \tag{2.28}$$

We do not present a construction aiming to a realistic profile, but simply discuss the origin of the scalings above, without optimizing the constants. The construction is illustrated in Fig. 10.

Proof. We first construct a curve of length 2 which has the desired geometric properties, and then re-parameterize it by arc length. To do this, we start from the piecewise affine map

$$\gamma_0(t) = \begin{cases} t\, e_2 & \text{if } t > 0 \\ t\,(\cos\varphi e_2 + \sin\varphi e_3) & \text{if } t \leqq 0 \end{cases}$$

and define γ_1 as the convolution of γ_0 with a smooth kernel whose support is contained in the interval $(-1/3, 1/3)$. The curve γ_1 is smooth, coincides with γ_0 for $|t| \geqq 1/3$, its derivative is φ-close to e_2, and its second derivative is controlled

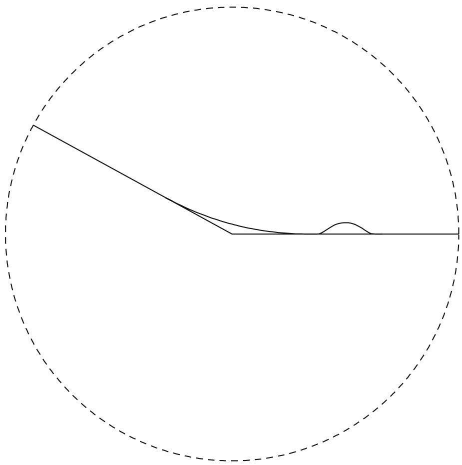

Fig. 10. Sketch of the construction of γ in Lemma 2.2. The curve γ_0 is composed of two segments: γ_1 is smoothed around the center, and γ_2 contains an additional hump on one of the radii

by $C\varphi$; hence it satisfies all requirements up to the one on the total length. The length of γ_1 can be estimated by

$$2 - C\varphi^2 \leqq \text{Length}(\gamma_1) \leqq 2.$$

To see this, it suffices to observe that for every point t the gradient γ_1' is a weighted average between the two values taken by γ_0'. Both have length 1, and their distance is controlled by φ, hence the length of any convex combination is between $1 - \varphi^2$ and 1.

We shall now modify γ_1 in the interval $(1/3, 2/3)$ in order to reach length 2. To do so, define for $\varepsilon \in \mathbb{R}$ the curve

$$\gamma_2(t) = \gamma_1(t) + \varepsilon\rho_{1/6}(t - 1/2)e_3,$$

where $\rho_{1/6}$ is a mollification kernel supported on $(-1/6, 1/6)$. The length of γ_2 is a continuous function of ε; for $\varepsilon = 0$ it is less than 2, and since the maximum of $\rho_{1/6}$ is larger than 3, for $\varepsilon = 1$ the length of γ_2 is larger than 6. This implies that there is a value of $\varepsilon \in (0, 1)$ such that the length is exactly 2.

It remains to show that for small φ also ε is small. This follows from the fact that for small ε the length of γ_2 behaves as the length of γ_1 plus a term of order $C\,\varepsilon^2$, which shows that ε scales as φ for small φ. This in turn implies that also the term $\varepsilon\rho_{1/6}$ satisfies the estimate on the second derivative, and that the difference between γ_2' and γ_1' (and hence e_2) is controlled by a constant times φ.

To conclude the construction, it suffices to reparameterize by arc length, and observe that by so doing we do not change our estimates, up to considering a bigger constant C in (2.27) and (2.28). Then (2.26) is deduced from (2.27) by integration, thanks to the fact that $\gamma(1) = e_2$. □

In the estimate for the stretching energy of u the integral of various quantities related to γ are involved. For small values of φ the curve γ comes closer to a straight segment, which has zero energy, hence we expect these quantities to be small. This is made quantitative in the following lemma.

Lemma 2.3. *The properties of* φ *stated in* Lemma 2.3 *imply that, for all* $t \in [-1, 1]$,

$$|\eta(t)| \leqq C\varphi, \quad |\zeta(t)| + |\zeta'(t)| + |\omega(t)| + |\xi(t)| \leqq C\varphi^2. \tag{2.29}$$

Proof. We recall that the quantities entering the statement are defined by $\eta = \gamma - t\gamma'$, $\zeta = \eta \cdot \gamma'$, $\omega(t) = \int_{-1}^t \zeta(s)\,\mathrm{d}s$, and $\xi = t\zeta - \omega$. To estimate η we use (2.26) and (2.27),

$$|\eta| = |\gamma - t\gamma'| \leqq |\gamma - te_2| + |\gamma' - e_2| \leqq C\varphi.$$

In order to estimate ζ, we first observe that $(\gamma')^2 = 1$ implies that γ' is orthogonal to γ'', and hence to $\eta' = -t\gamma''$. Therefore

$$\zeta' = \eta' \cdot \gamma' + \eta \cdot \gamma'' = \eta \cdot \gamma''.$$

Both factors are bounded by $C\varphi$, hence $|\zeta'| \leqq C\varphi^2$. Integrating (and using $\zeta(1) = 0$) we obtain the estimate for ζ, integrating again and recalling that $\omega(1) = 0$ we get the estimate for ω, and we conclude by $|\xi| \leq |\zeta| + |\omega|$.

We now pass to the choice of f.

Lemma 2.4. *For any* $\tau, l > 0$ *and* $\sigma \in (0, l/8)$ *there is* $f \in C^\infty([0, l])$ *such that*

(i) f *is even with respect to* $x = l/2$*, that is,* $f(x) = f(l-x)$*;*
(ii) f *and its first three derivatives are controlled on* $(0, l/2)$ *by*

$$f_0(x) = \tau\sigma^{1/3}(x+\sigma)^{2/3} - \tau\sigma,$$

in the sense that there is a universal constant C *such that, for* $x \in (0, l/2)$*,*

$$\frac{1}{C} f_0 \leqq f \leqq f_0, \quad |f'| \leqq C|f_0'|, \quad |f''| \leqq C|f_0''|, \quad \textit{and} \quad |f'''| \leqq C|f_0'''|. \tag{2.30}$$

Note that, since $0 \leqq f_0(x) \leqq \tau x$ for $x \geqq 0$, (i) and (ii) imply $0 \leqq f(x) \leqq \tau \min\{x, l-x\}$ for $x \in (0, l)$.

Proof. It suffices to take a smooth interpolation between $f_0(x)$ and $f_0(l-x)$. Precisely, let $\psi \in C^\infty([0, 1]; [0, 1])$ be such that $\psi = 1$ on $(0, 1/4)$ and $\psi(x) = 1 - \psi(1-x)$, and set

$$f(x) = \frac{1}{4}\left[\psi\left(\frac{x}{l}\right) f_0(x) + \psi\left(1 - \frac{x}{l}\right) f_0(l-x)\right].$$

It is clear that $0 \leqq f_0/4 \leqq f$ on $(0, l/2)$. The upper bound on f has to be checked only for $x \in [l/4, l/2]$. There, $l - x \leqq 3x$, hence $(l-x+\sigma)^{2/3} \leqq 3^{2/3}(x+\sigma)^{2/3}$. Analogously, since $\sigma \leqq l/8 \leqq x/2$, $3^{2/3}\sigma^{2/3} \leqq (x+\sigma)^{2/3}$. Combining these two estimates we obtain

$$(l-x+\sigma)^{2/3} - \sigma^{2/3} \leqq (1+3^{2/3})\left[(x+\sigma)^{2/3} - \sigma^{2/3}\right].$$

We conclude that on $[l/4, l/2]$ we have $f_0(l-x) \leqq 4f_0(x)$, and hence $f \leqq f_0$.

The remaining bounds are proven by observing that, in the range $[l/4, l/2]$, one has

$$\frac{f_0(x)}{Cl} \leqq |f_0'(x)| + |f_0'(l-x)| \leqq C\frac{f_0(x)}{l},$$

and analogously for the higher derivatives.

Proof of Lemma 2.1, last part. III. The choice of γ and f. We take γ and f to be the curves introduced in Lemmas 2.2 and 2.4, respectively. As a consequence of Lemma 2.3 and of the estimates (2.23), (2.24) and (2.25) for the stretching energy we find that

$$\begin{aligned}
&\frac{1}{C}\int_{\mathbb{R}^2} (u_x^2 - 1)^2 + (u_x \cdot u_y)^2 + (u_y^2 - 1)^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\leqq \varphi^4 \int_0^l f'(x)^4 f(x) + f''(x)^2 f(x)^3 \,\mathrm{d}x \qquad (2.31)\\
&\quad + \varphi^8 \int_0^l f'(x)^6 f(x) + f'(x)^2 f''(x)^2 f(x)^3 + f'(x)^8 f(x) + f''(x)^4 f(x)^5 \,\mathrm{d}x.
\end{aligned}$$

Using 2.4 we can explicitly estimate

$$\int_0^{l/2} (f')^4 f + (f'')^2 f^3 \, dx \leqq C\tau^5\sigma^{5/3} \int_0^{l/2} \frac{dx}{(x+\sigma)^{2/3}} \leqq C\tau^5\sigma^{5/3} l^{1/3},$$

$$\int_0^{l/2} (f')^6 f + (f')^2 (f'')^2 f^3 \, dx \leqq C\tau^7\sigma^{7/3} \int_0^{l/2} \frac{dx}{(x+\sigma)^{4/3}} \leqq C\tau^7\sigma^2,$$

$$\int_0^{l/2} (f')^8 f + (f'')^4 f^5 \, dx \leqq C\tau^9\sigma^3 \int_0^{l/2} \frac{dx}{(x+\sigma)^2} \leqq C\tau^9\sigma^2.$$

By symmetry the same holds on $(0, l)$. Thus from (2.31) we obtain the following general estimate for the stretching energy:

$$\int_{\mathbb{R}^2} (u_x^2 - 1)^2 + (u_x \cdot u_y)^2 + (u_y^2 - 1)^2 \, dx \, dy \leqq C\left[\varphi^4\tau^5 l^{1/3}\sigma^{5/3} + \varphi^8\tau^7(1+\tau^2)\sigma^2\right]. \tag{2.32}$$

IV. Computation of the bending energy. From (2.12), (2.13), (2.14), by taking into account the fact that $|y| \leqq f$ in Ω, and by using (2.28) and (2.29) we find that

$$|\nabla^2 u|^2 \leqq |\nabla^2 \beta|^2 + C\varphi^2 F_1, \quad F_1 = \frac{1 + (f')^4}{f^2} + (f'')^2.$$

Similarly from (2.21), (2.22) and (2.29) we deduce

$$|\nabla^2 \beta|^2 \leqq C\varphi^4 F_2, \quad F_2 = \frac{(f')^2(1 + (f')^4)}{f^2} + (1 + (f')^2)(f'')^2 + f^2(f''')^2.$$

Therefore we have

$$\int_{Q_\sigma} |\nabla^2 u|^2 \, dx \, dy \leqq C\left[\varphi^2 \int_\sigma^{l-\sigma} F_1 f \, dx + \varphi^4 \int_\sigma^{l-\sigma} F_2 f \, dx\right].$$

We first estimate the integral of F_1.

$$\int_\sigma^{l-\sigma} \frac{dx}{f(x)} \leqq \frac{C}{\tau\sigma^{1/3}} \int_\sigma^{l/2} \frac{dx}{(x+\sigma)^{2/3} - \sigma^{2/3}} \leqq \frac{C}{\tau\sigma^{1/3}} \int_\sigma^{l/2} \frac{2^{2/3} \, dx}{(x+\sigma)^{2/3}} \leqq C\frac{l^{1/3}}{\tau\sigma^{1/3}}.$$

At the same time, the bounds on f give

$$\int_\sigma^{l-\sigma} \frac{(f')^4}{f} \, dx + \int_0^l (f'')^2 f \, dx \leqq C\tau^3,$$

and we obtain

$$\int_\sigma^{l-\sigma} F_1 f \, dx \leqq C\frac{1}{\tau}\frac{l^{1/3}}{\sigma^{1/3}} + C\tau^3.$$

Concerning F_2 we remark that, by using the properties of f as we did in the previous part of the proof,

$$\int_{\sigma}^{l-\sigma} \frac{(f')^2}{f}\,\mathrm{d}x \leqq C\tau, \quad \int_0^l (f''')^2 f^3\,\mathrm{d}x \leqq C\tau^5,$$

so that

$$\int_{\sigma}^{l-\sigma} F_2 f\,\mathrm{d}x \leqq C\left(\tau+\tau^5\right).$$

Thus we come to the following general estimate for the bending energy:

$$h^2\int_{Q_\sigma} |\nabla^2 u|^2\,\mathrm{d}x\,\mathrm{d}y \leqq Ch^2\left[\frac{\varphi^2 l^{1/3}}{\tau\sigma^{1/3}}+\varphi^4(\tau^5+\tau)\right]. \tag{2.33}$$

Since $\tau \leqq 1$, in each term only the lowest power of τ is relevant. In particular (2.4) follows by adding up (2.32) and (2.33), and (2.5) follows by choosing $\sigma = h$. We have now to check that the deformation u realizes also (2.1), (2.2) and (2.3).

V. Further properties. We start from the estimate on the gradient. By using (2.10), (2.20), (2.29) and (2.30) we have

$$|u_x - e_1| \leqq |\beta_x - 1| + |f'||\eta| \leqq C(\tau^2\varphi^2 + \tau\varphi),$$

and analogously from (2.11), (2.20),

$$|u_y - e_2| \leqq |\beta_y| + |\gamma' - e_2| \leqq C(\tau\varphi^2 + \varphi).$$

Since $v_x = e_1$ and $|v_y - e_2| \leqq C\varphi$ we deduce (2.2).

For $(x, y) \in \Omega$, since $u = v$ on $\partial\Omega$ we estimate

$$|u(x,y) - v(x,y)| \leqq |y - f(x)| \sup_{\Omega} |u_y - v_y| \leqq C\varphi f(x) \leqq C\tau\varphi l^{2/3}\sigma^{1/3}.$$

This proves (2.1). The estimate (2.3) for the second gradient can be deduced by similar considerations. This concludes the proof of Lemma 2.1.

In concluding this Section, we briefly remark the implications of the above Lemma for the optimal energy of a single fold. In Lemma 2.1 we have constructed a test function which fulfills boundary values on the full boundary ∂Q, but which has "good" energy only in the smaller set Q_σ. In order to relate our result to estimates on a single fold we present here a modification of the construction, where boundary values are piecewise affine only in part of the boundary, but the energy is estimated on the entire boundary.

Lemma 2.5. *Let* $([abcd], v)$ *be a single fold with length* l*, angle* φ*, and slope* τ*. Then for every* $h < l\varphi$ *there exists a deformation* u_h *such that*

$$u_h \in C^2([abcd]; \mathbb{R}^3),$$
$$(u_h, \nabla u_h) = (v, \nabla v) \textit{ on } (\partial[abcd]) \setminus \left(B_{h/\varphi}(a) \cup B_{h/\varphi}(c)\right),$$

which obeys

$$\|u_h - v\|_{L^\infty([abcd];\mathbb{R}^3)} \leqq C_\tau\, \varphi^{2/3}\, l^{2/3}\, h^{1/3},$$
$$\|\nabla u_h - \nabla v\|_{L^\infty([abcd];\mathbb{R}^{3\times 2})} \leqq C_\tau\, \varphi,$$
$$\|\nabla^2 u_h\|_{L^\infty([abcd];\mathbb{R}^{3\times 2\times 2})} \leqq C_\tau\, \frac{\varphi^2}{h},$$

and with elastic energy bounded by

$$I_h(u_h, [abcd]) \leqq C_\tau\, \varphi^{7/3} l^{1/3} h^{5/3},$$

with C_τ is a nonincreasing function of τ.

Proof. The key idea is to modify the function u_σ constructed in Lemma 2.1 close to the endpoints a and c of the fold, for a suitable value of σ. Doing this by extension and mollification on scale σ, as is done in Section 3 below, does not lead to the optimal dependence on the angle φ. Indeed, if the interpolation does not use the specific structure of the problem, then the distance from $O(2, 3)$ turns out to be of order φ, as in (3.7) below; see also (2.2). Thus this kind of extension leads to a stretching energy of order $\sigma^2\varphi^2$ from the region $(B(a, \sigma) \cup B(c, \sigma)) \cap [abcd]$, which in the relevant case $\sigma = h/\varphi$ equals h^2. The latter is smaller than the desired $\varphi^{7/3} l^{1/3} h^{5/3}$ scaling only under the somewhat artificial condition $h \leqq l\varphi^7$. Therefore a more careful construction is needed.

We give here a direct construction based on a redefinition of the function f in the proof of Lemma 2.1 to make it constant close to the extrema. Precisely, let

$$\hat{f}_0(x) = \frac{1}{2}\tau\sigma^{1/3}(x + \sigma)^{2/3},$$

so that $\hat{f}_0(x) \leqq \tau x$ on $[\sigma, l/2]$ and $f_0(x) \geqq \tau\sigma/2$ on $[0, l]$ (as above, we can replace τ by $\max\{\tau, 1\}$ in the entire computation). Proceeding as in Lemma 2.4, we find a function $\hat{f} \in C^\infty([0, l])$ which obeys the same inequalities as in (2.30). We repeat the construction used in the proof of Lemma 2.1 with $\hat{f}$ replacing f, and working on the full domain Q; we denote the resulting function by $\hat{u}_\sigma$. All estimates for the stretching term are unchanged; some differences appear in the treatment of the bending term in Step IV. Precisely, we estimate

$$\int_0^l \frac{\mathrm{d}x}{\hat{f}(x)} \leqq C \int_0^{l/2} \frac{\mathrm{d}x}{\hat{f}_0(x)} \leqq C \frac{l^{1/3}}{\tau\sigma^{1/3}}.$$

Hence $\hat{f}$ obeys on $[0, l]$ the same estimate that f obeys on $[\sigma, l - \sigma]$. This is true also for the other combinations of derivatives used in Step IV above, as simple computations show. We conclude that

$$h^2 \int_Q |\nabla^2 \hat{u}|^2 \,\mathrm{d}x\,\mathrm{d}y \leqq Ch^2 \left[\frac{\varphi^2 l^{1/3}}{\tau\sigma^{1/3}} + \varphi^4(\tau^5 + \tau)\right],$$

and as a consequence

$$I_h(\hat{u}_\sigma, Q) \leqq C_\tau \left[\varphi^4 l^{1/3}\sigma^{5/3} + \varphi^8\sigma^2 + h^2 \frac{\varphi^2 l^{1/3}}{\sigma^{1/3}} + h^2\varphi^4\right].$$

We finally choose σ so that the sum of the first and third terms is minimal, which gives σ proportional to h/φ. For notational convenience we take $\sigma = h/(8\varphi)$. This choice is admissible if $h \leqq l\varphi$, and leads to the bound

$$I_h(\hat{u}_{h/8\varphi}, Q) \leqq C_\tau\, \varphi^{7/3} l^{1/3} h^{5/3}.$$

The second and fourth terms could be dropped since in the stated admissibility range we have $ch^2\varphi^6 \leqq h^2\varphi^4 \leqq h^{5/3} l^{1/3} \varphi^{4+1/3}$. The other estimates follow analogously. Notice that $\hat{u}_\sigma = v$ on the set $\{(x, y) : |y| > \hat{f}(x)\}$, which in particular includes $\{(x, y) : |y| = \tau x, x \in [\sigma, l/2]\}$. Therefore setting $u_h = \hat{u}_\sigma$ concludes the proof.

3. Origami maps as compositions of single folds

In this section we provide the details of the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $(\mathcal{T}, v)$ be an origami map defined on a domain Ω, as described in Section 1.6; let ν, $l_{\max}$, and $l_{\min}$ be the maximum number of triangles from $\mathcal{T}$ intersecting at a point, the maximum, and the minimum side length of a triangle from $\mathcal{T}$, respectively. Let also φ be the maximum jump of ∇v. Let h^* be such that, for any $p \in \Omega$, $B(p, 2h^*)$ intersects at most 2ν triangles of $\mathcal{T}$. We assume

$$h < \min\left\{h^*, \frac{1}{8} l_{\min}\right\}. \tag{3.1}$$

We note that, for the grids constructed in the next section, one can take h^* proportional to the side of the starting equilateral triangles.

The key idea of the construction is to apply Lemma 2.1 to each single fold defined by $(\mathcal{T}, v)$, as sketched in Fig. 11. Precisely, we subdivide each triangle in $\mathcal{T}$ into three triangles, by joining the barycenter with the vertices. Then, for every edge L of $\mathcal{T}$ we consider as the domain of the corresponding single fold the quadrilateral which has additional vertices in the barycenters in the two triangles of whom L is a side. These quadrilatera are disjoint. In each of them we replace v by the result of Lemma 2.1 for $\sigma = h$: let us denote by u be the resulting deformation.

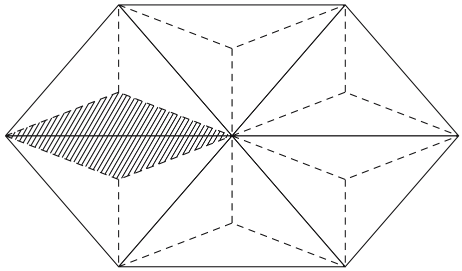

Fig. 11. Subdivision of a grid into single folds. The *full lines* separate the grid triangles, the *dashed lines* separate the quadrilatera composing the single folds. One of the rhombi is *dashed*

Since the construction of Lemma 2.1 preserves the boundary values, u is continuous, and—away from the vertices—has a continuous derivative and small energy:

$$\|u - v\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqq C_\tau \, \varphi \, l_{\max}^{2/3} \, h^{1/3}, \tag{3.2}$$

$$\|\nabla u - \nabla v\|_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} + h\|\nabla^2 u\|_{L^\infty(\Omega\setminus\Omega_h;\mathbb{R}^{3\times 2\times 2})} \leqq C_\tau \, \varphi, \tag{3.3}$$

$$I_h(u, \Omega \setminus \Omega_h) \leqq C_\tau \, |\mathcal{T}| \, \varphi^2 \, l_{\max}^{1/3} h^{5/3}. \tag{3.4}$$

Here Ω_h is the union of the balls $B(q_i, h)$, where $\{q_1, \dots, q_N\}$ denote the vertices of the triangulation, and C_τ denotes a nonincreasing function of the smallest slope of the various triangles from $\mathcal{T}$. Around the exterior edges, that is, those segments which are sides of only one grid triangle, we set $u = v$.

Let Ω_h^* denote the union of the balls $B(q_i, 2h)$, which are disjoint by (3.1). In each ball ∇v takes at most 2ν values, let F_i be one of them. Since the jumps of ∇v are less than φ, using (3.3) we have

$$|\nabla u - F_i| \leqq C_\tau \, \nu \, \varphi \quad \text{on } B(q_i, 2h). \tag{3.5}$$

We now define $\overline{u} : \Omega_h^* \to \mathbb{R}^3$ as

$$\overline{u} = u(q_i) + F_i(x - q_i) \quad \text{if } x \in B(q_i, 2h).$$

It is clear that $\nabla\overline{u}(x) \in O(2, 3)$ and $\nabla^2\overline{u} = 0$, and by (3.5) we have

$$\|\nabla\overline{u} - \nabla u\|_{L^\infty(\Omega_h^*;\mathbb{R}^{3\times 2})} \leqq C_\tau \, \nu \, \varphi, \quad \|\overline{u} - u\|_{L^\infty(\Omega_h^*;\mathbb{R}^3)} \leqq C_\tau \, \nu \, \varphi \, h. \tag{3.6}$$

Let now $\alpha \in C_c^\infty(B^2(0, 2); [0, 1])$ with $\alpha(p) = 1$ if $p \in B^2(0, 1)$, let

$$\alpha_h(p) := \sum_{i=1}^N \alpha\left(\frac{p - q_i}{h}\right),$$

and define

$$u_h := \alpha_h \overline{u} + (1 - \alpha_h) u.$$

It is clear that $u_h \in C^2(\Omega; \mathbb{R}^3)$. As $|u_h - v| \leqq \alpha_h|\overline{u} - u| + |u - v|$, by (3.2) and (3.6) we obtain (1.11). As $\nabla u_h - \nabla u = \alpha_h(\nabla\overline{u} - \nabla u) + (\overline{u} - u) \otimes \nabla\alpha_h$, from (3.3) and (3.6) we obtain

$$|\nabla u_h - \nabla v| \leqq C_\tau \, \nu \, \varphi,$$

which in turn implies

$$\|\mathrm{dist}(\nabla u_h, O(2, 3))\|_{L^\infty(\Omega)} \leqq C_\tau \, \nu \, \varphi. \tag{3.7}$$

The second derivatives are treated analogously: on the balls $B(q_i, h)$ we have $u_h = \overline{u}$, hence $\nabla^2 u_h = 0$, while outside these balls,

$$|\nabla^2 u_h| \leqq |\nabla^2 u| + \frac{C}{h}|\nabla\overline{u} - \nabla u| + \frac{C}{h^2}|\overline{u} - u|.$$

By combining (3.3) and (3.6) we obtain

$$h\|\nabla u_h^2\|_{L^\infty(\Omega;\mathbb{R}^{3\times2\times2})} \leqq C_\tau\, \nu\, \varphi. \tag{3.8}$$

It remains to estimate the energy. By (3.7) and (3.8) we have

$$I_h(u_h, \Omega_h^*) \leqq C_\tau\, |\mathcal{T}|\, h^2 \left(\nu^2\varphi^2 + \nu^2\varphi^2\right) \leqq C_\tau\, \nu^2\, l_{\max}^{1/3}\, |\mathcal{T}|\, \varphi^2\, h^{5/3}.$$

Since $I_h(u_h, \Omega) \leqq I_h(u, \Omega \setminus \Omega_h) + I_h(u_h, \Omega_h^*)$ combining with (3.4) we deduce (1.12) and conclude the proof.

Theorem 1.4 is then a simple consequence of Theorem 1.6:

Proof of Theorem 1.4. Let Q be a square of side $2\operatorname{diam}\Omega$ and containing Ω. We divide Q into many squares $Q^{(i)}$ of side length $l < r/\sqrt{2}$. The number of squares needed is proportional to $(\operatorname{diam}\Omega/r)^2$. Let $v_r \in W^{1,\infty}(Q;\mathbb{R}^3)$ be the map taking Q into rB^3 by folding along the sides of the $Q^{(i)}$'s, see Figs. 4 and 2b. Precisely, we set $v_r(x, y) = (\eta_l(x), \eta_l(y), 0)$, where η is the $2l$-periodic function defined by $\eta(t) = \min\{t, 2l - t\}$ for $t \in (0, 2l)$. By applying Theorem 1.6 to this particular origami map we conclude the proof.

4. Short maps as uniform limits of origami maps

In this section we are concerned with the proof of Theorem 1.8. Before going into this we state and prove two lemmas we shall need.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there is a sequence of diffeomorphisms $\Phi_k \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that*

$$\lim_{k\to\infty} \sup_{p\in\mathbb{R}^n} \left[|\Phi_k(p) - p| + |\nabla\Phi_k(p) - \mathrm{Id}_n|\right] = 0,$$

and $\Omega \subset\subset \Phi_k(\Omega)$.

Proof. We claim that it is sufficient to construct a single map $\Phi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $\Phi(p)\cdot\nu(p) > c > 0$ for almost every $p \in \partial\Omega$, where ν is the outer normal to $\partial\Omega$. Indeed, if such a Φ exists, it is easy to see that

$$\Phi_k(p) := p + \frac{1}{k}\Phi(p)$$

satisfies the statement, for k large enough.

To construct the function Φ, observe that by definition $\partial\Omega$ can be covered by a finite family of balls $B_i = B(p_i, r_i)$, such that in each of the double balls $B_i' = B(p_i, 2r_i)$ the set of Ω is the subgraph of a Lipschitz function. Precisely,

$$\Omega \cap B_i' = \left\{p \in B_i' : p\cdot e_i < f_i(P_i p)\right\},$$

where $e_i \in S^{n-1}$, P_i denotes orthogonal projection onto the space normal to e_i, and $f_i \in W^{1,\infty}(\mathbb{R}^{n-1};\mathbb{R})$. It is clear that $e_i\cdot\nu > c > 0$ on $B_i' \cap \partial\Omega$. Then it suffices to choose for each i a cutoff function $\psi_i \in C_0^\infty(B_i'; [0, 1])$ such that $\psi_i = 1$ on B_i, and set

$$\Phi(p) = \sum_i \psi_i(p)e_i.$$

This concludes the proof.

Let now $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ be such that $(\nabla u)^T \nabla u \leqq \lambda \mathrm{Id}$. If $[pq] \subset \Omega$ then integration of ∇u over the segment $[pq]$ leads to

$$|u(p) - u(q)| \leqq \lambda |p - q|.$$

Thus a λ-short map on Ω is also λ-Lipschitz on every convex component of Ω. For C^1 maps the converse inequality holds as well, for short segments. Precisely:

Lemma 4.2. *Let* $u \in C^1(\overline{\Omega}; \mathbb{R}^3)$ *be such that for some* $\lambda_1 > 0$

$$\lambda_1 \mathrm{Id}_2 \leq (\nabla u)^T \nabla u.$$

Then for every $\lambda_2 < \lambda_1$ *there exists* $l > 0$ *such that*

$$\lambda_2 |p - q| \leqq |u(p) - u(q)|, \quad \textit{whenever } [pq] \subset \Omega,\ |p - q| \leqq l.$$

Proof. As ∇u is continuous on $\bar{\Omega}$, there is a uniform modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to 0^+} \omega(t) = 0$ such that

$$\frac{|u(p) - u(q) - \nabla u(p)(p - q)|}{|p - q|} \leqq \omega(|p - q|)$$

for every $p, q \in \Omega$ with $[pq] \subset \Omega$. Thus

$$\begin{aligned} |u(p) - u(q)| &\geqq |\nabla u(p)(q - p)| - \omega(|p - q|)|p - q| \\ &\geqq (\lambda_1 - \omega(|p - q|))|p - q|. \end{aligned}$$

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. We start by explaining the strategy of the proof. We are given a short map u on Ω, and we want to approximate it uniformly with origami maps, that is, with piecewise affine isometries, which are affine on a finite number of triangles. In the first part we show that it suffices to consider the case in which u is a C^1 isometry defined on a domain strictly larger than Ω. In the second part we construct explicitly the approximating origami map.

I. Reduction to the case of a C^1 isometry. Let $\Phi_k : \mathbb{R}^2 \to \mathbb{R}^2$ be the sequence of diffeomorphisms given by Lemma 4.1, and define

$$v_k(p) = \frac{u\left(\Phi_k^{-1}(p)\right)}{\|\nabla \Phi_k^{-1}\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})}}, \quad p \in \Omega_k := \Phi_k(\Omega).$$

Then v_k is a sequence of short maps converging uniformly to u. By the Nash–Kuiper theorem, for every k we can find $u_k \in C^1(\Omega_k; \mathbb{R}^3)$ arbitrarily close to

v_k in $C^0(\Omega; \mathbb{R}^3)$, and such that $\nabla u_k \in O(2,3)$. If we can approximate u_k with origami maps, taking a diagonal sequence we can approximate v_k, and taking another diagonal sequence we can approximate u. Hence it suffices to consider C^1 isometries defined on a larger domain.

II. The explicit construction for a C^1 isometry. Let $u \in C^1(\Omega^*; \mathbb{R}^3)$ with $\nabla u \in O(2,3)$ and $\Omega \subset\subset \Omega^*$. For any $\delta \in (0, 1/2)$ we define

$$u_\delta := (1-\delta)u.$$

The triangles of $\mathcal{T}$ will be constructed at a scale where u_δ is approximately an affine contraction. We make this quantitative by fixing a parameter $\kappa \in (1, 3/2)$ (chosen below), and using Lemma 4.2 to see that there is $l(u_\delta, \kappa) > 0$ such that

$$(1-\kappa\delta)|p-q| \leqq |u_\delta(p) - u_\delta(q)| \leqq (1-\delta)|p-q| \tag{4.1}$$

for all pairs $p, q \in \Omega^*$ such that $|p-q| \leqq l(u_\delta, \kappa)$ and $[pq] \subset \Omega^*$. We choose

$$l \leqq \min\left\{\frac{1}{2}l(u_\delta, \kappa), \delta, \operatorname{dist}(\Omega, \partial\Omega^*)\right\}.$$

Consider a partition of $\mathbb{R}^2$ in equilateral triangles of side length l and let $\mathcal{T}_l$ be the triangulation defined by those which are contained in Ω^*. The choice of l ensures that $\Omega \subset \Omega_\mathcal{T}$, and that (4.1) holds for any pair of vertices of any triangle of $\mathcal{T}_l$, as well as for any pair of vertices of a couple of triangles of $\mathcal{T}_l$ which share a side.

We start by defining v on the vertices of $\mathcal{T}_l$, by setting

$$v(a) = u_\delta(a), \quad \text{if } a \text{ is a vertex of } \mathcal{T}_l. \tag{4.2}$$

Claim. We claim that we can find a subtriangulation $\mathcal{T}$ of $\mathcal{T}_l$, and values of v on the vertices of $\mathcal{T}$ that were not vertices of $\mathcal{T}_l$, so that the following holds: for every edge (r, s) of every triangle of $\mathcal{T}$, one has

$$|v(r) - v(s)| = |r - s|. \tag{4.3}$$

If the claim holds, then it suffices to define v in each triangle as the affine extension of the values in the corners to obtain an origami map. Further, since both v and u_δ are 1-Lipschitz on each of those triangles, they are equal on the vertices, and the diameter of the triangles is bounded by l, we have

$$\begin{aligned}\|v - u\|_{L^\infty(\Omega;\mathbb{R}^3)} &\leqq \|v - u_\delta\|_{L^\infty(\Omega;\mathbb{R}^3)} + \|u_\delta - u\|_{L^\infty(\Omega;\mathbb{R}^3)} \\ &\leqq 2l + \delta \operatorname{diam} \Omega \leqq C_\Omega \delta. \end{aligned} \tag{4.4}$$

Since δ is arbitrary, the claim implies the thesis.

Proof of the claim. This is the more technical part of the proof. To illustrate the construction we first present it for the particular case that u is an affine isometry; we show later how the extension to the general case can be obtained by a perturbation argument. The construction for this case is illustrated in Figs. 12, 13 and 14. Before starting, we remark that the construction is separated in two parts, and that each

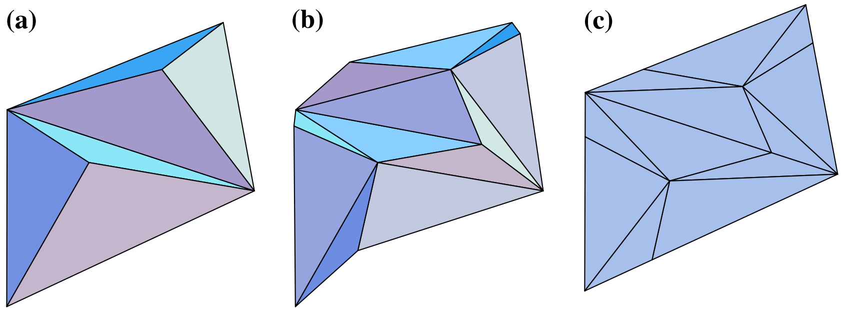

Fig. 12. Construction of the origami map for an affine compression. **a** First stage, where the midpoints of the triangles have been raised (only two triangles shown). **b** Final construction, where also a point in each edge has been raised. **c** Domain subdivision for the construction in (**b**)

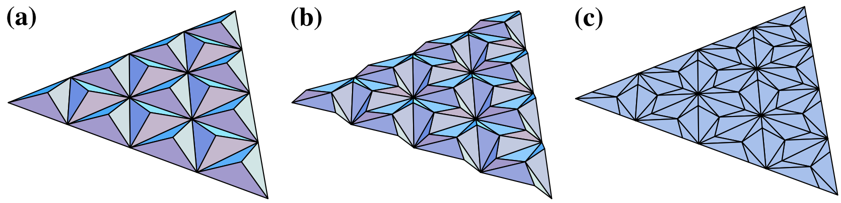

Fig. 13. Analogue of Fig. 12, for a larger portion of the domain

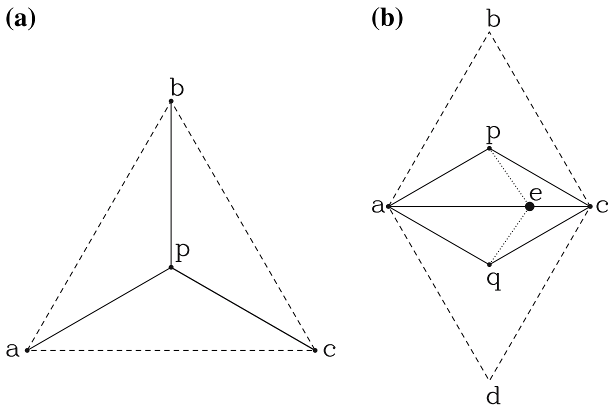

Fig. 14. Grid refinement. **a** One equilateral triangle $[abc]$, with side $|a-c| = l$, is subdivided into three using its midpoint p, with $|a-p| = l/\sqrt{3}$. **b** An edge between two neighboring equilateral triangles is subdivided using a point $e \in [ac]$

of them is done locally. Precisely, we first modify the map inside each of the triangles, without changing the value on the boundary. In this process we introduce new "interior" boundaries, which separate the edges of the original triangle. In the second step we work on each of the original edges separately, keeping the function constant on the boundary of an appropriate quadrilateral. This separation method permits us to consider only a small number of points in each construction step, and to adapt the local construction to the local structure of the map u to be approximated. This is remarkably different from the rigid structure of the Miura origami illustrated in Fig. 2a.

Part 0 (special case): Affine compression. Assume that u_δ is a uniform compression, that is it equals $1-\delta$ times an isometry. To approximate it with an origami map, we first raise the midpoint of each triangle, and then choose and raise a point on each edge. This is done in such a way that the resulting map obeys (4.3) on all edges of the new grid.

To see how this is done, consider for definiteness a triangle $[abc]$, and call p its midpoint. We have $v = u_\delta$ on a, b and c, and seek $v(p)$ so that v is isometric along the segments $[ap]$, $[bp]$, and $[cp]$, that is

$$|v(a) - v(p)| = |a - p|, \quad \text{and the same for } b \text{ and } c \tag{4.5}$$

(see Figs. 12a and 14a). Due to the symmetry of this situation it suffices to consider points of the form $v(p) = u_\delta(p) + \xi N$, where N is the normal to the plane defined by u_δ and $\xi > 0$. Since $v(a) = u_\delta(a)$, condition (4.5) is equivalent to

$$|u_\delta(a) - u_\delta(p)|^2 + \xi^2 = |a - p|^2.$$

We compute explicitly $|u_\delta(a) - u_\delta(p)|^2 = (1-\delta)^2|a-p|^2 = (1-\delta)^2 l^2/3$, which gives $\xi^2 = l^2(2\delta - \delta^2)/3$. In reducing to a one-dimensional problem we made use of the symmetry of this special case; existence of a solution for $v(p) \in \mathbb{R}^3$ is however a generic fact, as we shall show below. Having chosen $v(p)$, we replace the triangle $[abc]$ with the three triangles $[abp]$, $[bcp]$, and $[cap]$. At this stage, the map obtained by linear interpolation has the appearance illustrated in Figs. 12a and 13a. We remark that this intermediate map is not short. Indeed, it is easy to see that the distance of $v(p)$ from the midpoint of $[v(a), v(b)]$ is longer than the one of p from the midpoint of $[ab]$.

Now we treat the edges. Consider the edge $[ac]$, which is common to the two triangles $[abc]$ and $[adc]$, and let p and q be their midpoints, respectively (see Fig. 14b). We seek $e \in [ac]$ and $v(e) \in \mathbb{R}^3$ with the properties

$$\begin{aligned} |v(e) - v(a)| &= |e - a|, \quad |v(e) - v(c)| = |e - c|, \\ |v(e) - v(p)| &= |e - p|, \quad |v(e) - v(q)| = |e - q|. \end{aligned} \tag{4.6}$$

If this holds, then replacing $[acp]$ with the pair $[ape]$, $[cpe]$, and $[acq]$ with $[aqe]$, $[cqe]$, gives a grid which satisfies (4.3) on all edges.

Condition (4.6) implies that $v(e)$ has to lie in the plane of symmetry between $v(p)$ and $v(q)$, which contains $v(a)$ and $v(c)$. We can therefore set $e = \lambda c + (1-\lambda)a$, $v(e) = \eta v(c) + (1-\eta)v(a) + \zeta N$. A straightforward computation shows that

$$|v(e) - v(a)|^2 = |\eta(v(c) - v(a)) + \zeta N|^2 = \eta^2(1-\delta)^2 l^2 + \zeta^2$$

and so on, hence our problem reduces to finding $\eta, \zeta \in \mathbb{R}$ and $\lambda \in (0, 1)$ such that

$$\begin{aligned} \eta^2(1-\delta)^2 l^2 + \zeta^2 &= \lambda^2 l^2 \\ (1-\eta)^2(1-\delta)^2 l^2 + \zeta^2 &= (1-\lambda)^2 l^2 \\ \left(\frac{1}{2} - \eta\right)^2 (1-\delta)^2 l^2 + \frac{1}{12}(1-\delta)^2 l^2 + (\zeta - \xi)^2 &= \left(\frac{1}{2} - \lambda\right)^2 l^2 + \frac{1}{12} l^2. \end{aligned}$$

Subtracting the first equation from the other two, solving for η and ζ, and inserting back, leads, after some computations, to

$$\lambda^2 - \lambda + \frac{4-(1-\delta)^2}{16} = 0. \tag{4.7}$$

This has the two solutions

$$\lambda_1 = \frac{1+\delta}{4}, \quad \lambda_2 = \frac{3-\delta}{4}.$$

The relevant observation is that the quadratic equation (4.7) is soluble for all $\delta \in (0, 1)$, and that the resulting values of λ are uniformly bounded away from the endpoints 0 and 1. We shall show below that, for κ sufficiently close to 1, these properties are preserved in the general case.

For completeness, we also give the corresponding expressions for η and ζ:

$$\eta_1 = \frac{1-2\delta}{4-4\delta}, \quad \eta_2 = \frac{3-2\delta}{4-4\delta}, \quad \zeta_1 = \zeta_2 = \frac{3}{4}\xi = \frac{\sqrt{3}}{4}\sqrt{2\delta-\delta^2}l.$$

Part 1: Triangle centers. We now repeat the argument for a generic u_δ. Again, in the first step we focus on $[abc] \in \mathcal{T}$ with center p, assume that $v = u_\delta$ on the vertices, and seek $v(p) \in \mathbb{R}^3$ which obeys (4.5). This means that $v(p)$ must belong to the intersection of three spheres in $\mathbb{R}^3$,

$$v(p) \in \partial B^3\left(v(a), \frac{l}{\sqrt{3}}\right) \cap \partial B^3\left(v(b), \frac{l}{\sqrt{3}}\right) \cap \partial B^3\left(v(c), \frac{l}{\sqrt{3}}\right). \tag{4.8}$$

We further require $v(p)$ to lie on the side of the plane containing $[v(a)v(b)v(c)]$ determined by the normal N to u_δ (since u_δ is C^1, and $\nabla u_\delta \in (1-\delta)O(2, 3)$ has full rank, the normal is globally defined and continuous; since l is small the point of $[abc]$ on which we evaluate N does not matter).

We have already shown that for $\kappa = 1$ these conditions identify a single point, and it is easy to see that (for $\kappa = 1$) the three spheres intersect transversally. Therefore we can perturb the position of each of them by a small amount, and still have one transversal intersection around the original one. This means, that for $\kappa - 1$ small there will be a solution, and that the solution depends continuously on $v(a)$, $v(b)$, and $v(c)$. We write for brevity

$$v(p) := G(v(a), v(b), v(c)),$$

with G continuous.

Part 2: Edge points. Consider as above an edge $[ac]$, which is common to the two triangles $[abc]$ and $[adc]$, having midpoints p and q, respectively, (as in Fig. 14b). Given the points $v(a)$, $v(b)$, $v(c)$, and $v(d)$ in the domain $D(\delta, \kappa)$ of $(\mathbb{R}^3)^4$ defined by

$$\begin{cases} l(1-\kappa\delta) \leqq |v(a)-v(b)| \leqq l(1-\delta), \\ l(1-\kappa\delta) \leqq |v(a)-v(c)| \leqq l(1-\delta), \\ l(1-\kappa\delta) \leqq |v(b)-v(c)| \leqq l(1-\delta), \\ l(1-\kappa\delta) \leqq |v(a)-v(d)| \leqq l(1-\delta), \\ l(1-\kappa\delta) \leqq |v(c)-v(d)| \leqq l(1-\delta), \\ \sqrt{3}l(1-\kappa\delta) \leqq |v(b)-v(d)| \leqq \sqrt{3}l(1-\delta), \end{cases}, \tag{4.9}$$

and setting

$$v(p) := G(v(a), v(b), v(c)), \quad v(q) := G(v(a), v(c), v(d)), \tag{4.10}$$

as above, we seek $e \in [ac]$ and $v(e) \in \mathbb{R}^3$ which obey (4.6).

We need to consider b and d, even if they seem to play no role in the thesis, since they enter the definition of $v(p)$ and $v(q)$.

Again, we show that the general case can be obtained by the special case computed explicitly above by a continuity argument. In order to do this, it is convenient to restate the problem in suitable Cartesian coordinates. The points considered are

$$a = (0, 0), \quad c = (l, 0), \quad p = (l/2, l/2\sqrt{3}), \quad q = (l/2, -l/2\sqrt{3}).$$

Since $|v(a) - v(p)| = |v(c) - v(p)| = l/\sqrt{3}$, and the same for $v(q)$, we can assume that

$$\begin{aligned} v(a) &= (0, 0, 0), \quad v(c) = (\alpha, 0, 0)l, \\ v(p) &= \left(\frac{\alpha}{2}, \sqrt{\frac{1}{3} - \frac{\alpha^2}{4}}, 0\right) l, \\ v(q) &= \left(\frac{\alpha}{2}, -\sqrt{\frac{1}{3} - \frac{\alpha^2}{4}} \cos\psi, \sqrt{\frac{1}{3} - \frac{\alpha^2}{4}} \sin\psi\right) l. \end{aligned}$$

Here α and ψ are such that $(1 - \kappa\delta) \leqq \alpha \leqq (1 - \delta)$ and $\psi \in [0, 2\pi)$. In these coordinates the unknowns e and $v(e)$ can be written as

$$e = (\lambda l, 0) \text{ with } \lambda \in (0, 1), \quad v(e) = (x, y, z)l.$$

We compute explicitly

$$|p - e|^2 = |q - e|^2 = l^2(\lambda - 1/2)^2 + l^2/12 = l^2(1/3 - \lambda + \lambda^2),$$

and so on. The conditions (4.6) are therefore equivalent to the system

$$\begin{gathered} x^2 + y^2 + z^2 = \lambda^2 \\ (x - \alpha)^2 + y^2 + z^2 = (1 - \lambda)^2 \\ (x - \alpha/2)^2 + \left(y - \sqrt{\frac{1}{3} - \frac{\alpha^2}{4}}\right)^2 + z^2 = \frac{1}{3} - \lambda + \lambda^2 \\ (x - \alpha/2)^2 + \left(y + \sqrt{\frac{1}{3} - \frac{\alpha^2}{4}} \cos\psi\right)^2 + \left(z - \sqrt{\frac{1}{3} - \frac{\alpha^2}{4}} \sin\psi\right)^2 = \frac{1}{3} - \lambda + \lambda^2. \end{gathered}$$

We seek a solution $(x, y, z) \in \mathbb{R}^3$ and $\lambda \in (0, 1)$. Subtracting the first equation from the others we see that this is equivalent to

$$\begin{cases} x^2 + y^2 + z^2 = \lambda^2 \\ -2\alpha x + \alpha^2 = 1 - 2\lambda \\ -\alpha x - 2\sqrt{1/3 - \alpha^2/4}\, y = -\lambda \\ -\alpha x + 2\sqrt{1/3 - \alpha^2/4}\, (y \cos\psi - z \sin\psi) = -\lambda. \end{cases}$$

If $\sin\psi \neq 0$ we can solve the last three equations for x, y and z, giving

$$x = \frac{2\lambda - 1 + \alpha^2}{2\alpha}, \quad y = \frac{1-\alpha^2}{4\sqrt{1/3 - \alpha^2/4}}, \quad z = \frac{1+\cos\psi}{\sin\psi} y.$$

It remains to check that the first equation gives a solution for λ. Substituting these values, it becomes

$$\frac{1-\alpha^2}{\alpha^2}\lambda^2 - \frac{1-\alpha^2}{\alpha^2}\lambda + \frac{(1-\alpha^2)^2}{4\alpha^2} + \left[1 + \frac{(1+\cos\psi)^2}{\sin^2\psi}\right] \frac{(1-\alpha^2)^2}{16(1/3-\alpha^2/4)} = 0.$$

After multiplication by $\alpha^2(1-\alpha^2)^{-1}$ this is equivalent to

$$f(\lambda) := \lambda^2 - \lambda + \frac{1-\alpha^2}{4} + \frac{2+2\cos\psi}{\sin^2\psi} \frac{(1-\alpha^2)\alpha^2}{16(1/3-\alpha^2/4)} = 0,$$

which is the generalization of (4.7). It remains to be shown that, for κ sufficiently close to 1, this equation is a continuous perturbation of (4.7), and in particular that the two relevant properties (existence of the solution, uniform bounds) are preserved. Since this is a quadratic equation, where only the last term is nontrivial, this reduces to proving that, if δ and κ satisfy appropriate bounds, then $f(0)$ is a small perturbation of the value given in (4.7).

The key remark is that for $\kappa = 1$ the domain set $D(\delta, \kappa)$ defined in (4.9) shrinks to a point, up to isometries. Indeed, if $\kappa = 1$ it is easy to see that the map v equals $1-\delta$ times an affine isometry on the four points $\{a, b, c, d\}$. Then, we are exactly in the situation discussed in Part 0, and have constructed the same $v(p)$ and $v(q)$, hence in this case $f(0)$ has the value entering (4.7). In the following we call the values corresponding to $\kappa = 1$ (that is, to the construction given in Part 0) "rigid" values.

Assume now that $\kappa - 1$ is small, that is that $v(a)$, $v(b)$, $v(c)$, and $v(d)$ can move in a neighborhood of their rigid position (again, up to isometries). Then in particular α is close to $(1-\delta)l$ [precisely, $(1-\kappa\delta)l < \alpha < (1-\delta)l$]; $v(p)$ and $v(q)$, which are fixed via the continuous function G, are also close to their rigid positions, and therefore also the angle ψ is close to its rigid value. This proves that for small $\kappa - 1$ the value $f(0)$ has small deviation from its rigid value, hence that there is always a solution λ_1 close to the one obtained for the rigid case. In particular we can assume $\lambda_1 \in (1/4, 1/2)$. Note that the necessary smallness of $\kappa - 1$ can depend on δ, but not on l.

Both steps are local, in the sense that they do not modify the function outside a small (fixed) region, as explained at the beginning, hence we can perform them independently on the entire grid. The only exception are the edges that belong to $\partial\Omega_{\mathcal{T}}$, since there is no neighboring triangle. They are simply dropped from the construction. Precisely, if the point d (in the notation above, see Fig. 14b) is not a grid point, then we eliminate the triangle $[acp]$ from the grid. Since l was chosen small enough, the resulting grid still covers Ω.

5. Short maps and three-dimensional elasticity

Here we show how the upper bounds for the two-dimensional functional I_h can be translated into corresponding bounds on the three-dimensional functional E_h. This is done through the Kirchhoff–Love ansatz introduced in Section 1.3. The following estimate is now standard; we give the explicit construction for completeness.

Lemma 5.1. *Let $u \in W^{2,2} \cap W^{1,\infty}(\Omega; \mathbb{R}^3)$, and let $M \in \mathbb{R}$ be such that $\|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \leqq M$. Then there is $\psi \in W^{1,2}(\Omega \times (0,h); \mathbb{R}^3)$ such that $\psi(x, y, 0) = u(x, y)$ (in the sense of traces), $|\psi_z| \leqq 2$, and*

$$\mathrm{dist}(\nabla\psi, SO(3)) \leqq C(M)\left[\mathrm{dist}(\nabla u, O(2,3)) + h|\nabla^2 u|\right], \tag{5.1}$$

where $C(M)$ denotes a universal constant depending only on M.

Proof. Let us consider $\psi(x, y, z) := u(x, y)+zN(x, y)$, where $N := f(\nabla u)(u_x \wedge u_y)$ and $f : \mathbb{R}^{3\times 2} \to \mathbb{R}$ is defined by

$$f(F) = \min\left\{2, \frac{1}{|F_1 \wedge F_2|}\right\}.$$

Then we have

$$\nabla\psi = \nabla u + z\nabla N + N \otimes e_3,$$

where $\nabla u = u_x \otimes e_1 + u_y \otimes e_2$ and $\nabla N = N_x \otimes e_1 + N_y \otimes e_2$. Let $F : \Omega \to O(2,3)$ be such that

$$|\nabla u - F| = \mathrm{dist}(\nabla u, O(2,3)).$$

Then $G := F + (F_1 \wedge F_2) \otimes e_3$ belongs to $SO(3)$, and thus (5.1) follows from

$$|\nabla\psi - G|^2 \leqq C(M)\left[\mathrm{dist}^2(\nabla u, O(2,3)) + h^2|\nabla^2 u|^2\right]. \tag{5.2}$$

It remains to prove (5.2). We first compute

$$|\nabla\psi - G|^2 = |\nabla u + z\nabla N - F|^2 + |N - F_1 \wedge F_2|^2.$$

We observe that $|z\nabla N|^2 \leqq C(M)h^2|\nabla^2 u|^2$ almost everywhere. Indeed, on the set where $|u_x \wedge u_y| \geqq 1/2$ we have $N = (u_x \wedge u_y)/|u_x \wedge u_y|$ and thus

$$|\nabla N| \leqq C\frac{|\nabla u|\,|\nabla^2 u|}{|u_x \wedge u_y|} \leqq C(M)|\nabla^2 u|.$$

On the other hand, on the set $|u_x \wedge u_y| \leqq 1/2$, we have $N = 2u_x \wedge u_y$ and thus

$$|\nabla N| \leqq C|\nabla u||\nabla^2 u| \leqq C(M)|\nabla^2 u|.$$

To conclude the proof of (5.2) we have to estimate $|N - F_1 \wedge F_2|$. Again, we distinguish two cases. If $|u_x \wedge u_y| \geqq 1/2$,

$$|N - F_1 \wedge F_2| = \left| \frac{u_x \wedge u_y}{|u_x \wedge u_y|} - \frac{F_1 \wedge F_2}{|F_1 \wedge F_2|} \right| \leqq C \left(|u_x - F_1| + |u_y - F_2| \right).$$

On the other hand if $|u_x \wedge u_y| < 1/2$ then necessarily $|\nabla u - F| \geqq 1/C$, hence

$$|N - F_1 \wedge F_2| \leqq |N| + |F_1 \wedge F_2| \leqq 2|\nabla u|^2 + 1 \leqq C(M)|\nabla u - F|.$$

This concludes the proof of (5.2) and therefore of the lemma.

Now we are in a position to prove Theorems 1.2 and 1.1.

Proof of Theorem 1.2. Let $(\mathcal{T}, v)$ be an origami map, with φ the maximum jump of ∇v and $\Gamma(\mathcal{T})$ the degeneracy parameter of $\mathcal{T}$, and let $h_k \to 0$. For k sufficiently large we can apply Theorem 1.6 in order to find $u_k \in C^2(\Omega; \mathbb{R}^3)$ such that

$$\|v - u_k\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqq C_{\Gamma(\mathcal{T})}\, \varphi\, h_k^{1/3}, \quad I_{h_k}(u_k, \Omega) \leqq C_{\Gamma(\mathcal{T})}\, |\mathcal{T}|\, \varphi^2\, h_k^{5/3},$$

and

$$\mathrm{dist}(\nabla u_k, O(2,3)) + h_k |\nabla^2 u_k| \leqq C_{\Gamma(\mathcal{T})}\, \varphi. \tag{5.3}$$

We apply now Lemma 5.1 to each u_k, with $M = C_{\Gamma(\mathcal{T})} + \sqrt{2}$, and obtain maps $\psi_k \in W^{1,2}(\Omega \times (0, h_k); \mathbb{R}^3)$. Since

$$\begin{aligned} |\psi_k(x,y,z) - v(x,y)| &\leqq |u_k(x,y) - v(x,y)| + \int_0^z |(\psi_k)_z(x,y,t)|\, \mathrm{d}t \\ &\leqq C_{\Gamma(\mathcal{T})}\, \varphi\, h_k^{1/3} + 2h_k, \end{aligned}$$

the uniform convergence of ψ_k to v follows. From (5.1) we get

$$\frac{1}{h_k} \int_{\Omega \times (0,h_k)} \mathrm{dist}^2(\nabla \psi_k, SO(3)) \leqq C_{\Gamma(\mathcal{T})}\, I_{h_k}(u_k, \Omega) \leqq C_{\Gamma(\mathcal{T})}\, |\mathcal{T}|\, \varphi^2\, h_k^{5/3}.$$

This proves the first part of the statement. If $\varphi \leqq c_*/C_{\Gamma(\mathcal{T})}$, then by (5.1) and (5.3) we get $\mathrm{dist}(\nabla \psi_k, SO(3)) \leqq \tilde{c}_*$, and the proof is concluded by (1.2).

Proof of Theorem 1.1. For any $\delta > 0$, by Theorem 1.8 applied to the short map $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ there is an origami map $(\mathcal{T}_\delta, v_\delta)$ such that

$$\|u - v_\delta\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqq \delta, \quad |\varphi_\delta| \leqq \delta,$$

where φ_δ denotes the maximum jump of ∇v_δ in Ω and $C_{\Gamma(\mathcal{T}_\delta)} \leqq C$. Thus, for small enough δ, the product $C_{\Gamma(\mathcal{T}_\delta)}\, \varphi_\delta$ is smaller than c_*. Then, by the second part of Theorem 1.2 we find for every $k \geqq k(\delta)$ a map $\psi_{\delta,k} \in W^{1,2}(\Omega \times (0, h_k); \mathbb{R}^3)$ such that $\psi_{k,\delta}$ converges uniformly to v_δ as $k \to \infty$ and $E_{h_k}(\psi_{k,\delta}, \Omega) \leqq C\, |\mathcal{T}_\delta|\, \delta^2\, h_k^{5/3}$. We conclude the proof by taking a diagonal subsequence in $k \to \infty$ and $\delta \to 0$.

Proof of Theorem 1.3 (ii). Part (ii) of Theorem 1.3 follows immediately from Lemma 2.5 and Lemma 5.1.

6. The approximation of a single fold: lower bound

In this Section we prove that the construction presented in Section 2 for the energy of a single fold is optimal. We do this first in the simplified two-dimensional setting of the energy I_h, then we extend the argument to the three-dimensional functional E_h. The latter result is based on the quantitative rigidity result recently obtained by Friesecke, James and Müller [8].

We first sketch the main idea of the proof, which is common to the two- and three-dimensional arguments, with the help of Fig. 15. Let E be the total energy. The boundary data are satisfied in a strip $\sigma \leqq |y| \leqq 2\sigma$. For each such y (and fixed z, in three dimensions), we have a one-dimensional problem, describing deformation of a rod with clamped endpoints. This implies that it can deviate from a straight segment only by increasing its length, which is penalized by the stretching energy. More precisely, we have a map $\zeta : (\alpha, \beta) \to \mathbb{R}^3$, such that $|\zeta(\alpha) - \zeta(\beta)| = |\alpha - \beta|$. Let ε be the maximum deviation of ζ from the affine interpolation between the endpoints. The length of the curve $\zeta([\alpha, \beta])$ must be at least $|\alpha - \beta|(1 + c\varepsilon^2/|\alpha - \beta|^2)$ (for small ε), and hence the energy—on the segment—has to be at least $c\varepsilon^4/|\alpha - \beta|^3$. A more careful estimate (Lemma 6.3) leads to an L^2 bound on ζ', which implies the L^∞ estimate via a standard embedding. Since y can be chosen freely in an interval of size σ, we get $E \geqq \sigma\varepsilon^4/l^3$. Further, we can repeat the argument for four values y_1, y_2, y_3, y_4, two positive and two negative, and with all distances bounded from above and below by multiples of σ.

Consider now a vertical section, intersecting the four lines at constant y. The deformation must be in four points uniformly close to the sharp fold, that is, to a map with a corner, hence it cannot be straight (see Fig. 16). Precisely, if u is ε-close to a line with a kink of order φ, at four points with distances of order σ, then either $\varepsilon > c\varphi\sigma$, or the oscillation of u_y must be at least $c\varphi$. In the first case $E \geqq c\sigma^5\varphi^4/l^3$. In the second case, the squared L^2 norm of u_{yy} on the segment is necessarily at least $c\varphi^2/\sigma$, and integrating we obtain $E \geqq ch^2 l\varphi^2/\sigma$. Combining the two estimates, and choosing the optimal value $\sigma = (l^2h/\varphi)^{1/3}$,

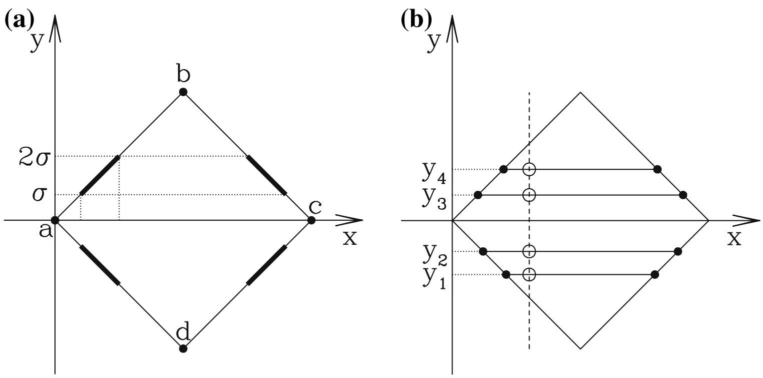

Fig. 15. Geometry for the proof of the lower bound: **a** represents the boundary conditions, **b** the values of y_i, and one cross-section at constant x (*dashed*). The *full dots* mark the points where the boundary values are used, the *open dots* the ones where the distance to v is small

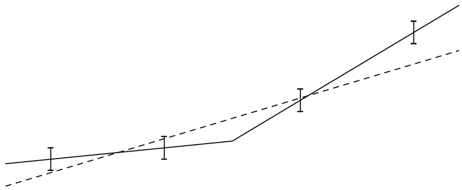

Fig. 16. No straight line can be in four points close to a curve with a kink. If the distances between the four points are at least σ, and the angle of the kink is φ, then the distance in at least one of the points must be at least $c\sigma\varphi$. (Note: three points would suffice, we use four to match the symmetry of our problem). The four points and the horizontal axis correspond to the *open dots* and the *dashed line* in Fig. 15

gives the result in two dimensions. In the three-dimensional case one needs an extra argument showing that a term analogous to the bending energy arises from the three-dimensional stretching energy, see below.

6.1. The lower bound in two dimensions

Let $([abcd], v)$ be a single fold of length $l > 0$, slope $\tau > 0$, and angle $\varphi \in (0, \pi)$. For simplicity we shall restrict ourselves to consider the case in which $[abcd]$ is a rhombus, so that after a change of coordinates we have

$$Q := [abcd] = \{(x, y) \in \mathbb{R}^2 : 0 \leqq x \leqq l,\ 0 \leqq |y| \leqq \tau \min\{x, l - x\}\}, \quad (6.1)$$

$$v(x, y) = \begin{cases} xe_1 + ye_2, & y > 0; \\ xe_1 + y\cos\varphi e_2 + y\sin\varphi e_3, & y < 0. \end{cases} \quad (6.2)$$

(statement and proof apply with minor changes to the general case).

Proposition 6.1. *Let $u_h \in W^{2,2}(Q; \mathbb{R}^3)$ be such that $u_h = v$ on $\{p \in \partial Q :$ $\mathrm{dist}(p, [ac]) \in (\sigma_h, 2\sigma_h)\}$, with*

$$\sigma_h = \frac{l^{2/3}h^{1/3}}{\varphi^{1/3}}, \quad h \leqq Cl\varphi\tau^3.$$

Then

$$I_h(u_h, Q) \geqq c\varphi^{7/3}l^{1/3}h^{5/3}.$$

Proof. We shall first work with generic values of $\sigma \in (0, l\tau/8)$, and show that if

$$u(x, y) = v(x, y), \quad \text{on } \partial Q \cap \{\sigma < |y| < 2\sigma\} \quad (6.3)$$

then

$$I_h(u, Q) \geqq c \min\left\{\frac{l\varphi^2h^2}{\sigma}, \frac{\varphi^4\sigma^5}{l^3}\right\}. \quad (6.4)$$

The Proposition will follow choosing $\sigma = \sigma_h$ as given in the statement, which makes this bound optimal. Choices with a different scaling would lead to a worse bound. In particular, if (6.3) holds with $\sigma = h^{1/3-\alpha}$ for some $\alpha \in (0, 1/3)$, then only a bound $I_h(u_h, Q) \geqq ch^{5/3+\alpha}$ is obtained (and this cannot be improved, as can be seen using the construction discussed before Remark 1.5).

It remains to prove that (6.3) implies (6.4). In order to do this, we use two lemmas, which correspond to the tension and bending part of the argument sketched above. Notice that $W^{2,2}$ functions in two dimensions have a continuous representative, hence there is no difficulty in speaking of point values.

Lemma 6.2. *Let v be as in* (6.2)*, $u \in W^{2,2}(Q)$, and y_1, y_2, y_3, and y_4 be such that*

$$-2\sigma < y_1 < y_2 < -\sigma, \quad \sigma < y_3 < y_4 < 2\sigma, \tag{6.5}$$

$$|y_2 - y_1| \geqq \frac{\sigma}{2}, \quad |y_4 - y_3| \geqq \frac{\sigma}{2}, \tag{6.6}$$

for some $\sigma < l\tau/8$. If

$$|u(x, y_i) - v(x, y_i)| \leqq \frac{\varphi\sigma}{32} \text{ for all } x \in (l/4, 3l/4), \quad i = 1, 2, 3, 4. \tag{6.7}$$

then

$$\int_Q |\nabla^2 u|^2 \geqq C \frac{l\varphi^2}{\sigma}.$$

Proof. We claim that (6.7) implies that for almost everywhere x in $(l/4, 3l/4)$ we have

$$\int_{-2\sigma}^{2\sigma} u_{yy}^2(x, y)\, \mathrm{d}y \geqq c_* \frac{\varphi^2}{\sigma} \tag{6.8}$$

for some universal constant c_* defined below. Integrating in x will give the result.

The claim is proven by contradiction. If for some x (6.8) does not hold, then

$$\int_{-2\sigma}^{2\sigma} |u_y - \gamma|^2\, \mathrm{d}y < c_* \varphi^2 \sigma$$

for some $\gamma \in \mathbb{R}^3$. Hence there is an affine function $\xi : \mathbb{R} \to \mathbb{R}^3$ such that

$$\begin{aligned} \|\xi - u(x, \cdot)\|_{L^\infty(-2\sigma,2\sigma)} &\leqq \|\gamma - u_y\|_{L^1(-2\sigma,2\sigma)} \\ &\leqq 2\sigma^{1/2} \|\gamma - u_y\|_{L^2(-2\sigma,2\sigma)} \leqq 2c_*^{1/2} \varphi\sigma. \end{aligned}$$

We choose c_* so that $2c_*^{1/2} = 1/32$, and combine with (6.7) to obtain

$$|\xi(y_i) - v(x, y_i)| \leqq \frac{\varphi\sigma}{16}, \quad i = 1, 2, 3, 4.$$

But this is impossible, as can be easily checked by applying the mean-value theorem to ξ first on (y_1, y_2) and then on (y_3, y_4) (see also Fig. 16).

We now present the second Lemma, which concerns the stretching part of the energy.

Lemma 6.3. *Let $\gamma \in W^{1,2}([a,b];\mathbb{R}^3)$, $e \in S^2$, and assume $g(b)-g(a) = (b-a)e$. Then*

$$\int_a^b |\gamma' - e|^2 \,\mathrm{d}t \leqq \int_a^b \Big| |\gamma'|^2 - 1 \Big| \,\mathrm{d}t.$$

Proof. For every $w \in \mathbb{R}^3$ and $e \in S^2$ we have $|w-e|^2 = |w|^2 - |e|^2 + 2e\cdot(e-w)$. Take $w = \gamma'(t)$ and integrate over (a,b).

We are finally ready to conclude the proof of Proposition 6.1. We distinguish two cases. If there are four points y_i such that the assumption of Lemma 6.2 holds, then we immediately have the result. If not, there is a set J of values of y of measure at least $\sigma/8$ such that for all $y \in J$

$$|u(x,y) - v(x,y)| > \frac{\varphi\sigma}{32} \quad \text{for some } x \in I^y = (|y|/\tau, l - |y|/\tau).$$

Since at the extrema of the interval I^y by the boundary condition $u = v$, we obtain

$$\frac{\varphi\sigma}{32} \leqq \|u_x - v_x\|_{L^1(I^y)} \leqq l^{1/2}\|u_x - e_1\|_{L^2(I^y)}.$$

The last term is in turn estimated in terms of the stretching energy using Lemma 6.3. We get

$$\int_{I^y} (u_x^2 - 1)^2 \,\mathrm{d}x \geqq \frac{1}{|I^y|} \left\| u_x^2 - 1 \right\|_{L^1(I^y)}^2 \mathrm{d}x \geqq \frac{1}{|I^y|} \|u_x - e_1\|_{L^2(I^y)}^4 \geqq C\frac{\varphi^4\sigma^4}{l^3}.$$

Integrating over $y \in J$ we obtain

$$I_h(u_h, Q) \geqq C\frac{\varphi^4\sigma^5}{l^3},$$

which concludes the proof of (6.4) and hence of Proposition 6.1.

6.2. The lower bound in three dimensions

We again consider the single fold defined in (6.1–6.2), and extend it to three dimensions via the Kirchhoff–Love ansatz as explained in Sect. 1.3.

Lemma 6.4. *Let $\psi_h \in W^{1,2}(Q \times (0,h);\mathbb{R}^3)$ be such that $\psi_h(x,y,z) = v(x,y) + Nz$ for $(x,y,z) \in \{p \in \partial Q : \mathrm{dist}(p,[ac]) \in (\sigma_h, 2\sigma_h)\} \times (0,h)$, with*

$$\sigma_h = \frac{l^{2/3}h^{1/3}}{\varphi^{1/3}}, \quad h \leqq Cl\varphi\tau^3, \quad h \leqq Cl\varphi^{-1/2}.$$

Then

$$E_h(\psi_h, Q) \geqq c\varphi^{7/3}l^{1/3}h^{5/3}.$$

The general lines of the argument are as above, however, the estimate of the bending energy is more complex. Therefore we replace Lemma 6.2 with the following.

Lemma 6.5. *Let* $T = (x_0, x_0 + h) \times (-2\sigma, 2\sigma) \times (0, h)$*,* $\sigma > 5h$*, and let* $\psi \in W^{1,2}(T; \mathbb{R}^3)$ *be such that*

$$\frac{1}{h^3} \int_{(x_0, y_i, 0)+(0,h)^3} |\psi(x, y, z) - v(x, y)| \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \leqq \frac{\varphi\sigma}{32}, \quad i = 1, 2, 3, 4, \tag{6.9}$$

where the y_i *satisfy* (6.5–6.6)*,* $y_4 < 2\sigma - 2h$*, and* $x_0 \in (l/4, 3l/4 - h)$*. Then*

$$\frac{1}{h} \int_T \mathrm{dist}^2(\nabla\psi, SO(3)) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \geqq c \frac{\varphi^2 h^3}{\sigma}.$$

Proof. Without loss of generality we assume $\sigma = Kh$ for some $K \in \mathbb{Z}$ (otherwise we restrict to a smaller domain). Consider the cubes

$$Q_k = (x_0, kh, 0) + (0, h)^3, \quad k \in \frac{1}{2}\mathbb{Z}, \quad -K \leqq k \leqq K - 1.$$

They are all contained in T. By the quantitative rigidity estimate of Friesecke, James, and Müller [8] there is a universal constant C such that for each cube one can find $R_k \in SO(3)$ with

$$\int_{Q_k} |\nabla\psi - R_k|^2 \leqq C \int_{Q_k} \mathrm{dist}^2(\nabla\psi, SO(3)). \tag{6.10}$$

Since each cube overlaps over half of its volume with the preceding one, it follows that

$$\begin{aligned} \frac{1}{2} h^3 |R_k - R_{k-1}|^2 &= \int_{Q_k \cap Q_{k-1}} |R_k - R_{k-1}|^2 \\ &\leqq \int_{Q_k \cap Q_{k-1}} |\nabla\psi - R_k|^2 + |\nabla\psi - R_{k-1}|^2 \\ &\leqq C \int_{Q_k \cup Q_{k-1}} \mathrm{dist}^2(\nabla\psi, SO(3)), \end{aligned}$$

where we used (6.10) on Q_k and on Q_{k-1}. Summing over all cubes we obtain

$$\sum_k \int_{Q_k} |\nabla\psi - R_k|^2 \leqq ChE \quad \text{and} \quad \sum_k |R_k - R_{k-1}|^2 \leqq C \frac{E}{h^2},$$

where

$$E = \frac{1}{h} \int_T \mathrm{dist}^2(\nabla\psi, SO(3)).$$

Let now $m : (-2\sigma, 2\sigma) \to \mathbb{R}^{3\times3}$ be defined by $m = R_k$ on $(kh, (k+1/2)h)$. Then $m \in BV$, and

$$\begin{aligned} \int_{(-2\sigma,2\sigma)} |Dm| &= \sum_k |R_k - R_{k-1}| \leqq (2K)^{1/2} \left(\sum_k |R_k - R_{k-1}|^2 \right)^{1/2} \\ &\leqq C \left(\frac{EK}{h^2} \right)^{1/2} = C \left(\frac{E\sigma}{h^3} \right)^{1/2}. \end{aligned}$$

Therefore there is $m_0 \in \mathbb{R}^{3\times 3}$ such that

$$\int_{-2\sigma}^{2\sigma} |m - m_0|^2 \leqq \sigma \left(\int |Dm| \right)^2 \leqq C \frac{E\sigma^2}{h^3}. \tag{6.11}$$

Consider now the function $g : (-2\sigma, 2\sigma) \to \mathbb{R}^3$ defined by averaging ψ over x and z,

$$g(y) = \frac{1}{h^2} \int_{x_0}^{x_0+h} \int_0^h \psi(x, y, z) \, dx \, dz.$$

By the convexity of the norm it follows that $g \in W^{1,2}$, and

$$\begin{aligned} \int_{-2\sigma}^{2\sigma} |g' - m \cdot e_2|^2 \, dy &\leqq \frac{1}{h^2} \int_T |(\nabla\psi(x, y, z) - m(y)) \cdot e_2|^2 \\ &\leqq \frac{1}{h^2} \sum_k \int_{Q_k} |\nabla\psi - R_k|^2 \leqq C \frac{E}{h}. \end{aligned}$$

Combining with (6.11) we obtain, since $\sigma \leqq h$,

$$\int_{-2\sigma}^{2\sigma} |g' - m_0 \cdot e_2|^2 \, dy \leqq C \frac{E\sigma^2}{h^3}.$$

Therefore there is an affine function $\xi(y) = (m_0 \cdot e_2)y + \xi_0$ such that

$$\|g - \xi\|_{L^\infty(-\sigma,\sigma)} \leqq \sigma^{1/2} \|g' - m_0 \cdot e_2\|_{L^2} \leqq C \left(\frac{E\sigma^3}{h^3} \right)^{1/2}.$$

At the same time, by averging (6.9) one has, since v is affine away from $y = 0$,

$$\left| g(y_i) - v\left(x_0 + \frac{h}{2}, y_i \right) \right| \leqq \frac{\varphi\sigma}{32}, \quad i = 1, 2, 3, 4.$$

Since there is no affine function ξ with

$$\left| \xi(y_i) - v\left(x_0 + \frac{h}{2}, y_i \right) \right| \leqq \frac{\varphi\sigma}{16},$$

we conclude that necessarily

$$\frac{\varphi\sigma}{32} \leqq C \left(\frac{E\sigma^3}{h^3} \right)^{1/2},$$

which is the thesis.

Proof of Lemma 6.4. We follow the same strategy used for Proposition 6.1, and prove that for all $\sigma \in (5h, l\tau/8)$ and all ψ which satisfy the boundary condition one has

$$E_h(\psi, Q) \geqq c \min\left\{\frac{l\varphi^2h^2}{\sigma}, \frac{\varphi^4\sigma^5}{l^3}, \frac{\varphi^2\sigma^3}{l}\right\} \tag{6.12}$$

(the last term will be irrelevant for the optimal σ). If there are four points y_i such that the assumption of Lemma 6.5 holds for all $x_0 \in (l/4, 3l/4-h)$, then (6.12) follows (indeed, there are $l/4h$ values of x_0 so that the corresponding Ts are disjoint, and all T are in the integration domain for E_h, since $\sigma < l\tau/8$). Otherwise, there is a set $J \in (-2\sigma, -\sigma) \cup (\sigma, 2\sigma)$ such that $|J| \geqq \sigma/8$ and for all $y_0 \in J$ there is an $x_0 \in (l/4, 3l/4 - h)$ such that

$$\frac{1}{h^3}\int_{(x_0,y_0,0)+(0,h)^3} |\psi(x, y, z) - v(x, y)| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z > \frac{\varphi\sigma}{32}. \tag{6.13}$$

For simplicity of notation, we focus on the case $y_0 > 0$ (which implies $y_0 > \sigma$). Using Lemma 6.3 for each fixed $(y, z) \in (y_0, y_0 + h) \times (0, h)$ we get

$$\|\psi - v\|^2_{L^\infty(I^y)} \leqq l\|\psi_x - e_1\|^2_{L^2(I^y)} \leqq l\|\psi_x^2 - 1\|_{L^1(I^y)},$$

where $I^y = (y/\tau, l - y/\tau)$. Averaging over $Q_0 = (x_0, y_0, 0) + (0, h)^3$ gives

$$\frac{1}{h^3}\int_{Q_0}|\psi - v| \leqq \frac{1}{h^2}l^{1/2}\int_0^h\int_{y_0}^{y_0+h}\left(\int_{I^y}\left|\psi_x^2 - 1\right|\mathrm{d}x\right)^{1/2}\mathrm{d}y\,\mathrm{d}z$$
$$\leqq \frac{l^{1/2}}{h}\left(\int_{K_y}\left|\psi_x^2 - 1\right|\right)^{1/2},$$

where $K_y = (Q \cap \{y_0 < y < y_0 + h\}) \times (0, h)$ is the three-dimensional corresponding integration domain. Comparing with (6.13) we get

$$\frac{l}{h^2}\int_{K_y}\left|\psi_x^2 - 1\right| \geqq C\varphi^2\sigma^2.$$

Finally, since $|J| \geqq \sigma/8$ there are at least $C\sigma/h$ values of $y \in J$ where this holds, and such that the corresponding K_y are disjoint. Summing we get

$$\int_{\cup K_y}\left|\psi_x^2 - 1\right| \geqq C\frac{\varphi^2\sigma^3h}{l}.$$

Finally, we estimate $|\cup K_y| \geqq ch\sigma l$ and $|\psi_x^2 - 1| \leqq C\,\mathrm{dist}(\nabla\psi, SO(3)) + C\,\mathrm{dist}^2(\nabla\psi, SO(3))$. Therefore

$$\int_{\cup K_y}\mathrm{dist}^2(\nabla\psi, SO(3)) + (h\sigma l)^{1/2}\left(\int_{\cup K_y}\mathrm{dist}^2(\nabla\psi, SO(3))\right)^{1/2} \geqq C\frac{\varphi^2\sigma^3h}{l},$$

that is,

$$\int_{\cup K_y} \operatorname{dist}^2(\nabla\psi, SO(3)) \geqq \min\left\{\frac{\varphi^2\sigma^3 h}{l}, \frac{\varphi^4\sigma^5 h}{l^3}\right\}.$$

which concludes the proof of (6.12). Finally, it suffices to insert in (6.12) the value of σ to prove the lemma.

Proof of Theorem 1.3 (i). Part (i) of Theorem 1.3 follows immediately from Lemma 6.4.

7. Notation

Constants. We denote by c and C generic positive and finite constants that can change from line to line and that are independent from the various parameters (h, τ, l, φ, r, diam Ω, etc.), but that can depend on the energy density W.

Vectors. The vector space $\mathbb{R}^3$ is spanned by an orthonormal system $\{e_1, e_2, e_3\}$, and its generic point is $(x, y, z) = xe_1 + ye_2 + ze_3$. The space $\mathbb{R}^2$ is the subspace generated by $\{e_1, e_2\}$, and its generic point is $(x, y) = xe_1 + ye_2$.

Tensors. The space $\mathbb{R}^{j\times k}$ of $j \times k$ matrices has $\{e_{j'} \otimes e_{k'}\}_{1\leqq j'\leqq j, 1\leqq k'\leqq k}$ as an orthonormal system. The set of linear isometries $\mathbb{R}^k \to \mathbb{R}^j$ is $O(k, j) := \{F \in \mathbb{R}^{j\times k} : F^T F = \mathrm{Id}_k\}$, where $\mathrm{Id}_k := \sum_{i=1}^k e_i \otimes e_i \in \mathbb{R}^{k\times k}$. A tensor $F \in \mathbb{R}^{j\times k}$ is said short if $(Fe)^2 \leqq e^2$ for every $e \in \mathbb{R}^k$, that is, if $F^T F \leqq \mathrm{Id}_k$. Further, $SO(3) := \{F \in O(3, 3) : \det F = 1\}$.

Domains. Given $a, b, c,$ and d in $\mathbb{R}^2$ or in $\mathbb{R}^3$ the symbols $[ab]$, $[abc]$ and $[abcd]$ denote the convex hulls of the sets $\{a, b\}$, $\{a, b, c\}$, and $\{a, b, c, d\}$, respectively. We use $p, q, a, b, c, d,$ and e to denote points of $\mathbb{R}^2$. Correspondingly, $\mathrm{d}x\,\mathrm{d}y$, $\mathrm{d}p$, and $\mathrm{d}q$ denote integration against the two-dimensional Lebesgue measure.

The letters Ω, Q, and variants are used to denote domains of $\mathbb{R}^2$; the corresponding three-dimensional domains have the form $\Omega \times (0, h) = \{p + ze_3 : p \in \Omega, z \in (0, h)\}$.

Maps. Three-dimensional deformations from $\Omega \times (0, h)$ to $\mathbb{R}^3$ are denoted by ψ, while we use u, v, and variants for two-dimensional deformations $\Omega \to \mathbb{R}^3$. Directional derivatives of ψ with respect to e_1, e_2, and e_3 are denoted by ψ_x, ψ_y, and ψ_z, and an analogous notation is used for u and v. By u_{xx}, u_{yy}, and u_{xy} we denote the second-order directional derivatives of $u : \Omega \to \mathbb{R}^3$, with respect to $\{e_1, e_2\}$. A deformation $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ is said short if $\nabla u(p) \in \mathbb{R}^{3\times 2}$ is a short tensor for almost every $p \in \Omega$.

Balls. The unit ball of $\mathbb{R}^k$ centered in the origin is denoted by B^k, while rB^k is the ball with radius r. More generally $B(\cdot, \cdot)$ is the ball of the ambient space of the first argument, having center in the first argument and radius equal to the second argument. In this way if $a \in \Omega$, $v : \Omega \to \mathbb{R}^3$, and $h > 0$ then $B(a, h)$ is the ball of $\mathbb{R}^2$ with center in a and radius h, while $B(v(a), h)$ is the ball of center $v(a)$ and radius h in $\mathbb{R}^3$.

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