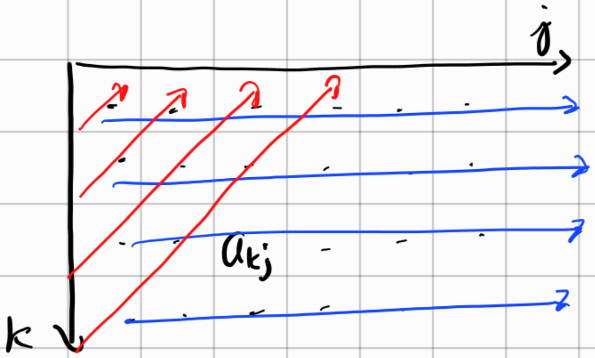
ANALISI MATEMATICA B
LEZIONE 30 – 28.11.2025

Lezione scorsa:
$$\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} a_{kj} = \sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} a_{kj}$$

se $\displaystyle\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} |a_{kj}| < +\infty.$

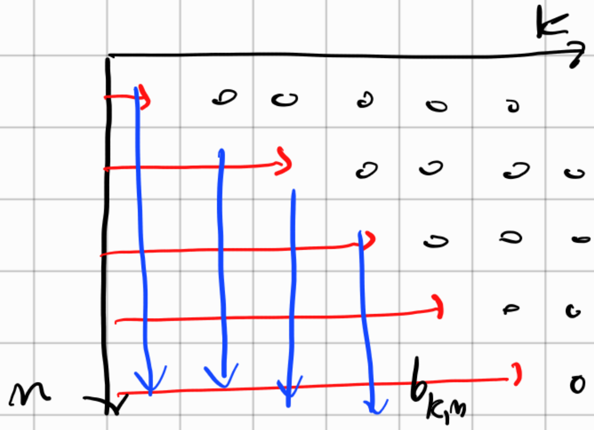
Corollario (somme alla Cauchy)

$$\underbrace{\sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_{k,n-k}} = \underbrace{\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} a_{k,j}}$$



se una delle due serie (e quindi entrambe) è assolutamente convergente

dim



$$b_{k,n} = \begin{cases} a_{k,n-k} & \text{se } k \le n \\ 0 & \text{se } k > n \end{cases}$$

$$\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} b_{k,n} = \sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty} a_{k,n-k} = \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} a_{k,j}$$

$$\sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty} b_{k,n} = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_{k,n-k}$$

□

NB: $0^0 = 1$

Esempio

$$f(z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!} \qquad \boxed{f(z) \stackrel{?}{=} e^z}$$

$$f(z+w) = \sum_{n=0}^{+\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{\binom{n}{k} z^k w^{n-k}}{n!}$$

$$\left[\binom{n}{k} = \frac{n!}{k!(n-k)!} \right] \qquad = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$$

$$= \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{z^k}{k!} \frac{w^j}{j!} = \sum_{k=0}^{+\infty} \left(\frac{z^k}{k!} \underbrace{\sum_{j=0}^{+\infty} \frac{w^j}{j!}} \right)$$

Somme alla Cauchy

$$= \left(\sum_{j=0}^{+\infty} \frac{w^j}{j!} \right) \cdot \left(\sum_{k=0}^{+\infty} \frac{z^k}{k!} \right)$$

$$= f(w) \cdot f(z).$$

è un omomorfismo da $(\mathbb{C}, +) \to (\mathbb{C} \setminus \{0\}, \cdot)$

$$\lim_{z \to 0} \frac{f(z) - 1}{z} \stackrel{?}{=} 1$$

$$\left[\begin{array}{l} \dfrac{e^x - 1}{x} \to 1 \quad (x \to 0) \\ \dfrac{\sin y}{y} \to 1 \quad (y \to 0) \end{array} \right.$$

$$\frac{f(z)-1}{z} = \frac{\sum_{k=0}^{+\infty} \frac{z^k}{k!} - 1}{z} = \frac{\cancel{1} + \sum_{k=1}^{+\infty} \frac{z^k}{k!} - \cancel{1}}{z}$$

$$= \sum_{k=1}^{+\infty} \frac{z^{k-1}}{k!} = \sum_{k=0}^{+\infty} \frac{z^k}{(k+1)!} = 1 + \sum_{k=1}^{+\infty} \frac{z^k}{(k+1)!}$$

$z^k \to 0$

VA DIMOSTRATO! (*) $\quad \downarrow z \to 0$

$$1 + 0 = 1$$

SERIE DI POTENZE

$$f(z) = \sum_{k=0}^{+\infty} a_k \cdot z^k$$ ← "polinomi di grado ∞"

$a_k \in \mathbb{C}$ (o $a_k \in \mathbb{R}$)

$f : A \to \mathbb{C}$ $\quad A = \{ z \in \mathbb{C} : \text{la serie converge} \}$

c'è convergenza assoluta?

$$\sum |a_k z^k| = \sum |a_k| \cdot |z|^k$$

uso il criterio della radice: $\sqrt[k]{|a_k|} \cdot |z|$

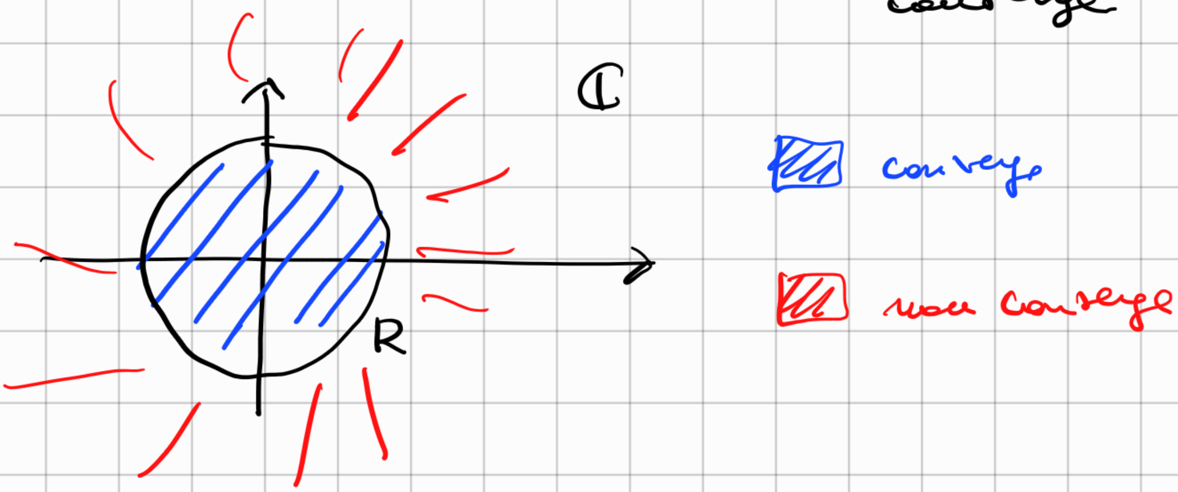
se $\sqrt[k]{|a_k|} \to \ell$ per $k \to +\infty$

allora $\sqrt[k]{|a_k|} \cdot |z| \to \ell \cdot |z| \overset{?}{<} 1$ sì se $|z| < \frac{1}{\ell}$

$R = \frac{1}{\ell}$ $\quad \ell = \lim_{k \to \infty} \sqrt[k]{|a_k|}$ (se il limite esiste)

se $|z| < R \Rightarrow$ la serie converge assolutamente

se $|z| > R \Rightarrow |a_k z^k| \to +\infty \Rightarrow$ la serie non converge



$$B_R \subseteq A \subseteq \overline{B_R}$$

$$B_R = \{z\in\mathbb{C} : |z|<R\} \qquad \overline{B_R} = \{z\in\mathbb{C} : |z|\le R\}$$

DETTAGLI DA PRECISARE:

(1) Se $\lim_{k\to+\infty}\sqrt[k]{|a_k|}$ non esiste posso prendere

$$\limsup_{k\to+\infty}\sqrt[k]{|a_k|}$$

esiste!

dove

$$\limsup_{k\to+\infty} b_k = \max\left\{ \lim_{j\to+\infty} b_{k_j} : \text{quando il limite esiste} \right\}$$

Es: $b_k=(-1)^k$

oppure

$$\limsup_{k\to+\infty} b_k = \underset{n\to+\infty}{\overset{(\inf)}{\lim}} \underbrace{\sup_{k\ge n}}\, b_k$$

esiste sempre   decrescente

Se $\limsup \sqrt[k]{|a_k|} = \ell$

$$\Rightarrow \begin{cases} \forall q<\ell & \sqrt[k]{|a_k|}>q \text{ frequentemente} \\ \forall q>\ell & \sqrt[k]{|a_k|}<q \text{ definitivamente} \end{cases}$$

$\Rightarrow$ vale il criterio della radice.

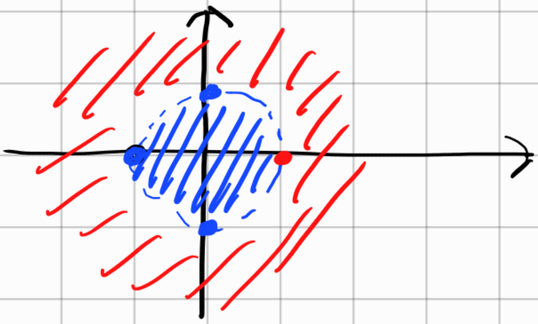
(2) se $\ell = 0 \Rightarrow R = +\infty \Rightarrow$ la serie converge assolutamente su tutto $\mathbb{C}$

se $\ell = +\infty \Rightarrow R = 0 \Rightarrow$ la serie converge solo per $z = 0$.

ES 1 la serie $\sum_{k=0}^{+\infty} \frac{z^k}{k}$ $\qquad a_k = \frac{1}{k}$

$$\ell = \lim_{k\to+\infty} \sqrt[k]{\frac{1}{k}} = 1 \qquad R = \frac{1}{\ell} = 1$$

converge assolutamente se $|z| < 1$



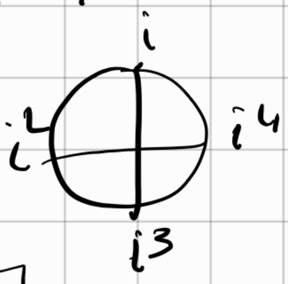
$z = 1 \qquad \sum \frac{1}{k} = +\infty$

$z = -1 \qquad \sum \frac{(-1)^k}{k}$

converge per Leibniz

ES Cosa succede se $|z| = 1$ e $z \notin \mathbb{R}$?

es: $z = i \qquad \sum \frac{i^k}{k} = ?$



$$i^k = \begin{cases} i & \text{se } k = 1, 5, 9, \dots \\ -1 & \text{se } k = 2, 6, 10, \dots \\ -i & \text{se } k = 3, 7, 11, \dots \\ 1 & \text{se } k = 4, 8, 12, \dots \end{cases}$$

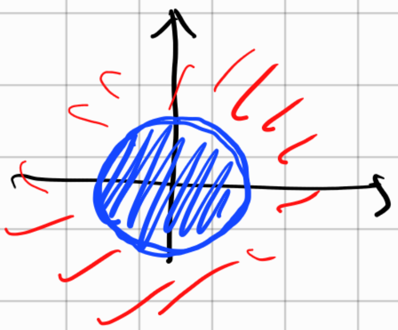
$$\sum_{k=1}^{2N+1} \frac{i^k}{k} = i \sum_{k=0}^{N-1} \frac{(-1)^k}{2k+1} + \sum_{k=1}^{N} \frac{(-1)^k}{2k} = \text{convergenti!}$$

Si può dimostrare che $\sum \frac{z^k}{k}$

converge per ogni $|z|=1$, $z\neq 1$.

usando il criterio di Abel che c'è negli appunti (generalizzazione di Leibniz)

ES 2



$$\boxed{\sum \frac{z^k}{k^2}}$$

converge assolutamente se $|z|\le 1$.

Teo (continuità delle serie di potenze).

Sia $f(z)=\sum_{k=0}^{+\infty} a_k z^k$ una serie di potenze con raggio di convergenza $R>0$

Allora $\lim_{z\to 0} f(z) = f(0)$.

dim

$$f(z) = a_0 + \sum_{k=1}^{+\infty} a_k z^k = f(0) + z\sum_{k=1}^{+\infty} a_k z^{k-1}$$

$$= f(0) + \underset{\downarrow\, 0}{z} \cdot \underbrace{\sum_{k=0}^{+\infty} a_{k+1} z^k}_{\text{è limitata?}}$$

$f(0)$

ha raggio di conv. R come quella originaria

Se $|z|\le\rho<R$ $\quad \left|\sum_{k=0}^{+\infty} a_{k+1} z^k\right| \le \sum_{k=0}^{+\infty} |a_{k+1}|\rho^k = \underline{M} < +\infty$ sì $\square$

Dunque $\sum_{k=0}^{+\infty} \frac{z^k}{k!} = e^z$.

In particolare: $e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$ $x \in \mathbb{R}$

se $z = iy$ $e^{iy} = \cos y + i \operatorname{sen} y$ $y \in \mathbb{R}$

ma $e^{iy} = \sum_{k=0}^{+\infty} \frac{(iy)^k}{k!} = \sum_{k=0}^{+\infty} \frac{i^k y^k}{k!}$

$i^{2k+1} = i \cdot i^{2k} = i(-1)^k$

$$= \sum_{k=0}^{+\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{i^{2k+1} y^{2k+1}}{(2k+1)!} = \underbrace{\sum_{k=0}^{+\infty} \frac{(-1)^k y^{2k}}{(2k)!}}_{\cos y} + i \underbrace{\sum_{k=0}^{+\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}}_{\operatorname{sen} y}$$

$$\cos y = \sum_{k=0}^{+\infty} (-1)^k \frac{y^{2k}}{(2k)!} \; / \quad \sin y = \sum_{k=0}^{+\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}$$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$$

$$\sin 1 = 1 - \frac{1}{6} + \frac{1}{120} \pm \frac{1}{600}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e = e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \pm \frac{1}{600}$$

$$\simeq 2{,}5 + 0{,}1\bar{6} + 0{,}04 \simeq 2{,}7$$

$$\frac{1}{6} = \frac{0{,}\bar{3}}{2} = 0{,}16\bar{6} \qquad \frac{1}{24} = \frac{0{,}1\bar{6}}{4}$$