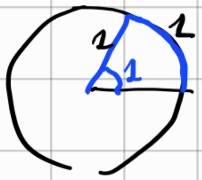
ANALISI MATEMATICA B
LEZIONE 24 – 14.11.2025

La definizione di π

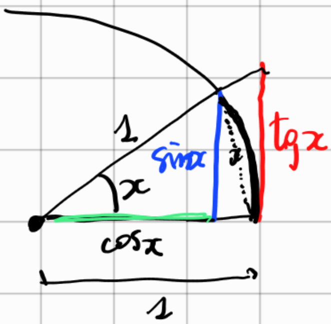
- $\dfrac{e^x-1}{x} \to 1$
- $\dfrac{\sin x}{x} \to 1$

per $x \to 0$

Geometricamente sia 2π la lunghezza della circonferenza unitaria prendiamo $\tau = 2\pi$ l'angolo unitario è il radiante cioè un angolo per cui l'arco di raggio unitario ha lunghezza 1.



Allora



$$\operatorname{tg} x = \frac{\sin x}{\cos x}$$

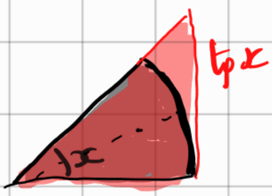
$$\sin x \overset{!}{\le} x \overset{?}{\le} \operatorname{tg} x$$

$$\Downarrow$$

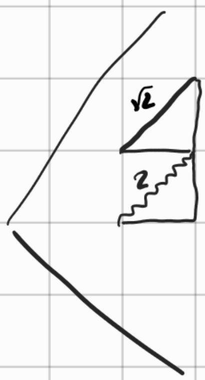
$$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x} \quad \text{per } x \to 0$$

$$\frac{x}{\sin x} \to 1, \qquad \frac{1}{\cos x} \to 1$$

(?) si può risolvere pensando alle aree:

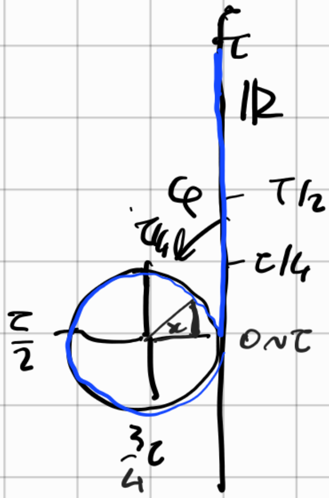


area settore: $\frac{1}{2}x \le \frac{1}{2}\operatorname{tg} x =$ area triangolo



In modo più formale

Abbiamo definito $\varphi : \mathbb{R} \to U \subseteq \mathbb{C}$ $\quad U = \{ z \in \mathbb{C} : |z| = 1 \}$



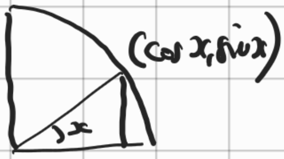
$$\varphi(x+y) = \varphi(x)\,\varphi(y)$$

↑ somma degli angoli

$$\varphi(\tau) = 1$$

$\operatorname{Im} \varphi$ crescente su $\left[0, \frac{\tau}{4}\right]$

$$\begin{cases} \cos x = \operatorname{Re} \varphi(x) \\ \sin x = \operatorname{Im} \varphi(x) \end{cases}$$



$$\underset{\uparrow}{\operatorname{tg} x = \frac{\sin x}{\cos x}} \qquad \operatorname{tg}(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

$$= \frac{\operatorname{tg} x + \operatorname{tg} y}{1 - \operatorname{tg} x \operatorname{tg} y}$$

Lemma se $n \in \mathbb{N}$, $x \geq 0$ $\quad nx < \frac{\tau}{4}$ $\quad \left(\frac{\tau}{4} = 90^\circ \right)$

allora $\quad n \sin x \leq \operatorname{tg} nx$.

dim Dimostriamo per induzione su n che:

$$nx < \frac{\tau}{4} \Rightarrow \underbrace{n \sin x \leq \operatorname{tg} nx}$$

ipotesi induttiva

- $n = 0$ (o $n = 1$) è banale.
- "$n \Rightarrow n+1$".

$(n+1)x < \frac{\tau}{4}$

$$\underbrace{\operatorname{tg}((n+1)x)}_{\geq 0} = \frac{\overbrace{\operatorname{tg} nx + \operatorname{tg} x}^{\geq 0}}{\underbrace{1 - \underbrace{\operatorname{tg} nx \operatorname{tg} x}_{\geq 0}}_{\leq 1}} \geq \operatorname{tg} nx + \operatorname{tg} x \geq n \sin x + \operatorname{tg} x$$

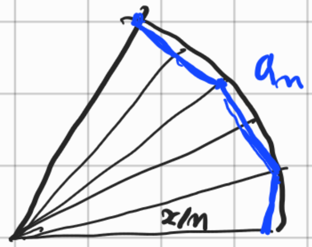
$$\geq n \sin x + \sin x$$

$$= (n+1) \sin x$$

□

Teorema se $0 \le x \le \frac{\pi}{2}$ la successione

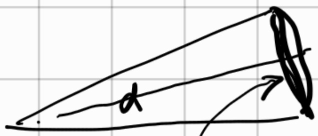
$a_n = n \sin_\pi \frac{x}{n}$ converge.



dim

Passo 1 dimostro che

a_n è crescente.



cioè

$$n \sin \frac{x}{n} \overset{?}{\le} (n+1) \sin \frac{x}{n+1}.$$

Prendo $t = \frac{x}{n(n+1)}$ $(n \ge 1)$

$$a_n = n \sin \frac{x}{n} = n \sin (n+1)t = n \sin(nt) \cos t + \underbrace{n \cos nt \sin t}_{\text{lemma}}$$

$$\le n \sin(nt) \overset{\le 1}{\cos t} + \cos nt \cdot \operatorname{tg} nt$$

$$\le n \sin(nt) + \sin(nt) = (n+1) \sin(nt)$$

$$= (n+1) \sin \frac{x}{n+1}. \qquad \square$$

[il lemma si applica se $nt < \frac{\pi}{4}$

noi sappiamo che $nt \le x \le \frac{\pi}{2}$

ma $n \cos nt \sin t \le \sin nt$ è ovvia se $\frac{\pi}{4} \le nt \le \frac{\pi}{2}$]

Dunque a_n ha limite finito o infinito.

ma $a_n = n \sin \frac{x}{n}$ $\quad a_{3n} = 3n \sin \frac{x}{3n} \underset{\text{lemma}}{\le} 3 \operatorname{tg} \frac{x}{3}$ $\quad \boxed{\frac{x}{3} < \frac{\tau}{4}.}$

a_{3n} è limitata.

dunque a_n non può tendere a $+\infty$. □

$n \sin_\tau \frac{x}{n}$ rappresenta la lunghezza dell'arco di misura x

Def $\boxed{\pi = \lim_{n\to\infty} n \sin_\tau \frac{\tau}{2n}}$.

$\frac{\tau}{2} = 180°$

Allora se scelgo $\tau = 2\pi$ ho

$$\lim_{n\to\infty} n \sin_{2\pi}\frac{\pi}{n} = \pi$$

cioè

$$\boxed{\lim_{n\to+\infty} \frac{\sin_{2\pi}\frac{\pi}{n}}{\frac{\pi}{n}} = 1}$$

D'ora in poi $\boxed{\tau = 2\pi}$ misuriamo gli angoli in RADIANTI!

Teorema (limite notevole)

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1.}$$

e, di conseguenza $\frac{1-\cos x}{x} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \overset{1}{\frac{\sin x}{x}} \cdot \underset{0}{\frac{\sin x}{1+\cos x}}$

$$\lim_{x\to 0} \frac{1-\cos x}{x}$$

cioè $\varphi(x) = \cos x + i \,\mathrm{sen}\, x$

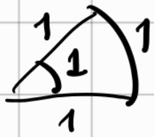
$$\frac{\Delta\varphi}{\Delta x} = \frac{\varphi(x)-\varphi(0)}{x-0} = \frac{\cos x + i\,\mathrm{sen}\, x - 1}{x}$$

$$= \frac{\cos x - 1}{x} + i\,\frac{\mathrm{sen}\, x}{x} \longrightarrow i$$

i è il vettore velocità istantanea del moto circolare uniforme $x \mapsto \varphi(x)$

l'arco con angolo x ha lunghezza x.

1 radiante = arco di lunghezza 1 sulla circonferenza di raggio 1



dim (teo.) se $x>0$ prendo $t=\frac{\pi}{x}$ $\quad t\to+\infty$

$$\frac{\sin(-x)}{-x} = \frac{\sin x}{x} = \frac{t}{\pi}\sin\frac{\pi}{t} \qquad \text{con } t\to+\infty$$

$$\frac{\lfloor t\rfloor}{\pi}\sin\frac{\pi}{\lceil t\rceil} \le \underbrace{\frac{t}{\pi}\sin\frac{\pi}{t}}_{\downarrow\ 1} \le \frac{\lceil t\rceil}{\pi}\sin\frac{\pi}{\lfloor t\rfloor} =$$

$$\| \qquad\qquad\qquad\qquad\qquad \|$$

$$\underbrace{\frac{\lfloor t\rfloor}{\lceil t\rceil}}_{\to 1}\ \underbrace{\frac{\lfloor t\rfloor}{\pi}\sin\frac{\pi}{\lceil t\rceil}}_{\to 1} \qquad\qquad \underbrace{\frac{\lceil t\rceil}{\lfloor t\rfloor}}_{\downarrow\ 1}\ \underbrace{\frac{\lfloor t\rfloor}{\pi}\sin\frac{\pi}{\lfloor t\rfloor}}_{\to 1}$$

$\square$

ESPONENZIALE COMPLESSO

$$\exp(ix) = \varphi(x) = \cos x + i \,\mathrm{sen}\, x$$

FORMULA DI EULERO

$$\boxed{e^{ix} = \cos x + i \,\mathrm{sen}\, x}$$

$$\exp(x+iy) = \exp x \cdot \exp(iy)$$

$z = x+iy$
$x, y \in \mathbb{R}$

$$\boxed{e^{z} = e^{x+iy} = e^{x} \underbrace{(\cos y + i \,\mathrm{sen}\, y)}_{1 \cdot 1 = 1}}$$

$\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$ $\qquad |e^{z}| = e^{x}$

$$\boxed{\exp(z+w) = \exp(z) \cdot \exp(w).}$$

infatti $\quad z = x+iy$
$\qquad\quad w = a+ib$

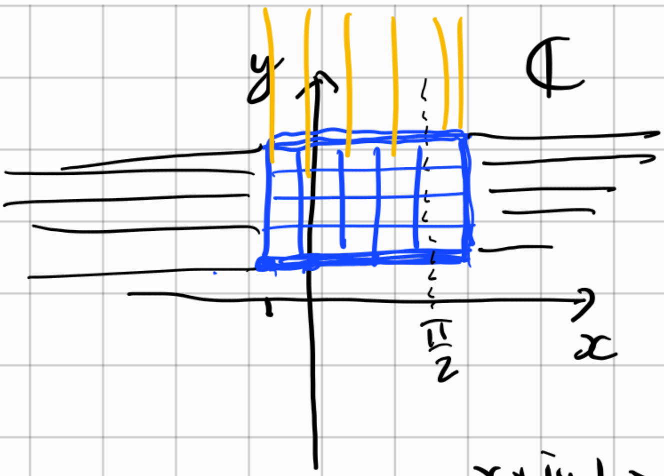
$$e^{z+w} = e^{a+x+i(b+y)} = e^{a+x}\varphi(b+y)$$

$$= e^{a} e^{x} \varphi(b)\varphi(y) = \underbrace{e^{a}\cdot\varphi(b)}\cdot\underbrace{e^{x}\varphi(y)}$$

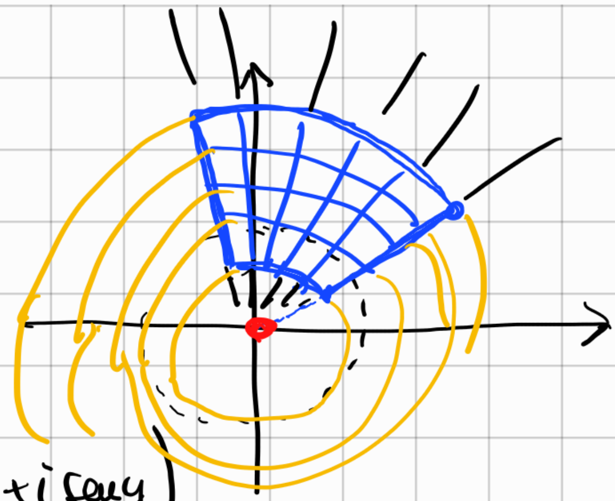
$$= e^{z} e^{w}.$$

Es 🏠 Immaginare come è fatta

$$\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$$



$\xrightarrow{\exp}$



$$x+iy \mapsto e^{x}(\underbrace{\cos y + i \operatorname{sen} y})$$

$$\overset{\shortparallel}{\rho}(\cos y + i \operatorname{sen} y) \qquad \rho = e^{x}$$

e^{x+iy} è $2\pi i$ - periodica