12 nov 2025 – LEZIONE 23

Risultato dimostrato il 10 nov

RECAPITOLAZIONE

$$\boxed{\lim_{x\to 0}\frac{e^x-1}{x}=1 \qquad \lim_{x\to 0}\frac{\log(1+x)}{x}=1}$$

$e^x-1\sim x$ per $x\to 0$

$e^x=1+x+o(x)$ per $x\to 0$

$$\lim_{x\to+\infty} x^\alpha e^{-x}=0$$

$$\lim_{x\to 0^+} x^\alpha \log x=0 \qquad \boxed{\forall\alpha>0}$$

$$\lim_{x\to +\infty}\frac{x^\alpha}{e^x}=0$$

$x^\alpha=o(e^x)$ per $x\to+\infty$

$\log x=o\left(\frac{1}{x^\alpha}\right)$

(*) $$\boxed{\lim_{x\to+\infty}\frac{\log x}{x^\alpha}=0 \quad \forall\alpha>0}$$

$$f(x)=o(g(x)) \text{ per } x\to\alpha \overset{\text{def}}{\iff} \lim_{x\to\alpha}\frac{f(x)}{g(x)}=0$$

$\boxed{\alpha>0}$

$$\lim_{x\to\infty}\frac{\log x}{x^\alpha}\underset{\boxed{\log x=y \iff y=e^x}}{=}\lim_{y\to\infty}\frac{1}{\alpha}\frac{\alpha y}{e^{\alpha y}}\underset{\boxed{t=\alpha y}}{=}\frac{1}{\alpha}\lim_{t\to\infty}\frac{t}{e^t}=0$$

$$\boxed{\lim_{x\to 0^+} x^x=1}$$

$$\boxed{x^{n+m}=x^n\cdot x^m \qquad \text{se } n=0 \quad x^0=1}$$

$$x^x=e^{x\log x}\xrightarrow{x\to 0}e^0=1$$

$$x\log x\xrightarrow[x\to 0^+]{}0$$

$$\boxed{a^x=e^{x\log a}}$$

$$\boxed{\lim_{x\to 0^+}\frac{x^x-1}{x}=-\infty}$$

$$x^x-1=e^{x\log x}-1\sim x\log x$$
per $x\to 0$

$$\frac{x^x - 1}{x} \sim \frac{\cancel{x} \log x}{\cancel{x}} \sim \log x \xrightarrow[x \to 0]{} -\infty$$

$$\boxed{\begin{aligned} &\underline{\text{Prop}}: \left.\begin{aligned} &\bullet\ f(x) \sim f_0(x) \\ &\bullet\ g(x) \sim g_0(x) \end{aligned}\right\} \text{ per } x \to \alpha \\ &\text{Allora } \lim_{x \to \alpha} \frac{f(x)}{g(x)} = \lim_{x \to \alpha} \frac{f_0(x)}{g_0(x)} \end{aligned}}$$

$\underline{\text{Dim}}$

$$\frac{f(x)}{g(x)} = \underbrace{\cancel{\frac{f(x)}{f_0(x)}}}_{\downarrow\ 1} \cdot \underbrace{\cancel{\frac{g_0(x)}{g(x)}}}_{\downarrow\ 1} \cdot \frac{f_0(x)}{g_0(x)} \quad \text{per } x \to \alpha$$

⚠ Non è lecito fare questo tipo di sostituzione quando ci sono operazioni $\underline{+}$

ESEMPIO

$$\lim_{x \to 0} \frac{\sqrt{1 + \frac{x}{3}} - 1 + 1 - \sqrt[3]{1 + x/2}}{x^2}$$

$$e^y - 1 \sim y \quad y \to 0$$

$$(1+x)^\alpha - 1 = e^{\alpha \log(1+x)} - 1 \sim \alpha \log(1+x) \sim \alpha x \quad \text{per } x \to 0$$

$$\left(1 + \frac{x}{3}\right)^{1/2} - 1 \sim \frac{x}{6} \qquad \sqrt[3]{1 + x/2} - 1 \sim \frac{x}{6}$$

$$\frac{\sqrt{1+\frac{x}{3}}-1+1-\sqrt[3]{1+x/2}}{x^2} \sim \frac{\cancel{x/6}-\cancel{x/6}}{x^2}=0$$

No!

$$\sqrt{1+\frac{x}{3}} = 1+\frac{x}{6}+o(x) \qquad \sqrt[3]{1+x/2} = 1+\frac{x}{6}+o(x)$$

$$L = \frac{\sqrt{1+\frac{x}{3}}-1+1-\sqrt[3]{1+x/2}}{x^2} = \frac{\cancel{x/6}-\cancel{\frac{x}{6}}+\boxed{o(x)}}{x^2}$$

?

Analizzo il numeratore

$$\boxed{\sqrt{1+\frac{x}{3}}-1+1-\sqrt[3]{1+x/2}} =$$

$$= \frac{2\cdot x/6}{\sqrt{1+\frac{x}{3}}+1} + \frac{\left(-3x/6\right)}{1+\sqrt[3]{1+\frac{x}{2}}+\sqrt[3]{(1+x/2)^2}} =$$

$$= \frac{x}{6}\cdot\frac{\boxed{2\left(1+\sqrt[3]{1+x/2}+\sqrt[3]{1+x+o(x)}\right)-3\left(\sqrt{1+\frac{x}{3}}+1\right)}^{\,N}}{\underbrace{\boxed{\left(\sqrt{1+\frac{x}{3}}+1\right)\left(1+\sqrt[3]{1+x/2}+\sqrt[3]{1+x+o(x)}\right)}}_{D}}$$

$$N = 2\left(1+1+\frac{x}{6}+o(x)+1+\frac{x}{3}+o(x)\right) - 3\left(1+\frac{x}{6}+o(x)+1\right)$$

tutti i termini costanti si elidono!

$$= 2\cdot\frac{x}{6}+2\cdot\frac{x}{3}-\frac{3x}{6}+o(x)$$

$$= x\left(\frac{2+4-3 \quad +o(1)}{6}\right) = x\left(\frac{1}{2}+o(1)\right)$$

$$L = \frac{1}{\cancel{x^2}}\frac{\cancel{x}}{6}\cdot\frac{\cancel{x}\left(\frac{1}{2}+o(1)\right)}{D} \xrightarrow[x\to 0]{} \frac{1}{72}$$

Formula di Taylor

$$D \xrightarrow[x\to 0]{} 6$$

Se avessimo strumenti più potenti potremmo conoscere subito lo sviluppo di ordine 2 e il calcolo si accorcia

$$\boxed{(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + o(x^2)} \quad \text{per } x\to 0$$

ESERCIZIO: usare questa formula per calcolare più agilmente il limite proposto

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$$\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) \qquad \varphi(x) = \frac{\log(1+x)}{x}$$

$$n\log\left(1+\frac{1}{n}\right) = \frac{\log(1+1/n)}{1/n} \qquad \frac{1}{n}\xrightarrow[n\to\infty]{} 0$$

$$\varphi\left(\frac{1}{n}\right) \xrightarrow[n\to\infty]{} 1$$

$a\neq 1$

$(a>0)$

$$\boxed{\lim_{n\to\infty} \sqrt[n]{a} = 1} \qquad e^x = 1 + x + o(x) \text{ per } x\to 0$$

$$\sqrt[n]{a} = e^{\frac{1}{n}\log a} = 1 + \frac{1}{n}\log a + o\left(\frac{1}{n}\right) \text{ per } n\to\infty$$

$$\boxed{\lim_{n\to\infty}\frac{\sqrt[n]{a}-1}{1/n}} = \lim_{n\to\infty} n\left(\frac{1}{n}\log a + o\left(\frac{1}{n}\right)\right) = \boxed{\log a}$$

$$\boxed{\lim_{n\to\infty} \left(\sqrt[n]{n}-1\right)n = +\infty}$$

$\log n = o(n)$

$$\sqrt[n]{n}-1 \sim e^{\frac{1}{n}\log n}-1 \sim \frac{\log n}{n}$$

$$\boxed{\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = e^2}$$

$$\left(\frac{n-1+2}{n-1}\right)^n = \left(1+\frac{2}{n-1}\right)^{n-1}\left(1+\frac{2}{n-1}\right)$$

$$\left(1+\frac{2}{n-1}\right)^{n-1} \to e^2 \qquad \left(1+\frac{2}{n-1}\right) \to 1$$

$$x_n = \frac{2n}{n-1} \to x = 2$$

$$\left(1+\frac{x_n}{n}\right)$$

$$\left(1+\frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^x$$

Prop: Se $x_n \to \ell \in \mathbb{R}$ allora

$$\boxed{\lim_{n\to+\infty} \left(1+\frac{x_n}{n}\right)^n = e^\ell}$$

Dim:

$$\left(1+\frac{x_n}{n}\right)^n = \exp\left(n\log\left(1+\frac{x_n}{n}\right)\right) \longrightarrow e^\ell$$

$$n\log\left(1+\frac{x_n}{n}\right) = \frac{\log\left(1+\frac{x_n}{n}\right)}{x_n/n}\cdot x_n$$

$$x_n \to \ell$$

$$\frac{\log(1+y)}{y} \longrightarrow 1$$

$$x_n \to \ell \Rightarrow y = \frac{x_n}{n} \to 0$$

E se $x_n \to \ell = +\infty$

$$\left(1 + \frac{x_n}{n}\right)^n \underset{\text{Bernoulli}}{\geq} 1 + \cancel{n}\frac{x_n}{\cancel{n}} \to +\infty$$

BERNOULLI

$$\lim_{x \to 0^+} x^{-1/x}$$

$$x^{-1/x} = e^{\underbrace{-\frac{1}{x}}_{-\infty}\,\underbrace{\log x}_{-\infty}} = e^{+\infty} = +\infty$$

$$\lim_{n \to \infty} \left(\frac{n^3 - n - 1}{n^3 - 5n + 6}\right)^n = 0$$

$$(\quad)^n = \left(1 + \frac{1}{n}\overbrace{\frac{4n^2 - 7n}{n^3 - 5n + 6}}^{x_n}\right)^n$$

$$\left(1 + \frac{1}{n} x_n\right)^n \to e^{\lim x_n}$$