10 nov 2025 — Lez 22

La volta scorsa ho scordato di dimostrare questa uguaglianza

TEOREMA 2

(i) $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x \overset{(1)}{=} e \overset{(2)}{=} \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$

(ii) $\lim_{t\to 0}(1+t)^{1/t} = e$

(i) Dimostrata

(i) $\Rightarrow$ (ii)

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1-\frac{1}{y+1}\right)^{-y-1}$$

$x = -(y+1)$

$x \to -\infty$

$y \to +\infty$

$$= \lim_{y\to+\infty}\left(\frac{y+1}{y}\right)^{y+1}$$

$$= \lim_{y\to+\infty}\underbrace{\left(1+\frac{1}{y}\right)^y}_{\downarrow\, e}\underbrace{\left(1+\frac{1}{y}\right)}_{\downarrow\, 1}$$

$$= e$$

$$2 < e < 4$$

$$\boxed{e < 3}$$

voglio dim questo

$$\boxed{\left(1+\frac{1}{n}\right)^{n+1} \searrow e}$$

Potrei usare questo fatto, ma la conv. è molto lenta

$$\left(1+\frac{1}{n}\right)^{n} = \sum_{k=0}^{n} \boxed{\binom{n}{k}\frac{1}{n^k}}$$

$$\binom{n}{k}\cdot\frac{1}{n^k} = \frac{1}{k!}\cdot\boxed{\frac{n!}{(n-k)!}\cdot\frac{1}{n^k}}$$

$$= \boxed{\frac{1}{k!}\cdot\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n\cdot n\cdot\ldots\cdot n}}$$

$$\boxed{\binom{n}{k}\frac{1}{n^k} \leq \frac{1}{k!}}$$

$$\left(1+\frac{1}{n}\right)^{n} \leq \sum_{k=0}^{n}\frac{1}{k!} \qquad (*)$$

$$\boxed{k! \geq 2^{k-1} \quad \forall k \in \mathbb{N}}$$

Da dim. per induzione

× eserc.

$$\sum_{k=0}^{n} \frac{1}{k!} = \boxed{1 + 1 + \frac{1}{2} + \frac{1}{6}} + \sum_{k=4}^{n} \frac{1}{k!}$$

$$\leq \boxed{\frac{8}{3}} + \sum_{k=4}^{n} 2^{-k+1}$$

$$\boxed{h = k-4}$$

$$= \frac{8}{3} + 2^{-3} \boxed{\sum_{h=0}^{n-4} 2^{-h}}$$

$k = 4+h$

2^{-4-h+1}

$= 2^{-3-h}$

$$\boxed{\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a}}$$

$$= \frac{8}{3} + \frac{1}{8} \cdot \boxed{\frac{1 - \left(\frac{1}{2}\right)^{n-3}}{1 - \frac{1}{2}}} \leq \frac{8}{3} + \frac{1}{8} \cdot \frac{1}{1-\frac{1}{2}}$$

$$\leq \frac{8}{3} + \frac{1}{4} < 3$$

$$\sum_{k=0}^{n} \frac{1}{k!} \leq \frac{8}{3} + \frac{1}{4} < 3$$

Questa stima è uniforme in n

Esercizio: Mostrare che
$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!} = e$$

Lo vedremo più avanti

$$2025^{2026} \qquad 2026^{2025}$$

chi è più grande?

Per quali valori di n vale che

$$n^{n+1} < (n+1)^n \; ?$$

$$\boxed{n < \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n}$$

$$\Rightarrow \quad n < 3$$

$n = 2025 > 3$ quindi

$$2025^{2026} > 2026^{2025}$$

MINI RIPASSO

$$\left(1+\frac{1}{n}\right)^n \longrightarrow e$$

$$\left(1-\frac{1}{n}\right)^n \longrightarrow \frac{1}{e}$$

$$\left(1+\frac{x}{n}\right)^n \longrightarrow e^x$$

$$1 \leq \underbrace{\left(1+\frac{1}{n^2}\right)^{n^2 \cdot \frac{1}{n}}}_{\downarrow\, 1} \leq e^{1/n} \longrightarrow 1$$

$$\left(1+\frac{1}{n}\right)^{n^2} \geq 1 + n^2 \cdot \frac{1}{n} \geq \underbrace{1+n}_{\downarrow\, \infty}$$

LIMITI FONDAMENTALI

$$\lim_{x \to 0} \frac{1}{x} \cdot \ln(1+x) = \lim_{x \to 0} \ln \underbrace{(1+x)^{1/x}}_{\to e}$$

$$= \ln e = 1$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{y}{\ln(1+y)} = 1$$

$$y = e^x - 1 \xrightarrow[x\to 0]{} 0$$

$$x = \ln(1+y)$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{e^{x\ln a} - 1}{x\ln a} \cdot \ln a = \ln a$$

$$\boxed{a^x = e^{x\ln a}}$$

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Esercizio: Calcolare $\lim_{x\to 0} \frac{\log_a(1+x)}{x}$

$\log_2$ $\boxed{\log_e}$ $\log_{10}$

$\uparrow$ INFORMATICI

$\uparrow$ MATEMATICI

$\uparrow$ FISICI CHIMICI INGEGNERI GEOLOGI

Se non diversamente specificato $\boxed{\log x}$ per noi vuol dire $\boxed{\log_e x}$

DISUGUAGLIANZE

$$e^x \geq 1+x \qquad \forall x \in \mathbb{R}$$

$$\uparrow$$

$$\left(1+\frac{x}{n}\right)^n \geq 1+n\frac{x}{n} = 1+x$$

$$e^{-x} \geq 1-x$$

$$e^x \leq \frac{1}{1-x}$$

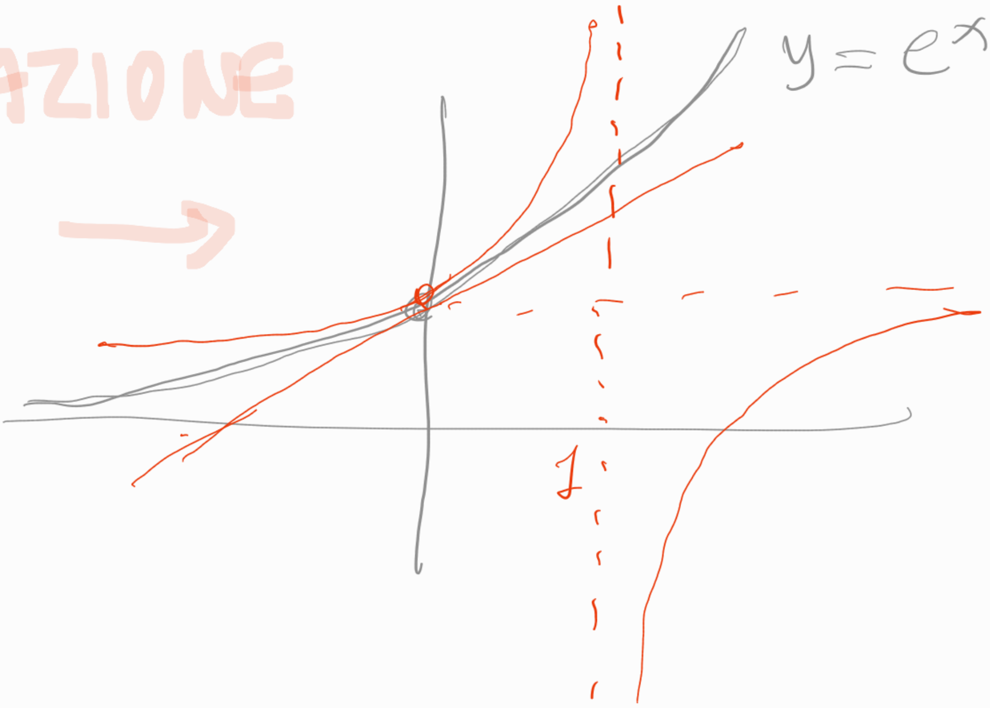
se $x<1$.

$\forall x<1$.

$$1+x \leq e^x \leq \frac{1}{1-x}$$

$\forall x \in \mathbb{R}$

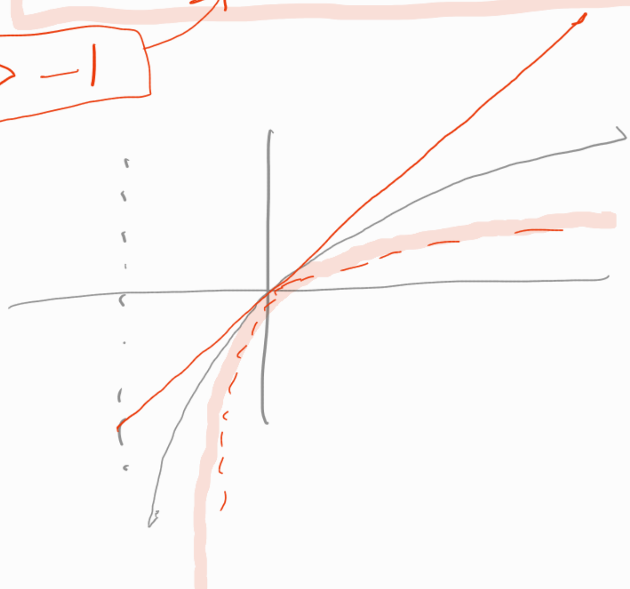
INTERPRETAZIONE GRAFICA →



$$\underbrace{e^x - 1}\doteq y \iff \boxed{x = \log(1+y)}$$

$$e^x \geq 1+x \iff \boxed{e^x - 1} \geq x$$

$$\boxed{y \geq \log(1+y)} \qquad \boxed{y > -1}$$



$y \qquad\qquad y$

$$\log\left(1 - \frac{x}{1+x}\right) \leq -\frac{x}{1+x}$$

$$\log\left(\frac{1}{1+x}\right) \leq -\frac{x}{1+x}$$

$$= -\log(1+x)$$

$$\boxed{\log(1+x) \geq \frac{x}{1+x}}$$

$$\boxed{\frac{x}{1+x} \le \log(1+x) \le x \qquad \forall x > -1}$$

$\boxed{x = \frac{1}{n}}$

$$\boxed{\frac{1}{n+1} \le \log\left(1+\frac{1}{n}\right) \le \frac{1}{n} \qquad \forall n \in \mathbb{N}_*}$$

SOMME ARMONICHE

$$H_n \doteq \sum_{k=1}^{n} \frac{1}{k}$$

$$H_n \geq \sum_{k=1}^{n} \log\left(1+\frac{1}{k}\right)$$

$$\geq \sum_{k=1}^{n} \left[\log(k+1) - \log k\right]$$

$$= \log(n+1) - \cancel{\log(1)}_{0}$$

$$H_n \geq \log(n+1) \xrightarrow[n\to\infty]{} +\infty$$

$$a_n \doteq H_n \cancel{-} \log n \qquad \boxed{a_1 = 1}$$

$$b_n \doteq H_n - \log(n+1) \qquad \boxed{b_1 = 1 - \log 2}$$

$$a_{n+1} - a_n = H_{n+1} - H_n \cancel{-} \left(\log(n+1) - \log n\right)$$

$$= \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right) \leq 0$$

$$\boxed{\frac{1}{n+1} \leq \log\left(1+\frac{1}{n}\right)}$$

$$a_{n+1} \leq a_n$$

a_n decrescente! $\quad a_n \searrow$

$$b_{n+1} - b_n = H_{n+1} - H_n - \log\left(1+\frac{1}{n+1}\right)$$

$$= \frac{1}{n+1} - \log\left(1+\frac{1}{n+1}\right)$$

$x \geq \log(1+x)$

$$\geq 0$$

$$b_{n+1} \geq b_n \implies b_n \nearrow \qquad b_n \geq \underset{\shortparallel}{b_1}$$

$$1 - \log 2 < b_n \leq a_n \leq a_1 = 1$$

$$\shortparallel$$

$$b_1$$

$$\implies a_n \searrow a_*$$

$$b_n \nearrow b_* \qquad b_* \leq a_*$$

$$a_n - b_n = \cancel{H_n} - \cancel{H_n} + \log(n+1) - \log(n)$$

$$= \log\left(1 + \frac{1}{n}\right) \xrightarrow[n \to \infty]{} 0$$

$$b_* = a_* = \gamma \leftarrow \text{costante } \gamma \text{ di Eulero-Mascheroni}$$

$$H_n - \log n \xrightarrow[n \to +\infty]{} \gamma = 0{,}5772156649\ldots$$

Nota: nessuno sa se γ sia o meno razionale

$$H_n - \log n = \gamma + \omega(n)$$

con $\omega(n) \xrightarrow[n\to\infty]{} 0$

$$H_n = \log n + \gamma + \omega(n)$$

"o piccolo"

$$H_n = \log n + \gamma + o(1) \quad \text{per } n \to \infty$$

$$2 \sum_{h=1}^{n} \frac{1}{2h}$$

$$A(n) = \sum_{k=1}^{2n} \frac{(-1)^k}{k} = -\sum_{k=1}^{2n} \frac{1}{k} + 2 \sum_{\substack{k=1 \\ k \text{ pari}}}^{2n} \frac{1}{k}$$

se k disp $\quad -\frac{1}{k}$

se k pari $\quad -\frac{1}{k} + \frac{2}{k} = \frac{1}{k}$

$$= -H_{2n} + H_n$$

$$= -\log 2n \cancel{-\gamma} - o(1) + \log n \cancel{+\gamma} + o(1)$$

$$= -\log 2 + o(1) \quad \text{per } n \to \infty$$

$$A(n) \xrightarrow[n \to +\infty]{} -\log 2$$

FORMULAZIONI EQUIVALENTI

$$\frac{\log(1+x)}{x} \longrightarrow 1 \qquad x \to 0$$

$$\boxed{\log(1+x) \sim x \qquad \text{per } x \to 0}$$

$$\boxed{\frac{\log(1+x)}{x} = 1 + o(1) \quad \text{per } x \to 0}$$

$$\boxed{\log(1+x) = x + o(x) \quad \text{per } x \to 0}$$

$$\frac{o(x)}{x} \xrightarrow[x \to 0]{} 0$$

DEF

$$f(x) \sim g(x) \quad \text{per } x \to \alpha$$

$$\Updownarrow \text{def}$$

$$\lim_{x \to \alpha} \frac{f(x)}{g(x)} = 1$$

f ASINTOTICAMENTE EQUIVALENTE a g per $x \to \alpha$

(NB: è una relazione di equivalenza)

DEF:

si legge "f è o-piccolo di g"

$$f(x) = o(g(x)) \quad \text{per } x \to \alpha$$

sse $$\lim_{x \to \alpha} \frac{f(x)}{g(x)} = 0$$

Questa condizione vuol dire che f è trascurabile rispetto a g quando $x \to \alpha$

OS

$$\lim_{x \to +\infty} \frac{x}{e^x} < \lim_{x \to +\infty} x \cdot \frac{4}{x^2} = 0$$

$$\left(1 + \frac{x}{n}\right)^n \nearrow e^x$$

$$\left(1+\frac{x}{2}\right)^2 < e^x$$

$$\vee \quad \leftarrow \text{se } x>0$$

$$\frac{x^2}{4}$$

$$\boxed{\frac{1}{e^x} < \frac{4}{x^2} \quad \text{se } x>0}$$

$$\lim_{x\to+\infty} \frac{x}{e^x} = 0$$

In altre parole ↓

$$\boxed{x = o(e^x) \quad \text{per } x \to +\infty}$$

$$\lim_{x\to+\infty} \frac{e^x}{x} = +\infty$$

$$\lim_{x\to+\infty} \frac{x^\alpha}{e^x} = 0 \qquad \forall \alpha>0$$

$$\boxed{x^\alpha = o(e^x) \text{ per } x\to+\infty}$$

$$\frac{x^\alpha}{e^x} = \left(\frac{x}{e^{x/\alpha}}\right)^\alpha$$

$$= \left(\frac{x/\alpha}{e^{x/\alpha}}\right)^\alpha \alpha^\alpha \xrightarrow{x \to +\infty} 0$$

$$\frac{x/\alpha}{e^{x/\alpha}} \xrightarrow[x\to\infty]{} 0$$

$$\lim_{x \to 0^+} x \log x = \lim_{t\to\infty} e^{-t}(-t)$$

$$\therefore 0$$

$$x = e^{-t} \qquad t = -\log x$$

$$t \to +\infty$$

$$\lim_{x\to 0^+} x^\alpha \log x = 0 \quad (\alpha > 0)$$

$$x^\alpha \log x = \frac{1}{\alpha} \underbrace{\boxed{x^\alpha \log x^\alpha}}_{\to 0}$$

$x \to 0$

Esercizio:

$$\lim_{x\to\infty} \frac{\log x}{x} = ?$$