ANALISI MATEMATICA B
LEZIONE 33 - 11.12.2024

Notazione di Landau

o-piccolo (O-grande)

Limiti notevoli: per $x \to 0$ • $\frac{\ln(1+x)}{x} \to 1$ • $\frac{e^x - 1}{x} \to 1$

• $\frac{\sin x}{x} \to 1$ • $\frac{\cos x - 1}{x} \to 0$

si possono anche scrivere così: $\begin{cases} \ln(1+x) \sim x \\ e^x - 1 \sim x \\ \sin x \sim x \end{cases}$ (per $x \to 0$)

$$\boxed{\ln(1+x) = x + o(x)}$$

$o(x) << x$, $\frac{o(x)}{x} \to 0$ per $x \to 0$

infatti $\ln(1+x) - x = o(x)$

$$\frac{\ln(1+x) - x}{x} = \frac{\ln(1+x)}{x} - 1 \to 1 - 1 = 0$$

$$\boxed{e^x = 1 + x + o(x)}$$ infatti $\frac{e^x - 1 - x}{x} \to 0$

$$\boxed{\sin x = x + o(x)}$$ ....

$$\boxed{\cos x = 1 + o(x)}$$ ...

Esempio $\lim_{x \to 0} \frac{\ln(1+x) + \sin x}{e^x - \cos x}$

$$\frac{\ln(1+x) + \sin x}{e^x - \cos x} = \frac{x + o(x) + x + o(x)}{1 + x + o(x) - (1 + o(x))} =$$

$$= \frac{2x + o(x)}{\cancel{1} + x - \cancel{1} + o(x)}$$

$$= \frac{2x + o(x)}{x + o(x)} = \frac{2 + \frac{o(x)}{x}}{1 + \frac{o(x)}{x}} \to \frac{2+0}{1+0} = 2 \qquad \square$$

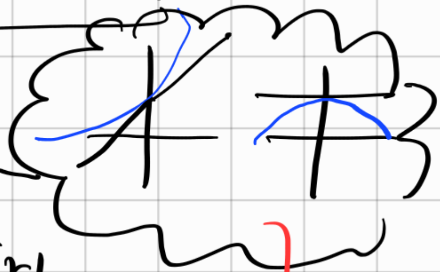
$o(x) + o(x) = o(x)$

$o(x) - o(x) = o(x)$

Esempio

$$\lim_{x \to 0} \frac{\sin x \cdot \ln(1+x)}{e^x \cos x - 1}$$

$$\frac{\sin x \cdot \ln(1+x)}{e^x \cos x - 1} \overset{\in}{=} \frac{\boxed{(x + o(x)) \cdot (x + o(x))}}{(1 + x + o(x))(1 + o(x)) - 1} =$$



$$\left[= \frac{x^2 + o(x) \cdot x + x \cdot o(x) + o(x) \cdot o(x)}{1 + x\,o(x) + o(x)\,o(x) + x + x\,o(x) + o(x) + o(x) \cdot o(x) - 1} \right]$$

$$= \frac{x^2 + o(x^2)}{\cancel{1} + x + o(x) + \cancel{o(x^2)} - \cancel{1}}$$

$$\frac{o(x) \cdot o(x)}{x^2} = \frac{o(x)}{x} \cdot \frac{o(x)}{x}$$

$$\frac{x \cdot o(x)}{x^2} = \frac{o(x)}{x} \to 0$$

$$x \cdot o(x) + o(x) \cdot x = x\,(o(x) + o(x)) = x \cdot o(x) = o(x^2)$$

$$= \frac{x^2 + o(x^2)}{x + o(x)} = \frac{\cancel{x^2}}{\cancel{x}} \cdot \frac{\left(1 + \frac{o(x^2)}{x^2}\right)}{\left(1 + \frac{o(x)}{x}\right)} \to 0$$

$o(x) + o(x^2) = o(x)$

$$\frac{o(x^2)}{x} = \frac{o(x^2)}{x^2} \cdot \cancel{x} \to 0$$

$o(x^2) \overset{\in}{=} o(x)$

ma ⚠ NO $o(x) \neq o(x^2)$

$$\frac{o(x^2)}{x^2} \to 0$$

In generale $\frac{o(g(x))}{g(x)} \to 0$

Formalmente $\{f : \frac{f}{x} \to 0\}$

$\sin x \in x + o(x)$

$\ln(1+x) \in x + o(x)$

$o(x^2) \subseteq o(x)$

Se raccolgo x^2?
$$\frac{x^2+o(x^2)}{x+o(x)} = \frac{1+\cancel{\frac{o(x^2)}{x^2}}^{\to 0}}{\frac{1}{x}+\frac{o(x)}{x^2}} = ?$$
??

Se raccolgo x?
$$\frac{x^2+o(x^2)}{x+o(x)} = \frac{x+o(x)}{1+o(1)} = \frac{o(1)}{1+o(1)} \to 0$$

$$\frac{o(x)}{x^2} = \frac{o(x)}{x} \cdot \frac{1}{x}$$
$$\frac{o(x^2)}{x} = o(x).$$
$$o(1) \to 0$$

NEW

ES $$\lim_{x\to 0} \frac{(e^x-1)\cdot \ln(1+x)}{x\cdot \sin x + x^2} = \frac{1}{2}$$

$e^x = 1+x+o(x)$ per $x\to 0$

ES $$\lim_{n\to\infty} \left(e^{\frac{1}{n}}-1\right)\cdot\sqrt{n^2+1}$$

$$\boxed{e^{\frac{1}{n}} = 1+\frac{1}{n}+o\left(\frac{1}{n}\right)}$$
per $n\to\infty$

$x=\frac{1}{n}$

$$\left(e^{\frac{1}{n}}-1\right)\sqrt{n^2+1} = \left(\cancel{1}+\frac{1}{n}+o\left(\frac{1}{n}\right)-\cancel{1}\right)\sqrt{n^2+1} = \frac{\sqrt{n^2+1}}{n} + o\left(\frac{\sqrt{n^2+1}}{n}\right)$$
$$= \sqrt{1+\frac{1}{n^2}} + o\left(\sqrt{1+\frac{1}{n^2}}\right)$$
$$\to 1$$
□

⚠

$x\to 0 \qquad o(x^2)\subseteq o(x)$

$n\to\infty \qquad o(n^2)\supseteq o(n)$

$$\frac{o(n)}{n^2} = \frac{o(n)}{n}\cdot\frac{1}{n}$$
$\to 0$ $\to 0$

$$o(x)\cdot o(x) \subseteq o(x^2)$$
$$\frac{o(x)\cdot o(x)}{x^2} = \frac{o(x)}{x}\cdot\frac{o(x)}{x} \to 0$$

⚠ dimostrare che $o(x^2) \subseteq o(x)\cdot o(x)$

$\sqrt{n^2+1} \sim n$ $\qquad \sqrt{n^2+1} = o(n^\alpha)$ $\quad \alpha > 1$

per $n \to \infty$

$\sqrt{n^2+1} \notin o(n)$

O-grande: $\sqrt{n^2+1} = O(n)$ per $n \to +\infty$

per $x \to x_0$ $\quad \dfrac{O(g(x))}{g(x)}$ è limitato in un intorno di x_0

oppure $\sqrt{n^2+1} = n + o(1)$ per $n \to \infty$

$$\sqrt{n^2+1} - n = \frac{\cancel{n^2}+1-\cancel{n^2}}{\left(\sqrt{n^2+1}+n\right)} \to 0$$

PUNTIAMO ALLE FORMULE DI TAYLOR.

Esempio

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

VEDREMO!

LA POTENZA della FORMULA DI TAYLOR

$$\lim_{x \to 0} \frac{(\sin x - x)^2}{(1-\cos x)^3}$$

$$\frac{(\sin x - x)^2}{(1-\cos x)^3} = \frac{\left(-\frac{x^3}{6} + o(x^3)\right)^2}{\left(\frac{x^2}{2} + \underbrace{o(x^2)}\right)^3} = \frac{\frac{x^6}{36} + o(x^6)}{\frac{x^6}{8} + o(x^6)} \to \frac{8}{36}$$

$$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

Teorema (Cauchy) $f, g : [a,b] \to \mathbb{R}$, continue su $[a,b]$, derivabili su (a,b), $g'(x) \neq 0 \ \forall x \in (a,b)$

Allora $g(b) \neq g(a)$ e

$$\exists c \in (a,b) : \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

dim (sbagliata)

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}} \overset{\text{Lagrange}}{\neq} \frac{f'(c_1)}{g'(c_2)}$$

NO!

dim (giusta)

Per Rolle se fosse $g(b) = g(a)$ $\exists c$ $g'(c) = 0$.

Devo mostrare $\exists c$:

$$g'(c)\big(f(b)-f(a)\big) - f'(c)\big(g(b)-g(a)\big) = 0 \quad (*)$$

$$H(x) = g(x)\big(f(b)-f(a)\big) - f(x)\big(g(b)-g(a)\big)$$

$$H(b) - H(a) = \big(g(b)-g(a)\big)\big(f(b)-f(a)\big) - \big(f(b)-f(a)\big)\big(g(b)-g(a)\big)$$

$$= 0$$

$\exists c : H'(c) = 0$ cioè $(*)$ $\square$

[Sugli appunti : Teorema di de L'Hospital]