ANALISI MATEMATICA B
LEZIONE 20 - 8.11.2024

Esercizio: dimostrare per induzione su n la **disuguaglianza di Bernoulli**:

$$\forall x > -1 \quad \forall n \in \mathbb{N} \qquad (1+x)^n \geq 1+nx.$$

- $n=0$ $\quad (1+x)^0 = 1 = 1+0\cdot x$ <u>ok!</u> N.B. in questo caso per def. $0^0 = 1$. perche' $0 \in \mathbb{N}$ è intero
- vale per n, vediamo per $n+1$

$(x>-1)$ $\quad (1+x)^{n+1} = (1+x)^n(1+x) \geq$

$\geq (1+nx)(1+x) =$

$= 1+x+nx+nx^2 \geq 1+x+nx = 1+(n+1)x \qquad \square$

$$(1+x)^n \geq 1+nx$$

$$(1+x)^{n+1} \overset{\Updownarrow}{\geq} 1+x(n+1)$$

$$(1+x)^n \cdot (1+x) \overset{\Updownarrow}{\geq} 1+xn+x \qquad \leftarrow$$

$$\cancel{1(1+x)^n} + \underbrace{(1+x)^n x} \overset{\Uparrow}{\geq} \cancel{1+xn} + \underbrace{n+x} \qquad (1+x)^n \overset{!}{\geq} 1+nx$$

$$\cancel{x}(1+x)^n \geq \cancel{x} \quad \rightarrow \quad x>0$$

$$(1+x)^n \geq 1$$

$$(1+x)^n \leq 1 \quad \rightarrow \quad x<0 \quad \ldots\ldots$$

LIMITI NOTEVOLI CON e

$$e = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \left(1+\frac{1}{n}\right)^n \qquad \text{(definizione di e)}$$

Teorema

$$\lim_{\substack{x\to 0 \\ x\in\mathbb{R}}} (1+x)^{\frac{1}{x}} = e$$

dim ① $\displaystyle\lim_{\substack{x\to+\infty \\ x\in\mathbb{R}}} \left(1+\frac{1}{x}\right)^x$

$$\left(1+\frac{1}{\lceil x\rceil}\right)^{\lfloor x\rfloor} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lceil x\rceil} \le \underbrace{\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor}}_{\to e,\ n=\lfloor x\rfloor} \cdot \underbrace{\left(1+\frac{1}{\lfloor x\rfloor}\right)}_{\to 1}$$

$$\left(1+\frac{1}{x}\right)^x \to e.$$

$$\left(1+\frac{1}{\lceil x\rceil}\right)^{\lfloor x\rfloor} = \frac{\left(1+\frac{1}{\lceil x\rceil}\right)^{\lceil x\rceil} \to e}{\left(1+\frac{1}{\lceil x\rceil}\right) \to 1}$$

$$① \Rightarrow \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e$$

② $\displaystyle\lim_{x\to-\infty} \left(1+\frac{1}{x}\right)^x = e$

ovvero $\displaystyle\lim_{x\to+\infty} \left(1-\frac{1}{x}\right)^{-x} = e$

$$\left(1-\frac{1}{x}\right)^{-x} = \left(\frac{x-1}{x}\right)^{-x} = \left(\frac{x}{x-1}\right)^x = \left(\frac{x-1+1}{x-1}\right)^x$$

$$= \left(1+\frac{1}{x-1}\right)^x = \underbrace{\left(1+\frac{1}{x-1}\right)^{x-1}}_{① \to e} \cdot \underbrace{\left(1+\frac{1}{x-1}\right)}_{\to 1} \to e$$

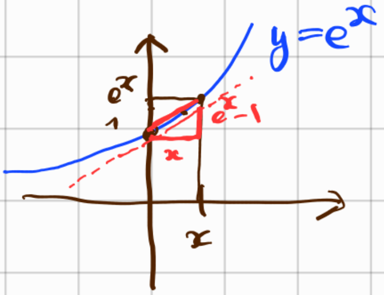
$\square$

Corollario (limiti notevoli)

$\ln x = \log_e x$

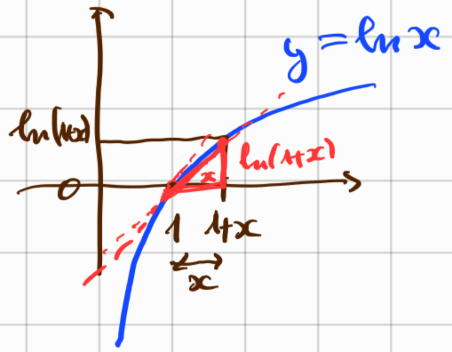
① $\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$

② $\boxed{\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1}$



$y=e^x$

$$\frac{e^x-1}{x} = \frac{\Delta y}{\Delta x}$$



Es

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \qquad a>0 \quad a \neq 1$$

$a = b^{\log_b a}$

$$\frac{a^x-1}{x} = \frac{e^{x \ln a}-1}{x} = \frac{e^{x \ln a}-1}{x \ln a} \ln a$$

$\frac{e^{x \ln a}-1}{x \ln a} \to 1$

$1 \leftarrow \frac{e^y-1}{y}$ $\qquad y = x \ln a$, $x \to 0$, $y \to 0$

dim (corollario)

② ($x \to 0$)

$$\frac{\ln(1+x)}{x} = \frac{1}{x} \cdot \ln(1+x) = \ln\left((1+x)^{\frac{1}{x}}\right) \to \ln e = 1$$

$(1+x)^{\frac{1}{x}} \to e$

① $$\frac{e^x-1}{x} = \frac{y}{\ln(1+y)} = \frac{1}{\frac{\ln(1+y)}{y}} \longrightarrow \frac{1}{1} = 1$$

$\square$

$y = e^x - 1 \qquad x \to 0, \quad y \to 0$

$y+1 = e^x \qquad x = \ln(1+y)$

CRITERI del RAPPORTO e della RADICE

<u>Teorema</u> Sia a_n una successione a termini positivi ($a_n > 0$)

RAPPORTO

Se $\frac{a_{n+1}}{a_n} \to \ell$, $\ell \in [0, +\infty]$

allora $\begin{cases} \text{se } \ell < 1 & \text{allora } a_n \to 0 \\ \text{se } \ell > 1 & \text{allora } a_n \to +\infty. \end{cases}$

RADICE

Se ($a_n \geq 0$)

$\sqrt[n]{a_n} \to \ell$ $\quad \ell \in [0, +\infty]$

allora $\begin{cases} \text{se } \ell < 1 & \text{allora } a_n \to 0 \\ \text{se } \ell > 1 & \text{allora } a_n \to +\infty. \end{cases}$

<u>Idea</u>. Caso speciale:
(successione geometrica) $a_n = q^n \xrightarrow[n \to \infty]{} \begin{cases} +\infty & \text{se } q > 1 \\ 0 & \text{se } q < 1 \end{cases}$

$$\frac{a_{n+1}}{a_n} = q$$

$$\sqrt[n]{a_n} = q$$

<u>dim</u> (RAPPORTO)

$\frac{a_{n+1}}{a_n} \to \ell$

$\ell > 1$

prendo q t.c. $\quad 1 < q < \ell$

definitivamente $\quad \frac{a_{n+1}}{a_n} \geq q$
($\exists n_0$ t.c. $\forall n \geq n_0$)

$$a_{n_0+1} \geq q \cdot a_{n_0}$$

$$a_{n_0+2} \geq q\, a_{n_0+1} \geq q^2 a_{n_0}$$

per induzione: $a_{n_0+k} \geq q^k \underline{a_{n_0}} \longrightarrow +\infty \qquad q>1$

quindi $a_n \to +\infty$ (località del limite)

(il limite di una successione non cambia se modifico/aggiungo/tolgo un numero finito di termini)

Stessa dimostrazione per il criterio della radice.

Oppure:

$\sqrt[n]{a_n} \to \ell \neq 1$

$\ln \sqrt[n]{a_n} \to \ln \ell \begin{cases} >0 & \text{se } \ell>1 \\ <0 & \ell<1 \end{cases}$

$$a_n = \left(\sqrt[n]{a_n}\right)^n = e^{n \ln \sqrt[n]{a_n}} \longrightarrow \begin{cases} e^{+\infty} & \text{se } \ell>1 \\ e^{-\infty} & \text{se } \ell<1 \end{cases}$$

$\square$

<u>Esercizio</u> Calcolare questi limiti:

$$\lim_{n\to\infty} \frac{n^7}{2^n}, \quad \lim_{n\to\infty} \frac{n!}{3^n}, \quad \lim_{n\to\infty} \frac{n^n}{n!}$$