ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 12 - 14.11.2024

Def [Caratheodory] Sia $L = L(x, y, z)$ $x, y, z \in \mathbb{R}$
Diremo che L è di Caratheodory se
(1) $\forall (y,z)$ $\quad x \longmapsto L(x,y,z)$ è lebesgue misurabile.
(2) $\tilde{\forall} x$ $\quad (y,z) \longmapsto L(x,y,z)$ è continua.

Teorema Sia $L = L(x,y,z)$ di Caratheodory e siano $u = u(x)$, $v = v(x)$ misurabili. Allora
$$x \longmapsto L(x, u(x), v(x))$$
è misurabile.

dim (vedi LEZIONE 11 anno scorso)

MINIMI DEBOLI

Def Sia $\mathcal{L} : V \subseteq W^{1,p}(a,b) \longrightarrow \mathbb{R}$, $V + C_c^\infty = V$.

Diremo che u è un minimo debole di $\mathcal{L}$ se $\exists\, \delta > 0$ tale che
$$\mathcal{L}(u) \leq \mathcal{L}(u+\varphi) \quad \forall \varphi \in C_c^\infty((a,b)) \text{ con}$$
$$\|\varphi\|_\infty + \|\varphi'\|_\infty \leq \delta.$$

OSS Se u è un minimo debole di $\mathcal{L}$ allora $\forall \varphi \in C_c^\infty((a,b))$ si ha che $\exists\, \varepsilon_0 > 0$ tale che
$$\mathcal{L}(u) \leq \mathcal{L}(u + \varepsilon\varphi) \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$
[Infatti $\|\varepsilon\varphi\|_\infty + \|\varepsilon\varphi'\|_\infty \leq \varepsilon_0 (\|\varphi\|_\infty + \|\varphi'\|_\infty) < \delta$]
se ε_0 piccolo abbastanza

OSS Se u è minimo assoluto

cioè $\qquad \mathcal{L}(u) \le \mathcal{L}(u+\varphi) \qquad \forall \varphi \in C_c^\infty((a,b))$

allora u è un minimo debole.

Teorema [E-L in $W^{1,p}$]

Sia $L = L(x,y,z)$, $x \in [a,b]$, $y \in \mathbb{R}$, $z \in \mathbb{R}$

Supponiamo che $\tilde{\forall} x \in [a,b]$: $\exists \frac{\partial L}{\partial y} = \frac{\partial L}{\partial y}(x,y,z)$ $\exists \frac{\partial L}{\partial z} = \frac{\partial L}{\partial z}(x,y,z)$

e (1) $L, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial z}$ sono di Carathéodory [IPOTESI di STRUTTURA]

(2) $\exists\, c > 0 \qquad \tilde{\forall} x \in [a,b], \forall y, \forall z$ [IPOTESI di CRESCITA]

$$\left|\frac{\partial L}{\partial y}(x,y,z)\right| + \left|\frac{\partial L}{\partial z}(x,y,z)\right| \le c\left(1 + |y|^p + |z|^p\right)$$

Consideriamo $\mathcal{I}(u) = \int_a^b L(x, u(x), u'(x))\, dx$

Sia $u \in W^{1,p}(a,b)$ tale che $\mathcal{I}(u) \in \mathbb{R}$, u minimo debole.

$\mathcal{I}(u)$ è finito.

Allora $\frac{\partial L}{\partial z}(x, u(x), u'(x)) \in W^{1,1}$

e
$$\boxed{\frac{d}{dx}\frac{\partial L}{\partial z}(x, u(x), u'(x)) = \frac{\partial L}{\partial y}(x, u(x), u'(x))}$$

EULERO-LAGRANGE

L^1

derivata debole

dim Sia $\varphi\in C_c^\infty((a,b))$, $\varepsilon\in\mathbb{R}$

$$\frac{1}{\varepsilon}\Big(\mathcal{L}(u+\varepsilon\varphi)-\mathcal{L}(u)\Big)=\int_a^b\frac{L(x,u(x)+\varepsilon\varphi(x),u'(x)+\varepsilon\varphi'(x))-L(x,u(x),u'(x))}{\varepsilon}\,dx$$

(per ipotesi $\mathcal{L}(u)\in\mathbb{R}$)

$$\frac{L(x,u(x)+\varepsilon\varphi(x),u'(x)+\varepsilon\varphi'(x))-L(x,u(x),u'(x))}{\varepsilon}=$$

$$(*)=\frac{L(x,u+\varepsilon\varphi,u'+\varepsilon\varphi')-L(x,u,u'+\varepsilon\varphi')}{\varepsilon}+\frac{L(x,u,u'+\varepsilon\varphi')-\underbrace{L(x,u,u')}_{\in L^1}}{\varepsilon}$$

Lagrange

$$=\underbrace{\frac{\partial L}{\partial y}(x,u+\theta\varphi,u'+\varepsilon\varphi')}_{\in L^1\,?\ \text{ok}}\cdot\underbrace{\varphi(x)}_{L^\infty}+\underbrace{\frac{\partial L}{\partial z}(x,u,u'+\tau\varphi')}_{\in L^1\,?\ \text{ok}}\underbrace{\varphi'(x)}_{L^\infty}\quad \in L^1\,?\ \text{ok}$$

con $|\theta|<|\varepsilon|$ $\quad|\tau|<|\varepsilon|$.

ipotesi di crescita

$$\left|\frac{\partial L}{\partial z}(x,u(x),u'(x)+\tau\varphi'(x))\right|\le C\Big(1+\underbrace{|u|^p}_{L^1}+|u'+\tau\varphi'|^p\Big)\le\dots$$

$L^1 \Leftarrow u\in L^p\Leftarrow u\in W^{1,p}$

[Oss $\quad |a+b|^p\le\begin{cases}|2a|^p & \text{se } |a|\ge|b|\\ |2b|^p & \text{se } |b|\ge|a|\end{cases}\le|2a|^p+|2b|^p=2^p(|a|^p+|b|^p)$]

$|\varepsilon|<1$

$$\left(|u'+\tau\varphi'|^p\le 2^p\big(|u'|^p+|\tau|^p|\varphi'|^p\big)\qquad |\tau|<|\varepsilon|<1\quad |\varphi'|\in L^\infty\right.$$

$$\left.\le C'(1+|u'|^p)\right)$$

$$\dots\le C''\Big(1+\underbrace{|u|^p}_{L^1}+\underbrace{|u'|^p}_{L^1}\Big)\in L^1(a,b)$$

$\Leftarrow u'\in L^p \Leftarrow u\in W^{1,p}$

stima uniforme in ε.

$|\theta|\le|\varepsilon|\le1$

Similmente:

$$\left|\frac{\partial L}{\partial z}(x,u+\theta\varphi,u'+\varepsilon\varphi')\right|\le\dots\le C'''\big(1+|u|^p+|u'|^p\big)\in L^1(a,b)$$

$$\underbrace{\mathcal{L}(u) \in \mathbb{R}, \qquad \int_a^b L(x, u+\varepsilon\varphi, u'+\varepsilon\varphi') - L(x,u,u') \in \mathbb{R}}_{\Rightarrow \ \mathcal{L}(u+\varepsilon\varphi) \in \mathbb{R}.}$$

$$\frac{\mathcal{L}(u+\varepsilon\varphi) - \mathcal{L}(u)}{\varepsilon} = \int_a^b (*) =$$

$|\theta| < |\varepsilon| \qquad |\tau| < |\varepsilon|$

$$= \int_a^b \Big[\overbrace{\frac{\partial L}{\partial y}(x, u+\theta\varphi, u'+\varepsilon\varphi)}^{\text{dominato } L^1} \varphi(x) + \overbrace{\frac{\partial L}{\partial z}(x, u+\varepsilon\varphi, u'+\tau\varphi')}^{\text{dominato } L^1} \varphi'(x) \Big] dx$$

$u+\theta\varphi \to u(x)$, $u'+\varepsilon\varphi \to u'(x)$, $u+\varepsilon\varphi \to u(x)$, $u'+\tau\varphi' \to u'(x)$

$\tilde{\forall} x$ per $\varepsilon \to 0$

$$\downarrow$$

$$\frac{\partial L}{\partial y}(x, u(x), u'(x))\varphi(x) + \frac{\partial L}{\partial z}(x, u(x), u'(x))\varphi'(x)$$

c'è convergenza dominata!

E-L FORMA INTEGRALE

$$\longrightarrow \boxed{\int_a^b \Big[\frac{\partial L}{\partial y}(x,u,u')\varphi(x) + \frac{\partial L}{\partial z}(x,u,u')\varphi'(x)\Big] dx = 0}$$

Ovvero

$$\frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = \qquad = 0$$

perché u è minimo debole

$\Downarrow$

$\mathcal{L}(u) \leq \mathcal{L}(u+\varepsilon\varphi)$

$\forall |\varepsilon| < \varepsilon_0.$

$\boxed{\text{Oss}}$ $\forall \varphi \in C_c^\infty$ $\quad \int_a^b [v(x)\varphi(x) + u(x)\varphi'(x)] = 0$

$u, v \in L^1$

$$\int_a^b u\varphi' = -\int_a^b v\varphi \iff \boxed{v = u'}$$

$u \in W^{1,1}$

$$\frac{\partial L}{\partial z}(x, u(x), u'(x)) \in W^{1,1}(a,b) \text{ e vale E-L} \quad \square$$

REGOLARITÀ

$$u \in W^{1,p} \overset{①}{\longrightarrow} u \in W^{1,\infty} \subseteq Lip \overset{②}{\longrightarrow} u \in C^1 \overset{③}{\longrightarrow} u \in C^k$$

$2 \leq k \leq +\infty$

① <u>Teorema</u> (regolarità lipschitz)

$L = L(x,y,z)$, $L \in C^0$, $u \in W^{1,p}(a,b)$, $p > 1$.

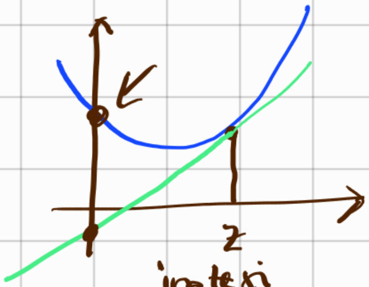
$\frac{\partial L}{\partial z}(x, u(x), u'(x)) \in W^{1,1}$ (tesi del teo. precedente)

$z \longmapsto L(x,y,z)$ convessa e derivabile $\forall x, \forall y$

e $L(x,y,z) \geq \alpha |z|^p - \varphi(y)$ | con $\alpha > 0$ e $\varphi \in C^0$.

Allora $u \in W^{1,\infty} \underset{\sim}{\subseteq} Lip$

<u>dim</u> $L(x,y,\cdot)$ convessa



$$L(x,y,0) \geq L(x,y,z) - z \cdot \frac{\partial L}{\partial z}(x,y,z)$$

ipotesi crescita

$$z \cdot \frac{\partial L}{\partial z}(x,y,z) \geq L(x,y,z) - L(x,y,0) \geq \alpha |z|^p - \varphi(y) - L(x,y,0)$$

$\underbrace{y = u(x)}_{C^0}$ $z = u'(x)$

$$u'(x) \frac{\partial L}{\partial z}(x, u(x), u'(x)) \geq \alpha |u'(x)|^p \underbrace{- \varphi(u(x)) - L(x, u(x), 0)}_{C^0([a,b]) \text{ è limitata}}$$

$$\geq \alpha |u'(x)|^p - c$$

$$c + |u'(x)| \cdot \overbrace{\left|\frac{\partial L}{\partial z}(x, u(x), u'(x))\right|}^{W^{1,1} \subseteq L^\infty} \geq \alpha |u'(x)|^p$$

Dove $|u'(x)| \geq 1$ $\quad |u'(x)| \cdot \left(C + \left\|\frac{\partial L}{\partial z}(x,u,u')\right\|_\infty\right) \geq \alpha |u'(x)|^p$

e quindi $|u'(x)|^{p-1} \leq C'$ $(p>1)$

dove $|u'(x)| \leq 1$ a maggior ragione $\leq C'$ se $C' \geq 1$.

$|u'(x)|$ è limitata $\Rightarrow u \in W^{1,\infty}$ □

② <u>Teo</u> (regolarità C^1) $L \in C^0$, $\exists \frac{\partial L}{\partial z} \in C^0$

$$z \mapsto \frac{\partial L}{\partial z}(x,y,z) \quad \text{iniettiva.}$$

$u \in Lip$ e $\frac{\partial L}{\partial z}(x, u(x), u'(x)) \in W^{1,1}$

Allora $u \in C^1([a,b])$.

<u>dim</u> (FUORI PROGRAMMA o forse lo vedremo più avanti)

③ <u>Teo</u> (Regolarità C^k)

Sia $L \in C^k$, $2 \leq k \leq \infty$, $u \in C^1$ soddisfa $\boxed{\text{E-L in senso debole.}}$

e $\frac{\partial^2 L}{\partial z^2}(x, u(x), u'(x)) > 0 \quad \forall x \in [a,b]$

Allora $u \in C^k([a,b])$.

$$\underbrace{\frac{d}{dx}\frac{\partial L}{\partial z}(x,u,u')}_{C^0} = \frac{\partial L}{\partial y}(x,u,u')$$

<u>dim</u> (sugli appunti oppure più avanti)