ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 11 - 12.11.2024

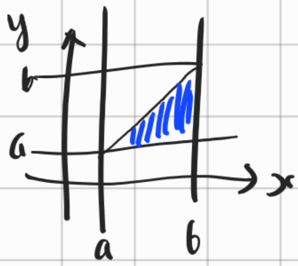
__Lemma 1__ [analogo del Teo. fondamentale del calcolo]

Sia $v \in L^1(a,b)$, $u(x) = \int_a^x v$

Allora $u' = v$. (Nota: $u \in C^0$)

derivata debole.

__dim__ Sia $\varphi \in C_c^\infty(a,b)$ $\left(\int_a^b v\cdot\varphi \overset{?}{=} -\int_a^b u\cdot\varphi' \right)$



$$\int_a^b u\varphi' = \int_a^b \left(\int_a^x v(y)\,dy \right) \varphi'(x)\,dx$$

$$= \iint_{\substack{(a,b)^2 \\ y \le x}} v(y)\varphi'(x)\,dx$$

$$= \int_a^b \left(\int_y^b \varphi'(x)\,dx \right) v(y)\,dy$$

$$= \int_a^b \left(\cancel{\varphi(b)} - \varphi(y) \right) v(y)\,dy = -\int_a^b \varphi\cdot v \qquad \square$$

__Lemma 2__ Se $u \in W^{1,1}(a,b)$

allora $\exists\, c$ t.c.

$$\tilde{\forall}\, x \in (a,b): \qquad u(x) = \underbrace{\left(\int_a^x u' \right)}_{w} + c$$

__dim__

$w(x) = \int_a^x u'$. Sappiamo che $w' = u'$

dobbiamo mostrare che $w - u$ è costante (q.o).

Idea: usare teo Du Bois-Reymonds.

Data $\varphi \in C_c^\infty(a,b)$:

u e w hanno derivata debole

$$\int_a^b (u-w)\cdot\varphi' = \int_a^b u\varphi' - \int_a^b w\varphi' = -\int_a^b u'\varphi + \int_a^b w'\varphi$$

$$= \int_a^b (\cancel{w'-u'})\varphi = 0$$

$w' = u'$

(Du Bois-Reymonds) $\Longrightarrow$ $u - w = c$

lemma 2 $\square$

Corollario $u \in W^{1,1}(a,b)$ $\quad u(x) = \underbrace{\int_a^x u'}_{\text{è } C^0.} + c$

q.o. $u = \tilde{u}$ $\quad \tilde{u} \in C^0$.

Vogliamo dire un po' di più.

RAPPRESENTAZIONE PUNTUALE DI $W^{1,1}$.

Def $u : [a,b] \to \mathbb{R}$ si dice essere <u>ASSOLUTAMENTE CONTINUA</u>

scritto: $u \in AC(a,b)$

se

$\forall \varepsilon > 0\ \exists \delta > 0$ tale che se

$I_k = (a_k, b_k)$ $\quad k = 1 \dots N$ intervalli <u>disgiunti</u>

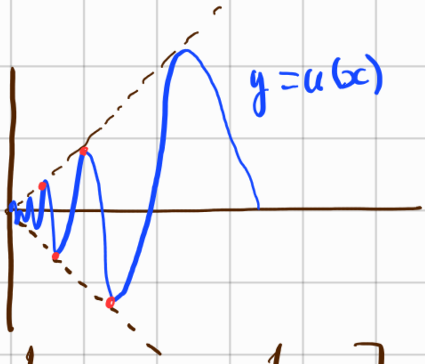
$$\sum_{k=1}^N |b_k - a_k| < \delta \Rightarrow \sum_{k=1}^N |u(b_k) - u(a_k)| < \varepsilon.$$

Se metto $N = 1$ ho l'uniforme continuità.

$$AC \subseteq C^0([a,b])$$

Esempio $\quad u \in C^0([0,1]) \setminus AC(0,1)$

$$u(x) = \begin{cases} x \cos \frac{1}{x} & \text{se } x \in (0,1] \\ 0 & \text{se } x = 0 \end{cases}$$



$$I_k = [a_k, b_k] = \left[\frac{1}{(2k+1)\pi}, \frac{1}{2k\pi} \right] \qquad \begin{cases} \cos \frac{1}{a_k} = -1 \\ \cos \frac{1}{b_k} = 1 \end{cases}$$

$$|b_k - a_k| = \frac{(2\cancel{k}+1)-(2\cancel{k})}{(2k+1)2k\pi} = \frac{1}{(4k^2+2k)\pi} \sim \frac{1}{4k^2\pi}$$

$$|u(b_k) - u(a_k)| = b_k + a_k = \frac{2k+1+2k}{(2k+1)2k\pi} = \frac{4k+1}{(4k^2+2k)\pi}$$

$$\sim \frac{1}{k\pi}$$

$$\sum_{k=1}^{\infty} |b_k - a_k| = \sum_{k=1}^{\infty} \frac{1}{(4k^2+2k)\pi} \quad \text{è convergente.}$$

$$\Rightarrow \forall \delta > 0 \quad \exists n \quad \underbrace{\sum_{k=n}^{\infty} \frac{1}{(4k^2+2k)\pi} < \delta}$$

ma $$\sum_{k=n}^{\infty} |u(b_k) - u(a_k)| = \sum_{k=n}^{\infty} \frac{4k+1}{(4k^2+2k)\pi} = +\infty$$

Scelto $\varepsilon = 1$. $\quad \exists N \quad \displaystyle\sum_{k=n}^{N} |u(b_k) - u(a_k)| > \varepsilon$

$\exists\, \varepsilon>0 \quad \forall \delta>0 \quad \exists\, [a_k, b_k] \quad k=1,\dots,N$

$\varepsilon=1$

$$\sum |a_k-b_k|<\delta \quad \text{ma} \quad \sum |u(b_k)-u(a_k)|>\varepsilon$$

$\square$

<u>Osservazione</u> $\quad u\in \mathrm{Lip}([a,b]) \Rightarrow u\in AC.$

$$|u(x)-u(y)|\le L\,|x-y|$$

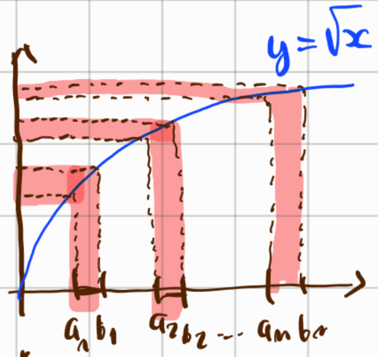
$$\sum_{k=1}^{n} |u(b_k)-u(a_k)| \le L\sum_{k=1}^{n} |b_k-a_k| \le L\,\delta<\varepsilon$$

se $\delta < \frac{\varepsilon}{L}$ $\square$

$$\mathrm{Lip} \subseteq AC \subseteq C^0$$

<u>ES</u> $\quad u\in AC([0,1])\setminus \mathrm{Lip}([0,1])$

$u(x)=\sqrt{x} \qquad u\notin \mathrm{Lip}([0,1])$



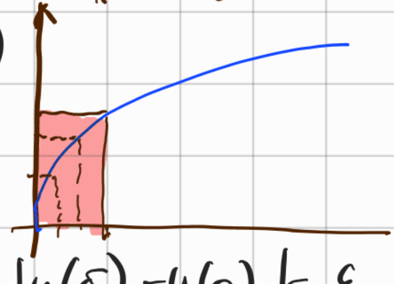
fissato h

$u(x+h)-u(x)$ è decrescente in x

$b_k>a_k$

$$\sum_{k=1}^{n} |u(a_k)-u(b_k)| \le u\Big(\sum (b_k-a_k)\Big)-u(0)$$

$$\le u(\delta)-u(0).$$



ma u è continua in $0 \Rightarrow |u(\delta)-u(0)|<\varepsilon.$

$\forall\varepsilon>0\ \exists\delta>0$ $\square$

Quello che è vero: $\mathrm{Lip} \subsetneq W^{1,\infty} \subseteq \underset{\substack{\cap \\ C^{0,\alpha}}}{W^{1,p}} \subsetneq W^{1,1} \simeq AC$ $\quad \alpha = 1-\frac{1}{p}$

Teo $u \in W^{1,p}(a,b) \Rightarrow \exists\, \tilde{u}$ t.c. $\tilde{u} = u$ q.o. e
$1 \le p \le +\infty$

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u' \qquad \forall x,y \in [a,b]$$

In particolare $\tilde{u} \in AC$.

dim $\quad \tilde{v}(x) = \int_a^x u' \overset{\text{lemma}}{\Rightarrow} \tilde{v}' = u' \overset{\text{lemma}}{\Rightarrow} \tilde{v} = u + k$ q.o.
$\quad \hookrightarrow \tilde{v} \in C^0$

$u = \tilde{v} - k = \tilde{u}$

$$\tilde{u}(y) - \tilde{u}(x) = \left(\int_a^y u' - \cancel{k}\right) - \left(\int_a^x u' - \cancel{k}\right)$$
$$= \int_x^y u' \quad \text{— c.v.d.}$$

$\tilde{u} \overset{?}{\in} AC \qquad I_k = [a_k, b_k]$

$$\sum_{k=1}^n |\tilde{u}(b_k) - \tilde{u}(a_k)| = \sum_{k=1}^n \left|\int_{a_k}^{b_k} u'\right| \le \sum_{k=1}^n \int_{a_k}^{b_k} |u'| = \int_{\cup I_k} |u'|$$

[Assoluta continuità dell'integrale di Lebesgue $f \in L^1$
$\forall \varepsilon > 0 \ \exists \delta > 0 \quad |A| < \delta \Rightarrow \int_A |f| < \varepsilon.$]

Se $\left|\bigcup_{k=1}^n I_k\right| = \sum_{k=1}^n |b_k - a_k| < \delta \Rightarrow \int_{\cup I_k} |u'| < \varepsilon. \quad \square$
$\qquad u' \in L^1$

Teorema $u \in AC \Rightarrow u \in W^{1,1}$ $\qquad$ SENZA DIMOSTRAZIONE.

Teorema $u \in W^{1,\infty}(a,b)$ allora $\exists\, \tilde{u} : \tilde{u} = u$ q.o., $\tilde{u} \in Lip([a,b])$

dim

$$\exists\, \tilde{u} \text{ t.c. } \quad \tilde{u}(y) - \tilde{u}(x) = \int_x^y u' \qquad u' \in L^\infty$$

$$|\tilde{u}(y) - \tilde{u}(x)| \leq \int_x^y |u'| \leq \|u'\|_\infty \cdot |y-x|$$

$\Rightarrow \tilde{u}$ è L-Lipschitz con $L = \|u'\|_\infty$. $\square$

Teorema $u \in Lip([a,b]) \Rightarrow u \in W^{1,\infty}(a,b)$ SENZA DIMOSTRAZIONE

Teorema Se $u \in W^{1,p}(a,b)$, $1 < p < +\infty$, allora

$\exists\, \tilde{u}$ t.c. $\tilde{u} = u$ q.o., $\tilde{u} \in C^{0,\alpha}([a,b])$, $\alpha = 1 - \frac{1}{p}$

$\uparrow$ α-Hölderiana

dim

$$|\tilde{u}(y) - \tilde{u}(x)| \overset{?}{\leq} c\, |y-x|^\alpha \qquad u' \in L^p$$

$\exists\, \tilde{u}$:

$$|\tilde{u}(y) - \tilde{u}(x)| = \left|\int_x^y u'\right| \leq \int_x^y |u'| = \|u'\|_{L^1(x,y)} \leq$$

$$\leq \|u' \cdot 1\|_1 \leq \|u'\|_{L^p(x,y)} \cdot \|1\|_{L^q} \leq \|u'\|_p \left(\int_x^y 1^q\right)^{\frac{1}{q}} =$$

Hölder $\quad \frac{1}{p} + \frac{1}{q} = 1 \qquad \alpha = \frac{1}{q} = 1 - \frac{1}{p}$

$$= \|u'\|_p \cdot |x-y|^\alpha$$

$\square$

In più variabili si ha:

parte intera

esponente di Hölder

$$\left[\begin{array}{lll} \Omega \subseteq \mathbb{R}^n & W^{m,p}(\Omega) \subset\subset\!\!\rightarrow C^{\lfloor k \rfloor, \{k\}} & \\ \text{aperto regolare} & & \text{Teorema } \left(\begin{array}{c}\text{immersioni di} \\ \text{Sobolev}\end{array}\right) \\ |\Omega| < \infty & \text{per } k < m - \frac{n}{p} & \end{array}\right]$$