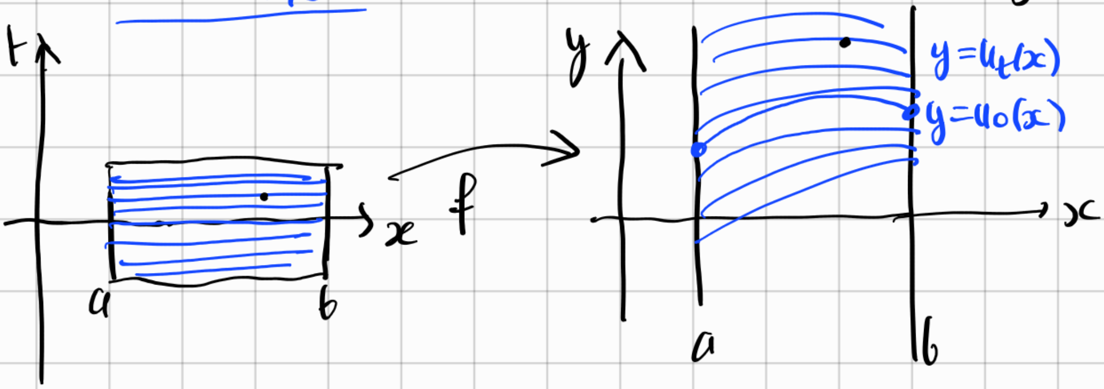
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 8 – 24.10.2024

Calibrazioni

$$\mathcal{L}(u) = \int_a^b L(x,u,u')$$



$$(x,t) \longmapsto f(x,t) = (x, u_t(x))$$

Passo 1 Basta mostrare che $\mathcal{M}(u_t)$ è minimo

con $\mathcal{M}(u) = \int_a^b M(x, u(x), u'(x))\, dx$

$$M(x,y,z) = L(x,y,\bar{z}(x,y)) + \frac{\partial L}{\partial y}(x,y,\bar{z}(x,y)) \cdot (z - \bar{z}(x,y))$$

con $\bar{z}(x, u_t(x)) = u_t'(x)$

Lagrangiana nulla (null-Lagrangian)

se $\exists\, S$ tale che $L(x, u(x), u'(x)) = \frac{d}{dx} S(x, u(x))$

diremo che L è una lagrangiana nulla

$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\,dx = \int_a^b \frac{d}{dx} S(x, u(x))\, dx$$

$$= \Big[S(x, u(x))\Big]_a^b = S(b, u(b)) - S(a, u(a))$$

dipende solo dal dato al bordo

$\mathcal{L}$ è costante sulla classe delle funzioni con dato al bordo assegnato.

ogni u è minimo di $\mathcal{L}$ (con il proprio dato al bordo)

Basta mostrare che M è una Lagrangiana nulla.

$$M(u) = \int_a^b L(x,u(x),\bar z(x,u(x))) + \frac{\partial L}{\partial z}(x,u(x),\bar z(x,u(x)))\,(u'(x)-\bar z(x,u(x)))\,dx \overset{?}{=} \frac{d}{dx} S(x,u(x))$$

$$= \int_\gamma \omega$$

$$\gamma(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix} = \begin{pmatrix} x \\ u(x) \end{pmatrix} \qquad \begin{aligned} dx &= dx \\ dy &= u'(x)dx \end{aligned}$$

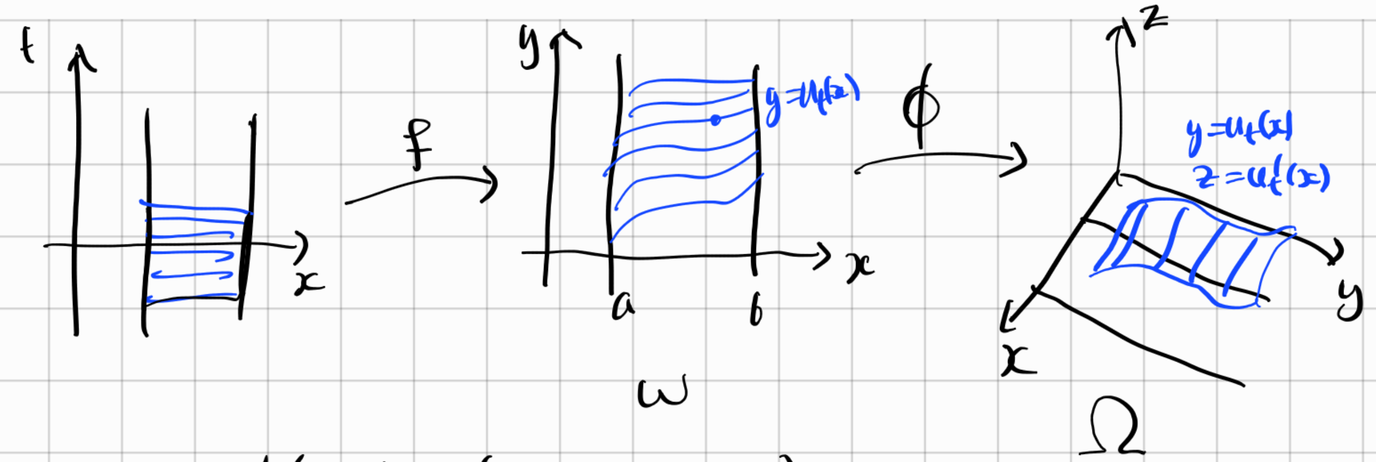
$$\gamma : [a,b] \to A \subseteq \mathbb{R}^2$$

$$\omega = \left[L(x,y,\bar z(x,y)) - \frac{\partial L}{\partial z}(x,y,\bar z(x,y))\cdot \bar z(x,y)\right]dx + \frac{\partial L}{\partial z}(x,y,\bar z(x,y))\,dy$$

M è una lagrangiana nulla $\iff$ ω è esatta: $\omega = dS$

$$M(u) = \int_\gamma \omega = S(\gamma(b)) - S(\gamma(a)) = S(a,u(a)) - S(b,u(b))$$

$$\omega = \alpha\,dx + \beta\,dy \qquad \text{con} \begin{cases} \alpha(x,y) = L(x,y,\bar z) - \bar z \dfrac{\partial L}{\partial z}(x,y,\bar z) \\ \beta(x,y) = \dfrac{\partial L}{\partial z}(x,y,\bar z) \end{cases}$$



$$\phi(x,y) = \left(x, y, \bar{z}(x,y)\right)$$

$$\Omega = \left(L(x,y,z) - z\frac{\partial L}{\partial z}(x,y,z)\right)dx + \frac{\partial L}{\partial z}(x,y,z)\,dy + 0\cdot dz$$

FORMA di BELTRAMI

$x = x \qquad dx = dx$

$y = y \qquad dy = dy$

$z = \bar{z}(x,y)$

$$dz = \frac{\partial \bar{z}}{\partial x}dx + \frac{\partial \bar{z}}{\partial y}dy$$

Nota $\underline{\underline{\omega = \Phi_* \Omega}}$

$$\Omega = \left(L - z\frac{\partial L}{\partial z}\right)dx + \frac{\partial L}{\partial z}dy$$

A semplicemente connesso

ω è esatta $\Longleftrightarrow$ ω è chiusa $\Longleftrightarrow$ $f_*\omega$ è chiusa

$$\left(\omega = f^{-1}_* f_* \omega\right)$$

$$f_*\omega = \left(\Phi \circ f\right)_* \Omega = ?$$

$$\boxed{\phi(f(x,t)) = \phi\begin{pmatrix} x \\ u_t(x)\end{pmatrix} = \begin{pmatrix} x \\ u_t(x) \\ u_t'(x)\end{pmatrix}}$$

$$\begin{cases} x = x \\ y = u_t(x) =: g(x,t) \\ z = u_t'(x) = \dfrac{\partial g}{\partial x}(x,t) \end{cases} \qquad \begin{cases} dx = dx \\ dy = \dfrac{\partial g}{\partial x}dx + \dfrac{\partial g}{\partial t}dt \\ dz = \dfrac{\partial^2 g}{\partial x^2}dx + \dfrac{\partial^2 g}{\partial x \partial t}dt \end{cases}$$

$$(\phi\circ f)_*\omega = \left[L(x, u_t(x), u_t'(x)) - \cancel{u_t'(x)\frac{\partial L}{\partial z}}(x, u_t(x), u_t'(x))\right]dx$$
$$+ \frac{\partial L}{\partial z}\left(x, u_t(x), u_t'(x)\right)\cdot\left(\cancel{\frac{\partial g}{\partial x}dx} + \frac{\partial g}{\partial t}dt\right)$$

$$= L(x, u_t, u_t')\, dx + \frac{\partial L}{\partial z}(x, u_t, u_t') \cdot \frac{\partial g}{\partial t}\, dt$$

è chiusa se:

$$0 \overset{?}{=} \underbrace{\frac{d}{dt} L(x, u_t, u_t')} - \frac{d}{dx}\left[\frac{\partial L}{\partial z}(x, u_t, u_t') \cdot \frac{\partial g}{\partial t}\right] =$$

$$\left[u_t \text{ soddisfa E-L}: \quad \boxed{\frac{\partial L}{\partial y}(x, u_t, u_t')} = \frac{d}{dx}\frac{\partial L}{\partial z}(x, u_t, u_t') \right]$$

$$(u_t = g) \qquad u_t' = \frac{\partial g}{\partial x}$$

$$= \underbrace{\cancel{\frac{\partial L}{\partial y}(x, u_t, u_t') \frac{\partial g}{\partial t}} + \cancel{\frac{\partial L}{\partial z}(x, u_t, u_t') \frac{\partial^2 g}{\partial x \partial t}}}$$

$$- \cancel{\boxed{\frac{\partial L}{\partial y}(x, u_t, u_t')} \cdot \frac{\partial g}{\partial t}} - \cancel{\frac{\partial L}{\partial z}(x, u_t, u_t') \cdot \frac{\partial^2 g}{\partial x \partial t}}$$

$$= 0$$

✓ $\square$

Esempio
$$\begin{cases} \mathcal{L}(u) = \frac{1}{2}\int_0^b \left(|u'(x)|^2 - |u(x)|^2\right) dx \to \min \\ u(0) = 0 \qquad\qquad a = 0 \\ u(b) = 0 \end{cases}$$

$$L(x, y, z) = \frac{1}{2} z^2 - \frac{1}{2} y^2$$

$$\frac{\partial L}{\partial z} = z$$

$$\frac{\partial L}{\partial y} = -y$$

E-L: $\quad -u = \frac{d}{dx} u'(x)$

$\underline{E-L}$: $u'' + u = 0$ $\lambda^2 + 1 = 0$

$\lambda_{12} = \pm i$

$$\boxed{u(x) = A\cos x + B\sin x} = C\cdot\sin(x - x_0)$$

$u(0) = 0 \Rightarrow A = 0$

$u(x) = B\sin x$

$\boxed{\text{caso } b < \pi}$

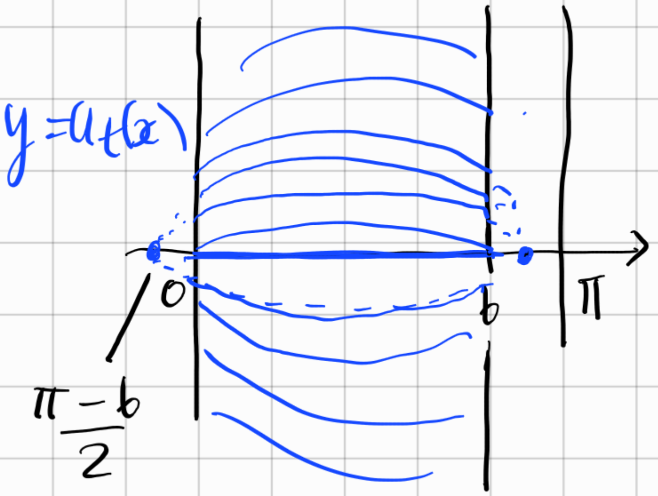
$u(b) = 0$

$\Downarrow$

$B = 0$

$u = 0$ è l'unica soluzione di EL.

$\mathcal{L}(0) = 0$ 0 è minimo?



$u_t(x) = t\sin(x - x_0)$

$x_0 = -\frac{\pi - b}{2}$

Ogni u_t è minimo di $\mathcal{L}$ con il suo dato al bordo

e u_0 è minimo di $\mathcal{L}$ con dato nullo.

$\mathcal{L}(u) \geq \mathcal{L}(u_0) = 0$ $\int_0^b |u'|^2 \geq \int_0^b |u|^2$ se $b < \pi$ $\forall u \in C_0^1([a,b])$

(DISUGUAGLIANZA DI POINCARÉ)

- $b = \pi$ Vale lo stesso
- $b > \pi$ $\inf \mathcal{L} = -\infty$

|| (appunti dell'anno scorso)