ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 4 – 8.10.2024

Es Meccanica Lagrangiana

$$\mathcal{L}(u) = \int L(x,u,u') \qquad L = E - V$$

$u(x)$ = posizione al tempo x

$$E = \frac{1}{2} m \left(u'(x)\right)^2 \qquad u'(x) = \text{velocità}$$

$$F(y) = -\nabla V(y) \qquad F(y) = \text{forza nel punto } y$$

$$L(x,y,z) = \frac{1}{2} m |z|^2 - V(y)$$

$$\nabla_y L = -\nabla V \qquad \nabla_z L = m z$$

E-L: $$-\nabla V(u(x)) = \frac{d}{dx} m u'(x) = m u''(x)$$

$$F = m a$$

Conservazione dell'energia

$$-\nabla V(u(x)) = m u''(x)$$

moltiplico ambo i membri per $u'(x)$

$$-\nabla V(u(x)) \cdot u'(x) = m \, u''(x) \cdot u'(x)$$

$$-\frac{d}{dx} V(u(x)) = \frac{1}{2} m \frac{d}{dx}\left(u'(x)^2\right)$$

$$\frac{d}{dx}\left[\frac{1}{2} m v^2 + V(u(x))\right] = 0$$

$$E + V = \text{costante}$$

ORDINE SUPERIORE

$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x), u''(x))\, dx$$

$$L = L(x, y, z, w)$$

$u \in C^2([a,b])$

$$\begin{cases} u(a) = y_a \\ u(b) = y_b \\ u'(a) = z_a \\ u'(b) = z_b \end{cases}$$

Potrei omettere alcune o tutte queste condizioni

Affinché $u+\varepsilon\varphi$ abbia le stesse condizioni: $\varphi \in C^2$

$$\varphi(a) = \varphi(b) = 0 \quad (*)$$
$$\varphi'(a) = \varphi'(b) = 0 \quad (**)$$

$$0 = \frac{d}{d\varepsilon} \mathcal{L}(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = \int_a^b \frac{d}{d\varepsilon} L(x, \underbrace{u+\varepsilon\varphi}_{y}, \underbrace{u'+\varepsilon\varphi'}_{z}, \underbrace{u''+\varepsilon\varphi''}_{w})\Big|_{\varepsilon=0} dx$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}(x,u,u',u'')\cdot\varphi(x) + \frac{\partial L}{\partial z}(x,u,u',u'')\varphi'(x) + \frac{\partial L}{\partial w}(x,u,u',u'')\varphi''(x)\right] dx$$

PER PARTI

$$= \int_a^b \left[\frac{\partial L}{\partial y}\cdot\varphi - \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right)\cdot\varphi - \frac{d}{dx}\left(\frac{\partial L}{\partial w}\right)\cdot\varphi'\right] +$$

$$+ \cancel{\left[\frac{\partial L}{\partial z}\varphi\right]_a^b}_{(*)} + \cancel{\left[\frac{\partial L}{\partial w}\cdot\varphi'\right]_a^b}_{(**)} =$$

$$= \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z} + \frac{d^2}{dx^2}\frac{\partial L}{\partial w} \right] \varphi \; +$$

$$+ \left[\cancel{\frac{d}{dx}\left(\frac{\partial L}{\partial w}\right)} \cdot \varphi \right]_a^b$$

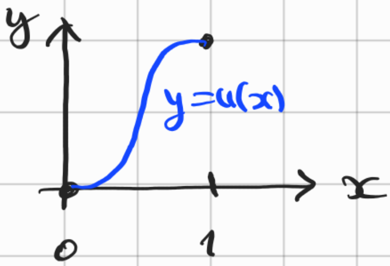
$(*)$

$$0 = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z} + \frac{d^2}{dx^2}\frac{\partial L}{\partial w} \right] \varphi(x)$$

$$\forall \varphi \in C_c^\infty((a,b)) \subset C_0^2([a,b])$$

Lemma fondamentale:

$$\boxed{\underline{\text{E-L}} \text{ (II ordine)}: \; \frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z} + \frac{d^2}{dx^2}\frac{\partial L}{\partial w} = 0}$$



Es (SPLINE)

$u : [0,1] \to \mathbb{R}$

$u(0) = 0, \; u(1) = 1, \; u'(0) = 0, \; u'(1) = 0$

$$I(u) = \frac{1}{2}\int_0^1 \left(u''(x)\right)^2 dx \longrightarrow \min$$

$$L(x,y,z,w) = L(w) = \frac{1}{2} w^2$$

$\frac{\partial L}{\partial w} = w$ E-L $\underline{\text{II}}$: $\boxed{\frac{d^2}{dx^2}\frac{\partial L}{\partial w} = 0}$

$$\frac{d^2}{dx^2} u'' = 0 \qquad u^{IV}(x) = 0$$

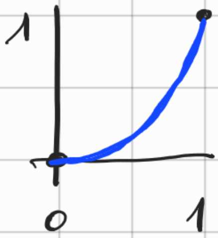
$$u(x) = ax^3 + bx^2 + cx + d$$
$$u'(x) = 3ax^2 + 2bx + c$$

$$b = 1 - a$$

$$u(0) = 0 \Rightarrow d = 0 \qquad u(1) = a + b + \overset{0}{\overset{\shortparallel}{c}} = a + b = 1$$
$$u'(0) = 0 \Rightarrow c = 0 \qquad u'(1) = 3a + 2b = 3a + 2 - 2a = 0$$
$$a = -2$$

$$\boxed{u(x) = -2x^3 + 3x^2} = x^2(3 - 2x)$$
$$u'(x) = -6x^2 + 6x = 6x(1 - x)$$

ESTREMI LIBERI



Es come sopra ma $u'(1)$ libero:

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u(1) = 1 \end{cases} \qquad \mathcal{L}[u] = \frac{1}{2}\int_0^1 (u''(x))^2\, dx.$$

IN GENERALE

$$\mathcal{L}[u] = \int_a^b L(x, u, u', u'')\, dx$$

$$0 = \left[\frac{d}{d\varepsilon}\mathcal{L}(u + \varepsilon\varphi)\right]_{\varepsilon=0} = \dots \text{ COME SOPRA } \dots =$$

non si cancella se $\varphi'(b) \neq 0$

$$= \int_a^b \left[\frac{\partial L}{\partial y} - \cancel{\frac{d}{dx}\left(\frac{\partial L}{\partial z}\right)} + \frac{d^2}{dx^2}\left(\frac{\partial L}{\partial w}\right)\right]\varphi(x)\, dx + \left[\frac{\partial L}{\partial w}\varphi'\right]_{x=b}$$

(*)

Ma posso comunque scegliere φ' con $\varphi'(b) = 0$

per ottenere la stessa eq. di prima:

$$\boxed{E-L\ II:\ (*)\ \frac{\partial L}{\partial y}-\frac{d}{dx}\frac{\partial L}{\partial z}+\frac{d^2}{dx^2}\frac{\partial L}{\partial w}=0}$$

Se poi scelgo φ con $\boxed{\varphi'(b)\neq 0}$

ottengo anche: $\left[\frac{\partial L}{\partial w}\right]_{x=b}=0$

$$\boxed{\left.\frac{\partial L(x,u,u',u'')}{\partial w}\right|_{x=b}=0}$$

Tornando all'esempio $L=L(w)=\frac{1}{2}w^2$

$$\frac{\partial L}{\partial w}=w$$

la condizione aggiuntiva è $u''(1)=0$

$$u(x)=ax^3+bx^2+cx+d$$
$$u'(x)=3ax^2+2bx+c \qquad u'(x)=3ax^2$$
$$u''(x)=6ax+2b$$

$$\begin{cases}u(0)=0\\u'(0)=0\\u(1)=1\\u''(1)=0\end{cases}\quad\begin{cases}d=0\\c=0\\a+b+c=1\\6a+2b=0\end{cases}\quad\begin{cases}b=-3a\\a-3a=1\\d=0\\c=0\end{cases}\quad\begin{cases}a=-\frac{1}{2}\\b=\frac{3}{2}\\c=0\\d=0\end{cases}$$

$$\boxed{u(x)=-\frac{1}{2}x^3+\frac{3}{2}x^2}$$

Se lascio libero anche $u'(0)$ noto che

$u(x) = x$ soddisfa le condizioni al bordo
$u''(x) = 0$ $\mathcal{I}(u) = 0$ $\Rightarrow$ $u(x) = x$ è certamente minimo.

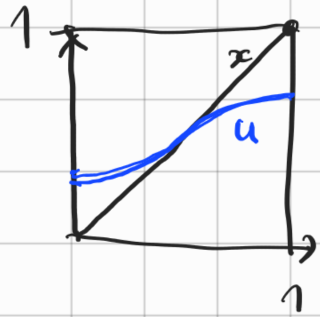
E' anche facile dimostrare che se $\exists x_0 : u''(x_0) \neq 0$

$\Rightarrow \mathcal{I}(u) > 0 \Rightarrow u(x) = cx + d$

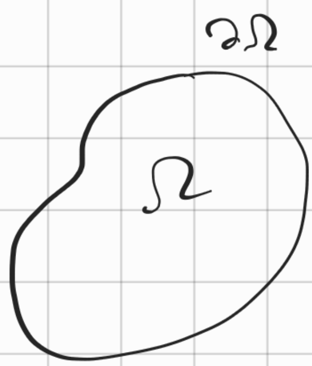
ma solo $c = 1$ $d = 0$ soddisfano le condizioni al bordo.

Es per casa
$$\mathcal{I}(u) = \frac{1}{2}\int_0^1 \left[|u'(x)|^2 + |u(x) - x|^2 \right] dx$$

$u \in C^1([0,1])$ $u(0)$ e $u(1)$ libero



ESEMPI IN PIÙ VARIABILI



$u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ $\underline{x} \in \Omega$ regolare limitato
$\underline{x} \mapsto u(\underline{x})$

$$\mathcal{I}(u) = \int_\Omega L(\underline{x}, u(\underline{x}), \nabla u(\underline{x}))\, d\underline{x}$$

$u \in C^1(\bar{\Omega})$ g data

$\mathcal{I}(u) \to \min$ $u(\underline{x}) = g(\underline{x})$ per $\underline{x} \in \partial\Omega$

Solito procedimento per trovare la condizione di stazionarietà $\quad u \rightsquigarrow u+\varepsilon\varphi \quad \varphi \in C^1_0(\bar{\Omega})$ ⊛

$$0=\left[\frac{d}{d\varepsilon}\mathcal{I}(u+\varepsilon\varphi)\right]_{\varepsilon=0}=\int_\Omega\left[\frac{d}{d\varepsilon}L(\underline{x},\underbrace{u+\varepsilon\varphi}_{y},\underbrace{\nabla u+\varepsilon\nabla\varphi}_{z})\right]_{\varepsilon=0}d\underline{x}=$$

$$\left[\quad L=L(\underline{x},y,\underline{z}) \qquad L:\underset{\underline{x}}{\overset{\subset\mathbb{R}^n}{\Omega}}\times\underset{y}{\mathbb{R}}\times\underset{\underline{z}}{\mathbb{R}^n}\quad\right]$$

$$=\int_\Omega\left[\frac{\partial L}{\partial y}(\underline{x},u,\nabla u)\cdot\varphi(\underline{x})+\nabla_z L(\underline{x},u,\nabla u)\cdot\nabla\varphi\right]d\underline{x}$$

⊛

$$=\int_\Omega\left[\frac{\partial L}{\partial y}-\mathrm{div}_x\left(\nabla_z L\right)\right]\varphi(\underline{x})\,d\underline{x}+\cancel{\int_{\partial\Omega}\varphi\nabla_z L\cdot\nu_\Omega\,d\sigma}$$

Teo della divergenza

$$\boxed{\int_\Omega \mathrm{div}\,\underline{f}=\int_{\partial\Omega}\underline{f}\cdot\underline{\nu}_\Omega\,d\sigma}$$

$g:\Omega\to\mathbb{R}$, $\underline{f}:\Omega\to\mathbb{R}^n$

$$\mathrm{div}(g\,\underline{f})=\sum_{k=1}^n\frac{\partial}{\partial x_k}(g\cdot f_k)=\sum_{k=1}^n\frac{\partial g}{\partial x_k}\cdot f_k+\sum_{k=1}^n g\cdot\frac{\partial f_k}{\partial x_k}$$

$$=\underbrace{\nabla g\cdot\underline{f}+g\,\mathrm{div}\,\underline{f}}$$

$$\int_\Omega\nabla g\cdot\underline{f}+\int_\Omega g\cdot\mathrm{div}\,\underline{f}=\int_\Omega\mathrm{div}(g\,\underline{f})=\int_{\partial\Omega}g\,\underline{f}\cdot\underline{\nu}_\Omega\,d\sigma$$

$$\int_\Omega\nabla g\cdot\underline{f}=-\int_\Omega g\,\mathrm{div}\,\underline{f}+\int_{\partial\Omega}g\,\underline{f}\cdot\underline{\nu}_\Omega\,d\sigma$$

$$0 = \int_\Omega \left[\frac{\partial L}{\partial y} - \mathrm{div}_x \nabla_z L \right] \varphi(x)$$

$$\forall \varphi \in C_0^1(\bar{\Omega})$$

Lemma fondamentale $\Rightarrow$

E-L: $\boxed{\frac{\partial L}{\partial y} = \mathrm{div}_x \nabla_z L}$

(vuol dire $\frac{\partial L}{\partial y}(\underline{x}, u(\underline{x}), \nabla u(x)) = \mathrm{div}_x \left(\nabla_z L (\underline{x}, u(\underline{x}), \nabla u(x)) \right)$

Esempio Integrale di Dirichlet

$$\mathcal{I}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_\Omega f(x)\, u(x)\, dx$$

$u \in C^1(\bar{\Omega})$ $\quad f \in C^0(\Omega)$ data

$$\begin{cases} \mathcal{I}(u) \to \min \\ u(\underline{x}) = g(\underline{x}) \text{ su } \partial\Omega \end{cases}$$

$g : C^0(\partial\Omega)$ data.

$$L(\underline{x}, y, \underline{z}) = \underbrace{\frac{1}{2} |\underline{z}|^2} + f(\underline{x}) \cdot y$$

$$\frac{\partial L}{\partial y} = f \qquad \nabla_z L = \underline{z}$$

E-L: $f(\underline{x}) = \operatorname{div}_x \nabla u(\underline{x}) = \Delta u(x)$

Equazione di Poisson: $\begin{cases} \Delta u = f & \text{su } \Omega \\ u = g & \text{su } \partial\Omega \end{cases}$

condizione di Dirichlet