ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 3 - 4.3.2024

Eq. di Beltrami $\qquad L(x,y,z) = L(y,z)$

$$\mathcal{L}(u) = \int_a^b L(u(x), u'(x))\,dx$$

E.L. $\qquad \dfrac{\partial L}{\partial y} = \dfrac{d}{dx}\dfrac{\partial L}{\partial z}$ $\qquad$ moltiplico per u'

$$u'\frac{\partial L}{\partial y} \overset{(*)}{=} u'\cdot\left(\frac{\partial L}{\partial z}\right)'$$

$$\frac{d}{dx}L(u(x),u'(x)) = u'\cdot\frac{\partial L}{\partial y} + u''\cdot\frac{\partial L}{\partial z}$$

$$\overset{(*)}{=} u'\left(\frac{\partial L}{\partial z}\right)' + u''\frac{\partial L}{\partial z}$$

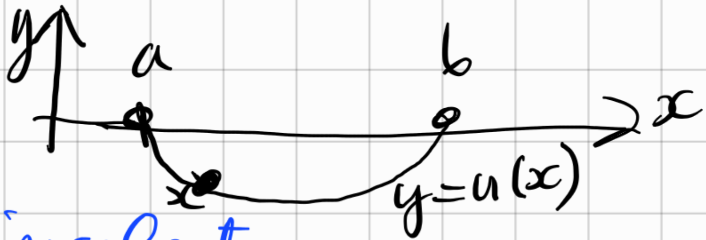
$$= \frac{d}{dx}\left(u'\cdot\frac{\partial L}{\partial z}\right)$$

$$\frac{d}{dx}\left(L - u'\frac{\partial L}{\partial z}\right) = 0$$

$$\boxed{L - u'\frac{\partial L}{\partial z} = \text{costante}}$$

identità di Beltrami.

Es. CATENARIA



è un problema vincolato

$$\mathcal{F}(u) = \int_a^b u(x)\cdot\sqrt{1+(u'(x))^2}\,dx$$

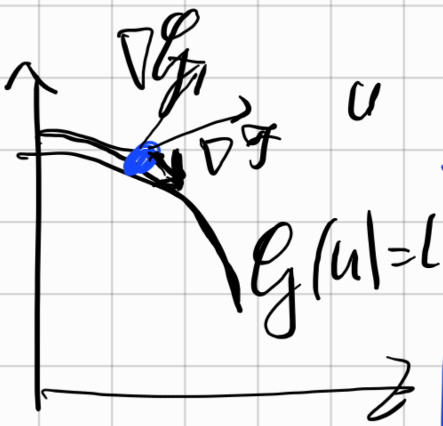
= energia potenziale della catena.

$$\mathcal{G}(u) = \int_a^b \sqrt{1+(u'(x))^2}\,dx$$

= lunghezza della catena.

$\mathcal{F}(u) \to \min$ $\qquad$ $u(a) = u(b) = 0$

$\mathcal{G}(u) = L$ fissato



INTUITIVAMENTE:

(moltiplicatori di Lagrange)

"$\nabla \mathcal{F} = \lambda \nabla \mathcal{G}$"

$$\mathcal{L}(u) = \mathcal{F}(u) - \lambda\,\mathcal{G}(u) \qquad \delta\mathcal{L} = 0$$

u (intuitivamente) dovrà essere un punto stazionario di $\mathcal{L}$. Cioè risolvere l'eq. di E.L:

$$\mathcal{L}(u) = \int_a^b \big(u(x) - \lambda\big)\sqrt{1+u'(x)^2}\,dx$$

$$= \int_a^b L(x, u(x), u'(x))\,dx$$

$$L(x,y,z) = (y-\lambda)\sqrt{1+z^2} \qquad \frac{\partial L}{\partial z} = \frac{(y-\lambda)\cdot z}{\sqrt{1+z^2}}$$

identità di Beltrami: $\quad L - u'\dfrac{\partial L}{\partial z} = c$

$$(u(x)-\lambda)\sqrt{1+(u'(x))^2} - u'(x)\cdot\frac{(u(x)-\lambda)\cdot u'(x)}{\sqrt{1+(u'(x))^2}} = c$$

$$(u-\lambda)\big(1+\cancel{(u')^2}\big) - \cancel{(u')^2(u-\lambda)} = c\sqrt{1+(u')^2}$$

$$u-\lambda = c\sqrt{1+(u')^2}$$

$$(u-\lambda)^2 = c^2\left(1+(u')^2\right) \qquad (u-\lambda)^2 = c^2 + c^2(u')^2$$

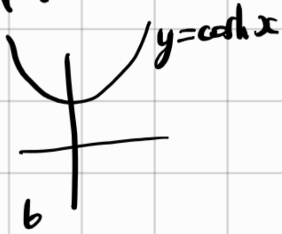
$$(u')^2 = \frac{(u-\lambda)^2 - c^2}{c^2} = \left(\frac{u-\lambda}{c}\right)^2 - 1$$

$$u' = \pm\sqrt{\left(\frac{u-\lambda}{c}\right)^2 - 1}$$

$$\frac{u'}{\sqrt{\left(\frac{u-\lambda}{c}\right)^2-1}} = \pm 1 \qquad \int \frac{du}{\sqrt{\left(\frac{u-\lambda}{c}\right)^2-1}} = \pm(x-x_0)$$

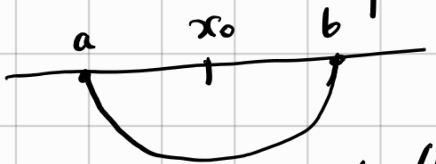
$$c\cdot \operatorname{settcosh}\left(\frac{u-\lambda}{c}\right) = \pm(x-x_0) \qquad \left(\begin{array}{l}\cosh^2 - \sinh^2 = 1 \\ \operatorname{settcosh}' t = \frac{1}{\sqrt{t^2-1}}\end{array}\right)$$

$$\frac{u-\lambda}{c} = \cosh\left(\frac{x-x_0}{c}\right)$$



$$u(x) = c\cdot\cosh\left(\frac{x-x_0}{c}\right) + \lambda$$

$$x_0 = \frac{a+b}{2}$$



$$0 = u(b) = c\cosh\left(\frac{b-x_0}{c}\right) + \lambda$$

$$\lambda = -c\cosh\left(\frac{b-x_0}{c}\right)$$

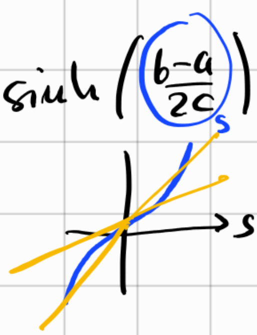
c si trova imponendo $G(u) = L$.

$$G(u) = \int_a^b \sqrt{1+(u'(x))^2}\,dx \qquad u'(x) = \sinh\left(\frac{x-x_0}{c}\right)$$

$$= \int_a^b \sqrt{1+\sinh^2\left(\frac{x-x_0}{c}\right)}\,dx = \int_a^b \cosh\left(\frac{x-x_0}{c}\right)dx$$

$$= \left[c\sinh\left(\frac{x-x_0}{c}\right)\right]_a^b = 2c\sinh\left(\frac{b-a}{2c}\right) = L$$

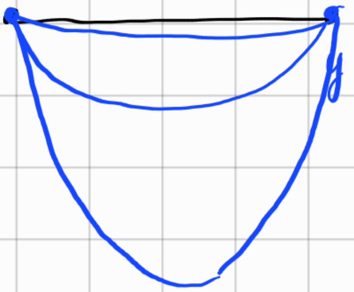
$$\sinh\left(\frac{b-a}{2c}\right) = \frac{L}{2c} = \frac{b-a}{2c}\,\frac{L}{b-a}$$



$$\sinh(s) = \frac{L}{b-a}\cdot s$$

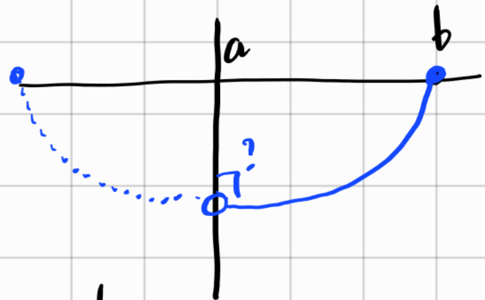
ha soluzione sse $\frac{L}{b-a} > 1$

$L > b-a$



$y = c\cosh\left(\frac{x-x_0}{c}\right)+\lambda$

ESTREMI LIBERI



$$\mathcal{L}(u) = \int_a^b L(x,u(x),u'(x))\,dx \qquad u\in C^1([a,b])$$

$\mathcal{L}(u)\to \min$, $\quad u(b)=y_b \quad$ $u(a)$ è libero.

$$\left[\frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\varphi)\right]_{\varepsilon=0}=0$$

$\forall\varphi\in C^1([a,b])$ con $\varphi(b)=0$

cosicché $(u+\varepsilon\varphi)(b)=u(b)=y_b$.

$$\shortparallel \int_a^b \frac{\partial L}{\partial y}(x,u,u')\cdot\varphi(x) + \frac{\partial L}{\partial z}(x,u,u')\,\varphi'(x)\,dx \qquad \underline{\varphi(b)=0}$$

$$=\int_a^b\left[\frac{\partial L}{\partial y}\cdot\varphi - \frac{d}{dx}\frac{\partial L}{\partial z}\cdot\varphi\right]dx + \left[\frac{\partial L}{\partial z}(x,u,u')\cdot\varphi\right]_a^b$$

Se $\varphi(a)=\varphi(b)=0$ $\qquad \int_a^b\left[\frac{\partial L}{\partial y}-\frac{d}{dx}\frac{\partial L}{\partial z}\right]\varphi=0$

$\Rightarrow$ vale EL $\boxed{\frac{\partial L}{\partial y}=\frac{d}{dx}\frac{\partial L}{\partial z}}$

se vale EL

$$=\int_a^b [\cancel{EL}]\cdot\varphi - \frac{\partial L}{\partial z}(a,u(a),u'(a))\cdot\underline{\varphi(a)} = 0$$

dunque $(*)\ \left[\frac{\partial L}{\partial z}(x,u(x),u'(x))\right]_{x=a}=0.$

Se fosse stata fissata in a e libera in b avrei avuto $(*)=0$ $x=b$.

Se fosse stata libera sia in a che in b avrei avuto $(*)=0$ sia in $x=a$ che in $x=b$.

Nell'esempio della catenaria con estremo libero in a

$$L(x,y,z) = (y-\lambda)\sqrt{1+z^2} \qquad \frac{\partial L}{\partial z} = \frac{(y-\lambda)\cdot z}{\sqrt{1+z^2}}$$

(4) $\frac{\partial L}{\partial z}(a, u(a), u'(a)) = 0$

$$(u(a)-\lambda)\frac{u'(a)}{\sqrt{1+(u'(a))^2}} = 0 \quad \begin{cases} u(a)=\lambda & \text{IMPOSSIBILE} \\ \underline{u'(a)=0} \end{cases}$$

$$u(x) = c\cdot\cosh\left(\frac{x-x_0}{c}\right) + \lambda \quad > \lambda \quad (< \lambda) \text{ se } c<0$$

$\cosh > 0$

$c > 0$

in quanto siamo in un minimo

ORDINE SUPERIORE

$\boxed{\begin{aligned} u(a) &= y_a \\ u(b) &= y_b \end{aligned}}$ $\quad \begin{aligned} \varphi(a) &= 0 \\ \varphi(b) &= 0 \end{aligned}$

$\vdots$?

$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x), u''(x))\,dx$$

$$L = L(x,y,z,w)$$

$$0 = \left[\frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\varphi)\right]_{\varepsilon=0} = \left[\int_a^b \frac{d}{d\varepsilon} L(x, u+\varepsilon\varphi, u'+\varepsilon\varphi', u''+\varepsilon\varphi'')\,dx\right]_{\varepsilon=0}$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}(x,u,u',u'')\varphi(x) + \frac{\partial L}{\partial z}(x,u,u',u'')\cdot\varphi'(x) + \frac{\partial L}{\partial w}(\dots)\cdot\varphi''(x)\right]dx$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}\varphi - \frac{d}{dx}\frac{\partial L}{\partial z}\varphi - \frac{d}{dx}\frac{\partial L}{\partial w}\cdot\varphi'\right] + \cancel{\left[\frac{\partial L}{\partial z}\varphi\right]_a^b} + \left[\frac{\partial L}{\partial w}\varphi'\right]_a^b$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}\varphi - \frac{d}{dx}\frac{\partial L}{\partial z}\varphi + \frac{d^2}{dx^2}\frac{\partial L}{\partial w}\cdot\varphi\right] + \left[\frac{\partial L}{\partial w}\varphi'\right]_a^b - \cancel{\left[\frac{d}{dx}\frac{\partial L}{\partial w}\cdot\varphi\right]_a^b}$$

$$= \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z} + \frac{d^2}{dx^2}\frac{\partial L}{\partial w}\right]\varphi + \underbrace{\left[\frac{\partial L}{\partial w}\varphi'\right]_a^b}_{\text{si annulla}}$$

Se $\varphi'(a) = \varphi'(b) = 0$

E.L.

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z} + \frac{d^2}{dx^2}\frac{\partial L}{\partial w} = 0}$$

Se imponiamo anche $u'(a)=z_a$ $\quad u'(b)=z_b$

allora $\varphi'(a)=\varphi'(b)=0$ ← deve essere

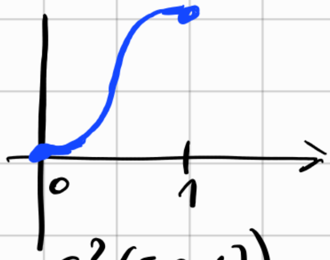
Se invece $u'(a)$ è libero: posso prendere $\varphi'(b)=0$, $\varphi'(a)\neq 0$

ottengo, per annullare ...

$$\boxed{\frac{\partial L}{\partial w}=0} \quad \text{in } x=a$$

... analogamente in b.

Esempio



$$\mathcal{L}(u)=\frac{1}{2}\int_0^1 \left(u''(x)\right)^2 dx$$

$\mathcal{L}(u)\to \min$

$$\left.\begin{array}{ll} u(0)=0, & u(1)=1 \\ u'(0)=0, & u'(1)=0 \end{array}\right]$$

$u\in C^2([0,1])$

$L(x,y,z,w)=\frac{1}{2}w^2$

E.L. $\quad \frac{\partial L}{\partial y}-\frac{d}{dx}\frac{\partial L}{\partial z}+\frac{d^2}{dx^2}\frac{\partial L}{\partial w}=0$

$\frac{\partial L}{\partial y}=0 \quad \frac{\partial L}{\partial z}=0$

$\frac{\partial L}{\partial w}=w$

$$\frac{d^2}{dx^2}u''(x)=0$$

$$u^{IV}(x)=0$$

$u(x)=ax^3+bx^2+cx+d$

$u(0)=0 \to d=0$

$u(1)=1 \qquad a+b+c=1 \qquad c=1-a-b$

$u'(x)=3ax^2+2bx+1-a-b$

$u'(0)=0 \qquad a+b=1 \qquad (c=0)\ \ b=1-a$

$u'(1)=0 \qquad 3a+2b=0$

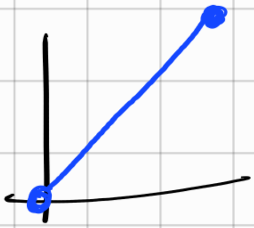
$3a+2(1-a)=0 \qquad a+2=0 \quad a=-2$

$u(x)=-2x^3+3x^2=x^2(3-2x)$

$u'(x)=6x-6x^2=6x(1-x)$

Stesso problema con: $u'(0)$ e $u'(1)$ liberi

$u(0)=0, u(1)=1$ come prima



E.L. è la stessa.

$$u(x) = ax^3 + bx^2 + (1-a-b)x$$

condizioni al bordo: $\dfrac{\partial L}{\partial w} = 0$ in $x=0$ e in $x=1$

$\dfrac{\partial L}{\partial w} = w$

$$u''(a) = 0 \,, \; u''(b) = 0$$

$$u'(x) = 3ax^2 + 2bx + (1-a-b)$$

$$u''(x) = 6ax + 2b$$

$u''(0) = 0 \Rightarrow b = 0$

$u''(1) = 0 \Rightarrow 6a = 0$

$$u(x) = (1-a-b)\cdot x = x.$$