ANALISI MATEMATICA B
LEZIONE 64 — 9-3-2022

Integrali di funzioni elementari:

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx \overset{\text{per parti}}{=} x \cdot \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \underbrace{x \cdot \ln x - x}$$

Verifica $\ln x + \cancel{x} \cdot \frac{1}{\cancel{x}} - \cancel{1}$

$$\int \operatorname{arctg} x \, dx = \int 1 \cdot \operatorname{arctg} x \, dx \overset{\text{parti}}{=} x \cdot \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= x \cdot \operatorname{arctg} - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \cdot \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2)$$

Dettagli

$$\int \frac{2x}{1+x^2} \, dx = \int \frac{1}{y} \, dy = \ln|y|$$

$$y = 1+x^2$$
$$\underbrace{dy = 2x \, dx}$$

$$= \ln(1+x^2)$$

$$df(x) = f'(x) \, dx$$

$$\int \frac{2x}{1+x^2} \, dx = \int \frac{d(1+x^2)}{1+x^2} = \ln(1+x^2)$$

Esempi di funzioni di cui non sappiamo scrivere le primitive tramite funzioni elementari.

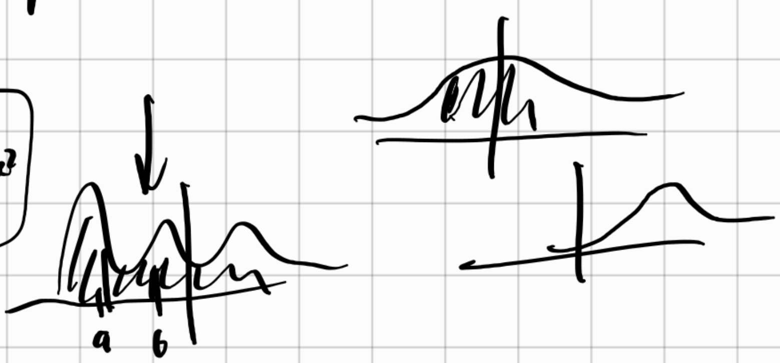
$$\rightarrow \boxed{\int e^{-x^2} \, dx}$$

$$\downarrow$$

$$F(x) = \int_0^x e^{-t^2} \, dt$$

$\operatorname{erf}(x) \leftarrow$

$$\boxed{\begin{array}{l} F(0) = 0 \\ F'(x) = e^{-x^2} \end{array}}$$



$$\int \boxed{x^2 e^{-x^2}}\, dx = \int x \cdot \underbrace{x e^{-x^2}}\, dx = \boxed{} - D \int \boxed{e^{-x^2}}\, dx$$

= non posso trovare una primitiva esplicita

Es 2. $\int \frac{1}{\ln x}\, dx \simeq li(x)$

Es 3 $\int \frac{\sin x}{x}\, dx \simeq si(x)$ $\quad \int \frac{\cos(x)}{x} \simeq ci(x)$.

PRIMITIVE DI FUNZIONI RAZIONALI

$$f(x) = \frac{P(x)}{Q(x)}$$ P, Q polinomi.

Es 0: $\int \frac{1}{x}\, dx = \ln|x|$ $\quad \int \frac{1}{1+x^2}\, dx = \arctg x$

Esempio 1 $\int \frac{x^4+1}{x-1}\, dx$

idea 1 se $\deg P \geq \deg Q$ faccio la divisione tra polinomi.

$$\begin{array}{l|l} x^4 + 1 & x-1 \\ \cline{2-2} x^4 - x^3 & x^3 + x^2 + x + 1 \\ \cline{1-1} x^3 + 1 & \\ x^3 - x^2 & \\ \cline{1-1} x^2 + 1 & \\ x^2 - x & \\ \cline{1-1} x + 1 & \\ x - 1 & \\ \cline{1-1} 2 & \end{array}$$

$$x^4 + 1 = \underbrace{(x^3 + x^2 + x + 1)(x-1) + 2}$$

$$\int \frac{x^4+1}{x-1}\,dx = \int \left[x^3+x^2+x+1 + \frac{2}{x-1}\right] dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + 2\ln|x-1|$$

Esempio 2 $\displaystyle\int \frac{x}{x^2-1}\,dx$

idea: scomposizione in fratti semplici

Teo fondamentale dell'algebra

Se $P \in \mathbb{C}[z]$, se $\deg P > 0$ Allora $\exists z \in \mathbb{C}$: $P(z) = 0$

$P(z) = c\cdot(z-\lambda_1)(z-\lambda_2) \dots (z-\lambda_n)$

$\lambda_1, \dots \lambda_n \in \mathbb{C}$

$n = \deg P$.

$$\frac{x}{x^2-1} = \frac{x}{(x-1)\cdot(x+1)} \overset{?}{=} \frac{A}{x-1} + \frac{B}{x+1}$$

$\mathbb{R}_{\le n}[x]$

$$= \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} \overset{!}{=} \frac{x}{(x-1)(x+1)}$$

$x+1$ e $x-1$ sono indipendenti

$\alpha(x+1) + \beta(x-1) = 0 \quad \forall x$

$\Rightarrow \alpha = \beta = 0.$

$$\boxed{A(x+1) + B(x-1) = x} \qquad \forall x.$$

$$x\cdot[A+B] + A - B = 1\cdot x + 0$$

$$\begin{cases} A+B=1 \\ A-B=0 \end{cases} \quad \begin{cases} 2A=1 \\ A=B \end{cases} \quad \begin{cases} A=\frac{1}{2} \\ B=\frac{1}{2} \end{cases}$$

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1|$$

$$= \ln\sqrt{|x^2-1|}$$

<u>Esempio 3</u> $\displaystyle\int \frac{x}{(x-1)^2}\,dx$

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B\cdot 1}{(x-1)^2}$$

$x-1$ e 1 sono indipendenti

$$A(x-1)+B = x \qquad \boxed{\begin{array}{l} B=1 \\ A=1 \end{array}}$$

$$\int \frac{x}{(x-1)^2}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \ln|x-1| - \frac{1}{x-1}$$

<u>Esempio 4</u> $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$

$$\frac{\Delta}{4} = -1-2 = -3 < 0$$

idea 1 ricondursi a $\int \frac{1}{1+x^2}\,dx$

$$\int \frac{1}{\underbrace{d+(x+c)^2}}\,dx$$

idea 2 togliere a numeratore la derivata del denominatore

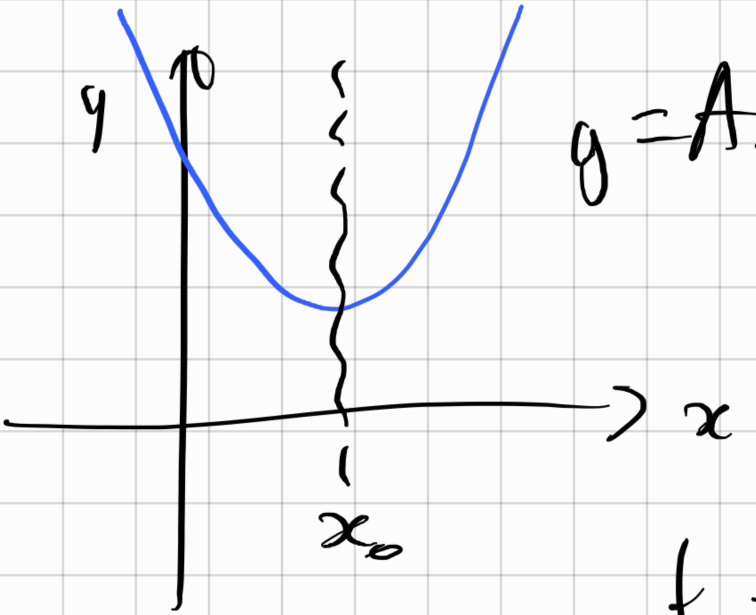
$$\frac{1}{x^2+2x+2} = \frac{1}{(x+1)^2+1} = D \operatorname{arctg}(x+1)$$

completamento del quadrato

$$\int \frac{x}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+2}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{1}{x^2+2x+2}\,dx$$

$$= \frac{1}{2} D \ln\left|x^2+2x+2\right| - D \operatorname{arctg}(x+1)$$

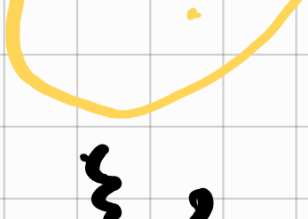


$$y = Ax^2 + Bx + C$$

$$t = x - x_0$$

$$y = a + bt^2$$

$$\int x e^{x^2 - x}$$



ξ

e^{t^2}