ANALISI MATEMATICA B
LEZIONE 50 - 2.2.2022

f derivabile quante volte vogliamo

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

polinomio di Taylor di f centrato in x_0 di ordine n.

Formula di Taylor: Resto di Peano

$$\boxed{\lim_{x \to x_0} \frac{f(x) - P(x)}{(x-x_0)^n} = 0} \quad (*)$$

$P^{(n-1)}(x) =$ pol. di Tay. di $f^{(n-1)}$ di ordine $1 = f^{(n-1)}(x_0) + f^{(n)}(x_0)(x-x_0)$

di Taylor.

Osservazione: Se P pol. di ordine n per f allora P' pol. di Taylor di ordine $n-1$ per f'

Ricordiamo: $P^{(k)}(x_0) = f^{(k)}(x_0)$ per $k = 0, 1, \dots, n$. $\Uparrow$

Inoltre il pol. di Taylor P di ordine n è l'unico polinomio di grado $\leq n$ tale che $(*)$ vale.

dim Sia Q un polinomio di grado $\leq n$ tale che:

$$① \quad \frac{f(x)-Q(x)}{(x-x_0)^n} \to 0 \quad \text{per } x \to x_0.$$

So che

$$② \quad \frac{f(x)-P(x)}{(x-x_0)^n} \to 0 \quad \text{per } x \to x_0$$

$$\frac{P(x)-Q(x)}{(x-x_0)^n} \to 0 \quad \text{per } x \to x_0$$

$R(x) = P(x) - Q(x)$ $\qquad$ $\deg R \leq n.$

ipotesi di comodo $x_0 = 0$

$$\frac{R(x)}{x^n} \to 0 \quad \text{per } x \to 0$$

$$R(x) = a_0 + a_1 x + \dots + a_n x^n$$

$$\frac{\overset{0}{\overset{\shortparallel}{a_0}} + \cancel{a_1 x} + \dots + \cancel{a_n x^n}}{x^n} \to 0 \quad \text{per } x \to 0$$

Se fosse $a_0 \neq 0$ il limite sarebbe $\pm\infty$.

$\Rightarrow a_0 = 0$

$$\frac{a_1 + \cancel{a_2 x} + \dots + \cancel{a_n x^{n-1}}}{x^{n-1}} \to 0$$

$\dots\dots \quad a_2 = 0 \dots\dots \quad a_n = 0 \quad \Rightarrow R = 0.$

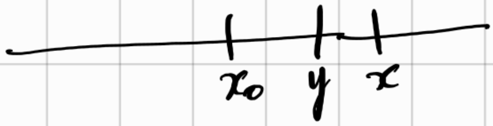
$\Downarrow$

$Q = P$ ∎

Teorema (formula di Taylor con resto di Lagrange)

f derivabile $(n+1)$ volte in un intervallo contenente x_0, P pol. di Taylor di f centrato in x_0 di ordine n

$$f(x) = \underbrace{P(x)} + \boxed{\frac{f^{(n+1)}(y)}{(n+1)!}}(x-x_0)^{n+1} \qquad \text{con } y \in \boxed{(x_0,x) \cup (x,x_0)}$$



dim ($x > x_0$)

Cauchy $\exists\, x_1 \in (x_0,x)$

$x_2 \in (x_0,x_1)$
$\vdots$
$x_n \in (x_0,x_{n-1}) \subset (x_0,x)$

$$\boxed{} \overset{?}{=} \frac{f(x)-P(x)}{(x-x_0)^{n+1}} \overset{\text{Cauchy}}{=} \frac{f'(x_1)-P'(x_1)}{(n+1)(x_1-x_0)^{n}} = \dots$$

$$\dots = \frac{f^{(n)}(x_n)-P^{(n)}(x_n)}{\underbrace{(n+1)\cdot n\cdot(n-1)\cdots 2}_{n \text{ fattori}}\;\underbrace{(x_n-x_0)}} \overset{\text{Cauchy}}{=} \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}$$

$y = x_{n+1}$. □

Se $n=0$ $\quad P(x) = f(x_0)$

$$f(x) = f(x_0) + f'(y)(x-x_0)$$

$$\frac{f(x)-f(x_0)}{x-x_0} = f'(y).$$

è il teorema di Lagrange.

Esempio calcolare $\sqrt{e}$ $\qquad x_0 = 0.$

$$f(x) = e^x \qquad P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$\sqrt{e} = f\left(\tfrac{1}{2}\right) = P\left(\tfrac{1}{2}\right) + \frac{f(y)}{n!}\left(\frac{1}{2}\right)^n = P\left(\tfrac{1}{2}\right) + \frac{e^y}{n!\,2^n}$$

$$0 < y < \frac{1}{2} \qquad e^y < \sqrt{e} < \sqrt{3}$$

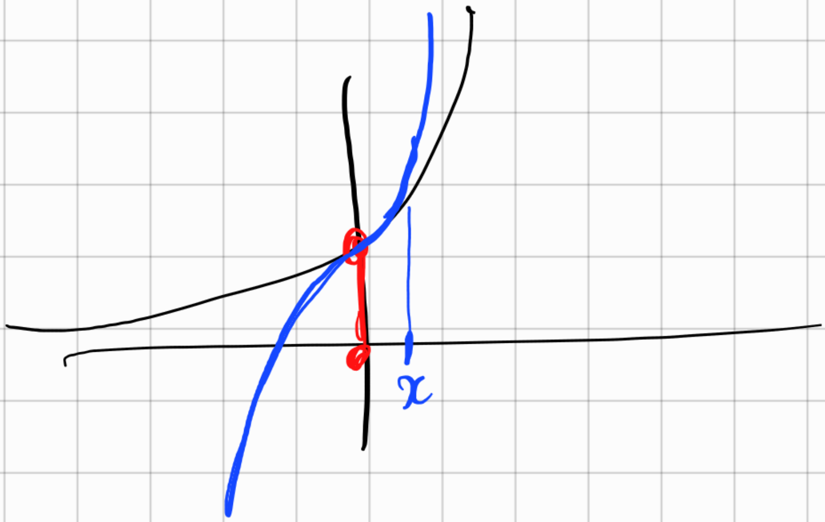
$$P\left(\tfrac{1}{2}\right) < \sqrt{e} < P\left(\tfrac{1}{2}\right) + \frac{\sqrt{3}}{n!\,2^n} \quad \varepsilon$$

$$n = 3 \qquad P\left(\tfrac{1}{2}\right) = 1 + \frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{6}\cdot\left(\frac{1}{2}\right)^3$$

$$= \frac{48 + 24 + 6 + 1}{48} = \frac{79}{48} = 1.6458\overline{3}$$

$$\sqrt{e} \approx 1.64872\dots$$

$$\sqrt{e} = \frac{79}{48} \pm \varepsilon \qquad 0 < \varepsilon = \frac{\sqrt{3}}{3!\cdot 2^3} < \frac{2}{6\cdot 8} = \frac{1}{24} =$$



Esempio $\quad f(x) = \ln x, \qquad x_0 = 1$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = -x^{-2}$$

$$f'''(x) = 2x^{-3}$$

$$f^{IV}(x) = -\underbrace{3\cdot 2}\, x^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1} (n-1)!\, x^{-n}$$

$$f^{(n)}(1) = (-1)^{n+1}(n-1)!$$

$$P_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots + (-1)^{n+1}\frac{1}{n}(x-1)^n$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$$\boxed{\begin{array}{c}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \\ n \to +\infty \\ x=2\end{array}}$$

$$P_n(2) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

Se $\boxed{f(2) - P_n(2) \to 0 \text{ per } n \to +\infty}$

allora $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = f(2) = \ln 2.$$

$$\left(\boxed{\ln x \overset{?}{=} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k} \right)$$

$$f(2) = P_n(2) + \frac{f^{(n+1)}(y)}{(n+1)!}(2-1)^{n+1}$$

$$f(2) - P_n(2) = \frac{(-1)^n n!\, y^{-(n+1)}}{(n+1)!} = \frac{(-1)^n}{(n+1)y^{n+1}} \to 0$$

$1 < y < 2$

pour $n \to +\infty$

$\ln(1+x)$

$$\lim_{x\to 0} \frac{\cos x \cdot \sin^2 x - 2 + 2\cos x}{(x \cdot \operatorname{tg} x)^2}$$

$f(x) = \sin x = x - \frac{x^3}{3!} + R_1(x)$

$\boxed{\frac{R_1(x)}{x^3} \to 0}$ pour $x \to 0$, $x_0 = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f''''(x) = f(x)$

$(A+B+C)^2 = A^2 + 2AB + B^2 + 2AC + C^2 + 2BC$

$$\sin^2 x = \left(x - \frac{x^3}{6} + R_1(x)\right)^2$$

$$= x^2 - \frac{x^4}{3} + \underbrace{\frac{x^6}{36} + 2x R_1(x) - \frac{x^3}{3} R_1(x) + R_1^2(x)}_{R_2(x)}$$

$$= x^2 - \frac{x^4}{3} + R_2(x)$$

$\frac{R_2(x)}{x^4} \to 0$?

$$\frac{R_2(x)}{x^4} = \frac{\frac{x^6}{36} + 2x R_1(x) - \frac{x^3}{3} R_1(x) + R_1^2(x)}{x^4}$$

$$= \underbrace{\cancel{\frac{x^2}{36}}}_{0} + 2\,\underbrace{\cancel{\frac{R_1(x)}{x^3}}}_{0} - \underbrace{\frac{R_1(x)\,x^2}{3x \cdot x^2}}_{0} + \underbrace{\left(\frac{R_1(x)}{x^2} \cdot \frac{x}{x}\right)^2}_{0} \longrightarrow 0$$

$$\cos x = \underbrace{1 - \frac{x^2}{2}}_{n=3} + R_3(x) \qquad \frac{R_3(x)}{x^3} \to 0$$

$$\operatorname{tg} x = \underbrace{x}_{n=2} + R_4(x) \qquad \frac{R_4(x)}{x^2} \to 0$$

$$\frac{\cos x \cdot \sin^2 x - 2 + 2\cos x}{(x \operatorname{tg} x)^2} =$$

$$= \frac{\left(1 - \frac{x^2}{2} + R_3(x)\right)\left(x^2 - \frac{x^4}{3} + R_2(x)\right) \cancel{-2} + 2\left(\cancel{1} - \frac{x^2}{2} + R_3(x)\right)}{\left(x \cdot (x + R_4(x))\right)^2}$$

$$= \frac{\cancel{x^2} - \frac{x^4}{2} - \frac{x^4}{3} + \frac{x^6}{6} + \left(x^2 - \frac{x^4}{3}\right)R_3 + \left(1 - \frac{x^2}{2}\right)R_2 + R_3 \cdot R_2 \cancel{-2} \cancel{+2} - \cancel{x^2} + 2R_3}{\left(x^2 + xR_4(x)\right)^2}$$

$$= \frac{-\frac{5}{6}x^4 + R_5(x)}{\left(x^2 + xR_4(x)\right)^2} \longrightarrow -\frac{5}{6} \qquad \frac{R_5(x)}{x^4} \overset{??}{\longrightarrow} 0$$