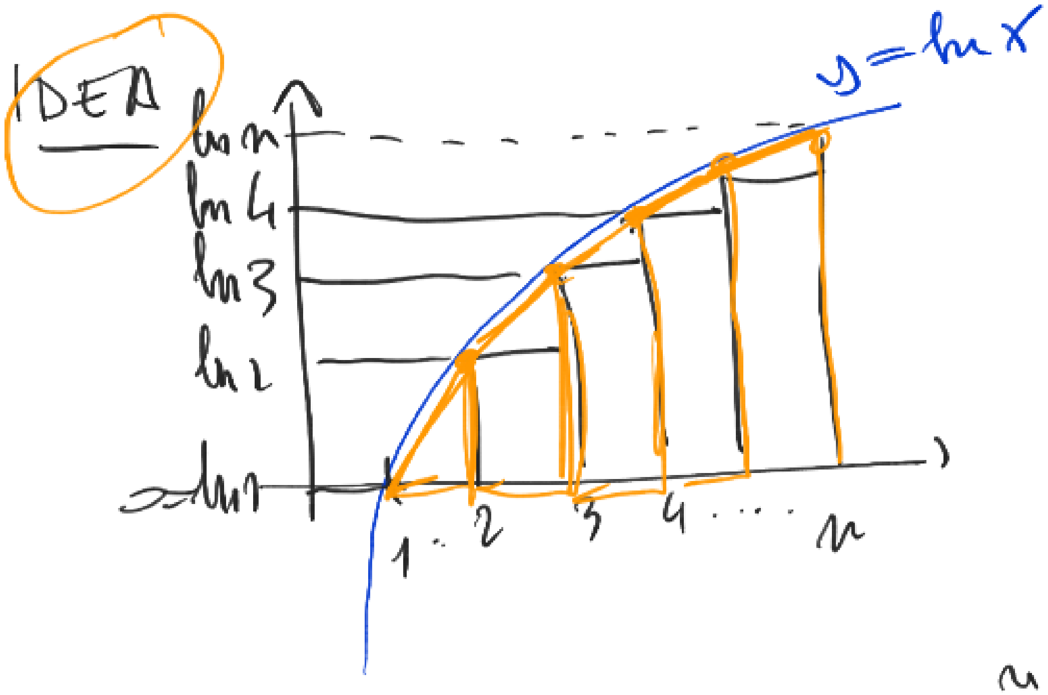
LA FORMULA DI STIRLING

$$n! \sim \sqrt{2\pi n}\cdot\frac{n^n}{e^n} \quad \text{per } n\to+\infty$$



$$\ln(n!) = \sum_{k=1}^{n} \ln k \quad \longleftrightarrow \quad \int_1^n \ln x\,dx$$

$$\int_1^n \ln x\,dx = \Big[x\ln x - x\Big]_1^n$$

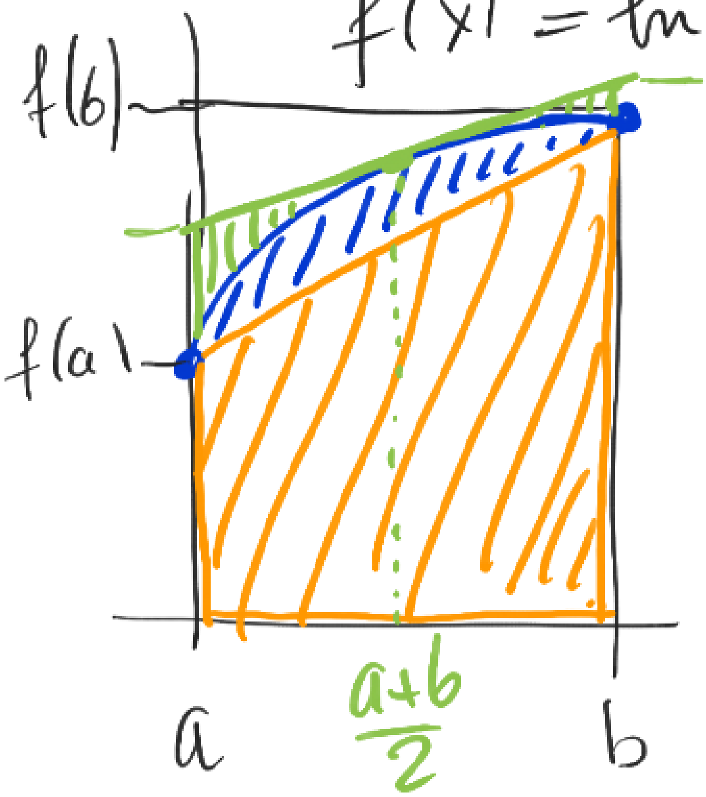
$$n\ln n - n + 1$$

$$e \qquad \frac{n^n}{e^n}\cdot e$$

e·

Approssimazione dell'area sotto il grafico di una funzione concava.

$f(x) = \ln x$ è concava



$$(b-a)\,\frac{f(a)+f(b)}{2} \le \int_a^b f(x)\,dx \le (b-a)\, f\left(\frac{a+b}{2}\right)$$

$a = k, \quad b = k+1, \quad f(x) = \ln x$

$$\frac{\ln k + \ln(k+1)}{2} \le \int_k^{k+1} \ln x \, dx \le \ln\left(k + \frac{1}{2}\right)$$

$$A_k \le B_k \le C_k$$

$$a_k = B_k - A_k$$

$$0 \le a_k \le \underline{C_k - A_k}.$$

$$a_k \le C_k - A_k = \ln\left(k+\frac{1}{2}\right) - \frac{\ln k + \ln(k+1)}{2}$$

$$= \frac{\ln \frac{\left(k+\frac{1}{2}\right)^2}{k(k+1)}}{2}$$

$$= \frac{1}{2}\ln \frac{k^2+k+\frac{1}{4}}{k^2+k}$$

$$= \frac{1}{2}\ln\left(1+\frac{1}{4(k^2+k)}\right)$$

$$\sim \frac{1}{2}\,\frac{1}{4(k^2+k)} \sim \frac{1}{8k^2}. \quad \text{per } k \to +\infty$$

$$\ln(1+x) \sim x$$

La serie $\sum_k a_k$ è convergente.

Poniamo $S = \sum_{k=1}^{+\infty} a_k$. $0 \le S < +\infty$.

$n-1$

$$S = \lim_{n\to+\infty} \sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n-1} B_k - \sum_{k=1}^{n-1} A_k$$

$$= \sum_{k=1}^{n-1} \int_k^{k+1} \ln x \, dx - \sum_{k=1}^{n-1} \frac{\ln k + \ln(k+1)}{2}$$

$$= \int_1^n \ln x \, dx - \frac{\sum_{k=1}^{n-1} \ln k + \sum_{k=2}^{n} \ln k}{2}$$

$$= \Big[x\ln x - x\Big]_1^n - \left(\sum_{k=2}^{n-1} \ln k + \frac{0 + \ln n}{2}\right)$$

$$= n\ln n - n + 1 - \sum_{k=1}^{n} \ln k + \frac{1}{2}\ln n$$

$$= \ln(n^n) - \ln e^n + \ln e - \ln(n!) + \ln\sqrt{n}$$

$$\longrightarrow S \quad \text{per } n \to +\infty$$

facendo l'exp di ambo i membri

$$\frac{n^n \cdot e \cdot \sqrt{n}}{e^n \cdot n!} \longrightarrow e^{S'}$$

$$\boxed{n! \sim \frac{n^n \cdot e \sqrt{n}}{e^S \cdot e^n} \overset{*}{=} C\,\frac{\sqrt{n}\cdot n^n}{e^n}}$$

La costante C si determina utilizzando il prodotto di Wallis. Anzi il binomiale centrale:

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$$

$$\|$$

$$\frac{(2n)!}{(n!)^2} \overset{*}{\sim} \frac{\cancel{C}\sqrt{2\cancel{n}}\cdot (2n)^{2n}\,\cancel{e^{2n}}}{\cancel{e^{2n}}\cdot C^2 (\sqrt{n})^{\cancel{2}}\, n^{2n}}$$

$$= \frac{\sqrt{2}\;2^{2n}\,\cancel{(n^n)^2}}{C\,\sqrt{n}\,\cancel{(n^n)^2}}$$

$$= \frac{4^n}{\sqrt{n}} \cdot \frac{\sqrt{2}}{C}$$

Allora $\frac{\sqrt{2}}{C} \longrightarrow \frac{1}{\sqrt{\pi}}$

$$C = \sqrt{2\pi}. \qquad \square$$