RICEVIMENTO

Titolo nota 15/11/2007

$$\lim_{n\to+\infty} n^4 \left\{ \sqrt[4]{\frac{16n^2-8}{n^2+2}} - \sqrt[3]{\frac{8n^2+1}{n^2+2}} \right\}$$

$$\sqrt[4]{\frac{16n^2-8}{n^2+2}} = \sqrt[4]{\frac{16\cancel{n^2}\left(1-\frac{1}{2n^2}\right)}{\cancel{n^2}\left(1+\frac{2}{n^2}\right)}} = 2\left(1-\frac{1}{2n^2}\right)^{\frac{1}{4}}\left(1+\frac{2}{n^2}\right)^{-\frac{1}{4}}$$

$$\left(1-\frac{1}{2n^2}\right)^{\frac{1}{4}} \qquad (1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+o(x^2)$$

$$\left(1-\frac{1}{2n^2}\right)^{\frac{1}{4}} = 1-\frac{1}{8n^2}-\frac{3}{32\cdot 4}\frac{1}{n^4}+o\left(\frac{1}{n^4}\right)$$

$$(1+x)^{\frac{1}{4}} = 1+\frac{1}{4}x-\frac{3}{32}x^2+o(x^2)$$

$$(1+x)^{-\frac{1}{4}} = 1-\frac{1}{4}x+\frac{5}{32}x^2+o(x^2)$$

$$\left(1+\frac{2}{n^2}\right)^{-\frac{1}{4}} = 1-\frac{1}{2n^2}+\frac{5}{32}\frac{4}{n^4}+o\left(\frac{1}{n^4}\right)$$

$$2\left(1-\frac{1}{2n^2}\right)^{\frac{1}{4}}\left(1+\frac{2}{n^2}\right)^{-\frac{1}{4}} = 2\left(1-\frac{1}{8n^2}-\frac{3}{128n^4}\right)\left(1-\frac{1}{2n^2}+\frac{5}{32}\frac{4}{n^4}\right)+o\left(\frac{1}{n^4}\right)$$

$$\sqrt[3]{\frac{8n^2+1}{n^2+2}} = \sqrt[3]{\frac{8\cancel{n^2}\left(1+\frac{1}{8n^2}\right)}{\cancel{n^2}\left(1+\frac{2}{n^2}\right)}} = 2\left(1+\frac{1}{8n^2}\right)^{\frac{1}{3}}\left(1+\frac{2}{n^2}\right)^{-\frac{1}{3}}$$

$$(1+x)^{\frac{1}{3}} = 1+\frac{1}{3}x-\frac{1}{9}x^2+o(x^2)\ ;\ (1+x)^{-\frac{1}{3}} = 1-\frac{1}{3}x+\frac{2}{9}x^2+o(x^2)$$

$$\left(1+\frac{1}{8n^2}\right)^{\frac{1}{3}} = 1+\frac{1}{24n^2}-\frac{1}{9}\frac{1}{64n^4}+o\left(\frac{1}{n^4}\right)$$

$$\left(1+\frac{2}{n^2}\right)^{-\frac{1}{3}} = 1-\frac{2}{3n^2}+\frac{8}{9}\frac{1}{n^4}+o\left(\frac{1}{n^4}\right)$$

—o—o—

$a_{n+1} = n \cdot a_n^3$ $\quad a_1 = \frac{1}{2}$ $\quad a_2 = \frac{1}{8}$ $\quad a_3 = 2 \cdot \frac{1}{8^3} = \frac{1}{2^8}$

$\boxed{\text{PIANO}}$ (i) $0 \leq a_n \leq \frac{1}{2^n}$

(ii) $a_n \to 0$ (CARABINIERI)

$a_n \leq \frac{1}{2^n}$ $\quad n=1$ $\quad \frac{1}{2} \leq \frac{1}{2}$

$\boxed{\text{P.I}}$ ipotesi: $a_n \leq \frac{1}{2^n}$ $\quad$ tesi: $a_{n+1} \leq \frac{1}{2^{n+1}}$

$$a_{n+1} = n \cdot a_n^3 \underset{\substack{\uparrow \\ \text{uso} \\ \text{ipotesi}}}{\leq} n \cdot \left(\frac{1}{2^n}\right)^3 = n\,\frac{1}{2^{3n}} \underset{\substack{\uparrow \\ \text{spero}}}{\leq} \frac{1}{2^{n+1}}$$

$$\frac{n}{2^{3n}} \overset{?}{\leq} \frac{1}{2^{n+1}} \qquad \frac{n}{\cancel{2^n}\cdot 2^{2n}} \overset{?}{\leq} \frac{1}{2\cdot \cancel{2^n}}$$ resta $2^{2n} \geq 2n$

A suo tempo si è fatto $2^n \geq n+1$, quindi

$$2^{2n} \geq 2n+1 \geq 2n$$

— o —

$$\sum_{n=3^{28}}^{\infty} (-1)^n \underbrace{\boxed{\frac{1}{\log(\log(\log n))}}}_{a_n}$$ LEIBNITZ

(i) $a_n \to 0$; (ii) $a_n \geq 0$ definitiv.

(iii) $a_{n+1} \leq a_n$ $$\frac{1}{\log(\log(\log(n+1)))} \overset{?}{\leq} \frac{1}{\log(\log(\log n))}$$

$$\cancel{\log}(\cancel{\log}(\cancel{\log}\, n)) \overset{?}{\leq} \cancel{\log}(\cancel{\log}(\cancel{\log}(n+1)))$$

$$\sum_{n=1}^{\infty} \left\{ \log(1+n+n^4) - 4\log n \right\}$$

$$\log(1+n+n^4) - 4\log n = \log\left(\frac{1+n+n^4}{n^4}\right) = \log\left(1+\frac{1+n}{n^4}\right)$$

$$\sim \frac{1+n}{n^4} \sim \frac{1}{n^3}$$

$\Rightarrow$ CONVERGE

$$\sum_{n=\infty}^{+\infty} (n+3)\sin\frac{1}{n^3+3}$$

$$\underbrace{\boxed{(n+3)\sin\frac{1}{n^3+3}}}_{a_n} \sim (n+3)\frac{1}{n^3+3} \sim \underbrace{\boxed{\frac{1}{n^2}}}_{b_n} \Rightarrow \text{CONVERGE}$$

$$\frac{a_n}{b_n} = n^2(n+3)\sin\frac{1}{n^3+3} = \boxed{(n^3+3n^2)}\ \boxed{\frac{\sin\frac{1}{n^3+3}}{\frac{1}{n^3+3}}} \cdot \frac{1}{\boxed{n^3+3}} \to 1 \begin{array}{l}\neq 0\\ \neq +\infty\end{array}$$

$$\frac{\sin x}{x} \to 1$$

$$a_{n+1} = a_n^3 - 6a_n^2 + 12a_n - 6 \qquad a_0 = 0$$

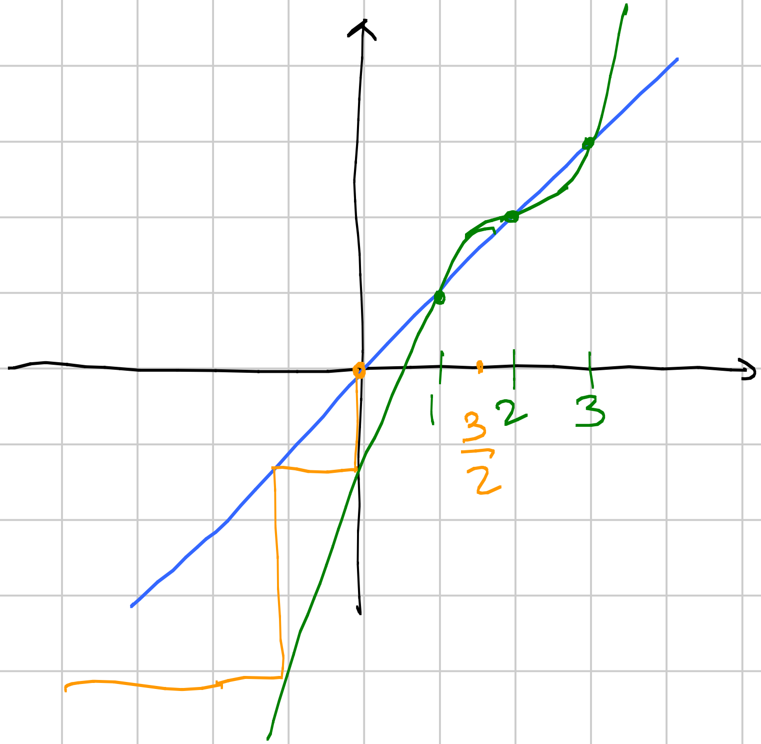
$$f(x) = x^3 - 6x^2 + 12x - 6$$

$$f(x) \geq x$$

$$x^3 - 6x^2 + 12x - 6 \geq x$$

$$x^3 - 6x^2 + 11x - 6 \geq 0$$

$x = 1$ è una radice



$$\begin{array}{l|l} x^3 - 6x^2 + 11x - 6 & x-1 \\ \hline -x^3 + x^2 & x^2 - 5x + 6 \\ \hline // \quad -5x^2 + 11x - 6 & \\ \quad +5x^2 - 5x & \\ \hline // \quad 6x - 6 & \end{array}$$

$$x^3 - 6x^2 + 11x - 6 =$$

$$= (x-1)(x^2 - 5x + 6)$$

$$= (x-1)(x-3)(x-2)$$

		1		2		3	
	− − −		+ +		+ +		+ + + +
	− − −		− −		+ +		+ + + +
	− − −		− −		− −		+ + + +
	⊖		⊕		⊖		⊕

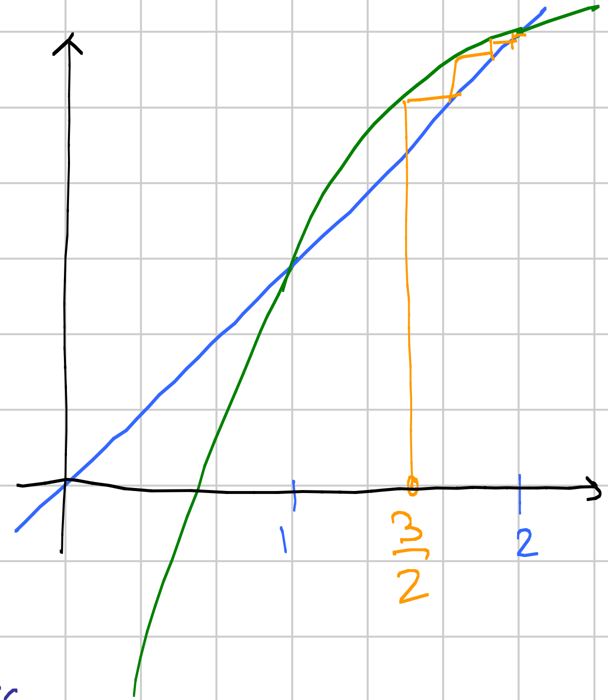
Studio la monotonia di $f(x)$

$f'(x) = 3x^2-12x+12 = 3(x^2-4x+4) = 3(x-2)^2 \geq 0$ sempre

Con partenza da $a_0=0$ tende a $-\infty$ decrescendo.

Con partenza da $a_0 = \frac{3}{2}$

tende a 2 crescendo



$\boxed{\text{PIANO con } a_0=0}$

(i) $a_n \leq 0 \quad \forall n \in \mathbb{N}$

(ii) $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

(iii) $a_n \to \ell \in \mathbb{R} \cup \{-\infty\}$ (ii) + Teo. succ. monotone

(iv) $\ell = -\infty$

se fosse $\ell \in \mathbb{R}$ avrei

$\ell = f(\ell) \Rightarrow \ell = 1, 2, 3$ incompatibile per (i)

$\boxed{\text{Dim (i)}}$ Per induzione $\boxed{n=0}$ $a_0 \leq 0$ $0 \leq 0$ ok

$\boxed{\text{P.I.}}$ ipotesi: $a_n \leq 0$ tesi: $a_{n+1} \leq 0$

Dim. se $a_n \leq 0$ applico f (che so essere strett. cresc.).
Ottengo $f(a_n) \leq f(0)$

$$\underset{a_{n+1}}{\overset{\shortparallel}{f(a_n)}} \leq \underset{-6}{\overset{\shortparallel}{f(0)}} \leq 0$$

$a_{n+1} \leq -6 \leq 0$

$\boxed{1^\circ \text{ modo}}$

$\boxed{\text{Dim (ii)}}$ $a_{n+1} \overset{?}{\leq} a_n$ $f(a_n) \leq a_n$ ci si riduce alla disuguaglianza $f(x) \leq x$ che è soddisfatta per $x \in (-\infty, 1] \cup [2, 3]$

Ma sappiamo che $a_n \leq 0$, quindi ok

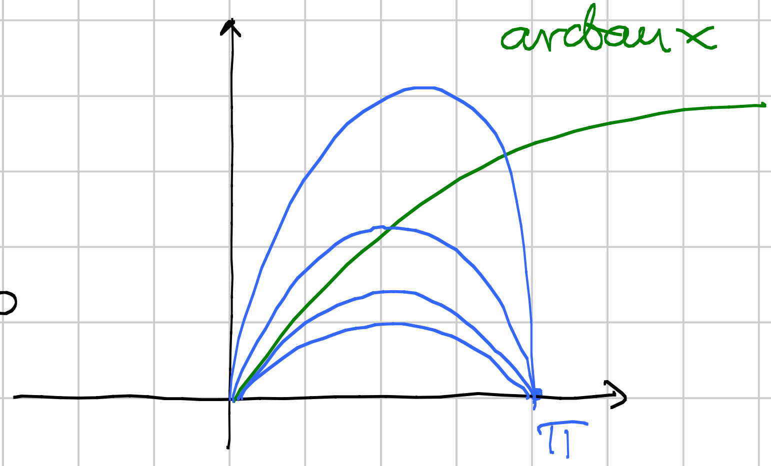
$\boxed{2^\circ \text{ modo}}$ induzione $\boxed{n=0}$ $a_1 \overset{?}{\leq} a_0$ $-6 \leq 0$ ok

$\boxed{\text{P.I.}}$ Ipotesi: $a_{n+1} \leq a_n$ Tesi: $a_{n+2} \leq a_{n+1}$

Dim.: prendo l'ipotesi e applico f.

$$\{ \lambda \in [0,4] : \arctan x \geq \lambda \sin(2x) \quad \forall x \in [0,\pi] \}$$

Ci interessano i valori di λ per cui il blu sta sotto il verde



Esamino le 2 funzioni vicino a $x=0$

$\arctan x \sim x$

$\lambda \sin(2x) \sim 2\lambda x$

Se voglio blu ≤ verde deve succedere che $2\lambda \leq 1$ $\quad \lambda \leq \frac{1}{2}$

Ora bisogna dimostrare che

$$\arctan x \geq \frac{1}{2} \sin(2x) \qquad \forall x \in [0,\pi]$$

Si dimostra che $\arctan x \geq \underbrace{x - \frac{x^3}{3}}_{Q(x)}$ (Taylor-Lagrange) e analogamente

$$\frac{1}{2}\sin(2x) \le \underbrace{\frac{1}{2}\left(2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120}\right)}_{P(x)}$$

(Taylor Lagrange)

Allora $\quad \frac{1}{2}\sin(2x) \le P(x) \le Q(x) \le \arctan x$

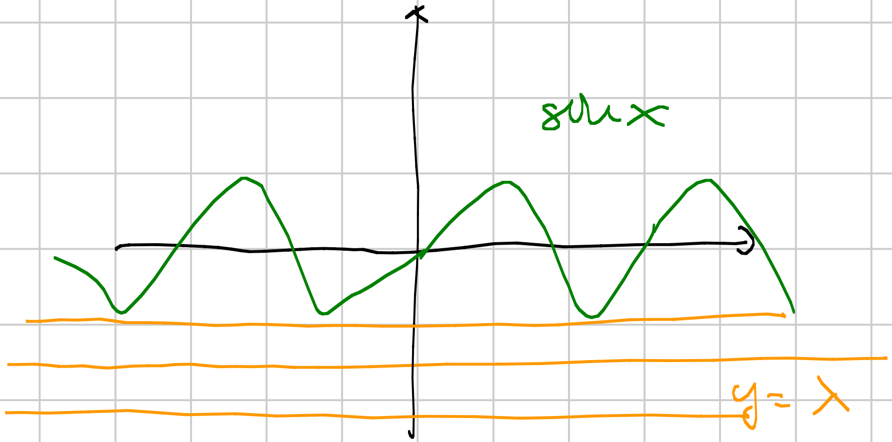
T.L.   confronto polinomiale, dunque fattibile   T.L.

— o —

$$\{\lambda \in \mathbb{R} : \sin x \ge \lambda \quad \forall x \in \mathbb{R}\}$$

$$= (-\infty, -1]$$

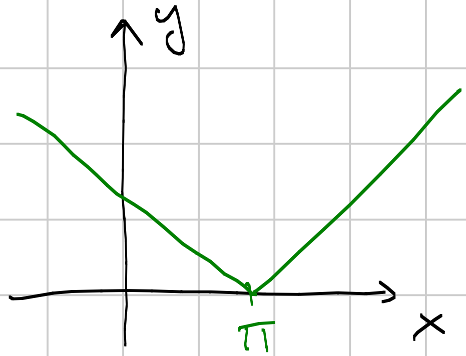
i λ che vanno bene sono tutti e soli quelli sotto il min della funzione.



$\{\, |x-\pi| \, : \, x \in \mathbb{R} \,\}$

$\min = \inf = 0$ $\qquad \sup = +\infty$

$\qquad\qquad\qquad\quad \max$ N.E.

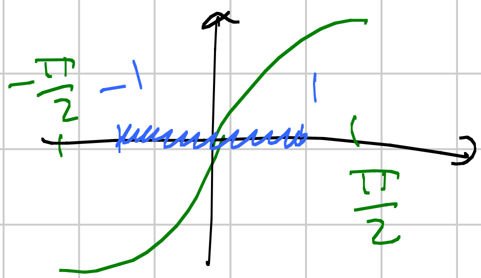


$$\lim_{n\to+\infty} \{\sin(\cos(n!+3))\}^n$$

$$[\sin(\cos(\text{Mostro}))]^n$$

$$-1 \le \cos(\text{Mostro}) \le 1$$

Applico $\sin x$ che è strett. cresc. tra -1 e 1



$$-\sin 1 \le \sin(\cos(\text{Mostro})) \le \sin 1$$

$$\underbrace{\boxed{0}}_{\downarrow\, 0} \le \underbrace{|\sin(\cos(\text{Mostro}))|^n}_{\to\, 0} \le \underbrace{(\sin 1)^n}_{\downarrow\, 0}$$

$$\left\{ x \in [0, 10] : \underbrace{x^3 - 17 \sin x}_{f(x)} \geq 0 \right\} = A$$

$0 \in A \Rightarrow \min = 0$, $10 \in A \Rightarrow \max = 10$

$$f(x) \sim -17x$$