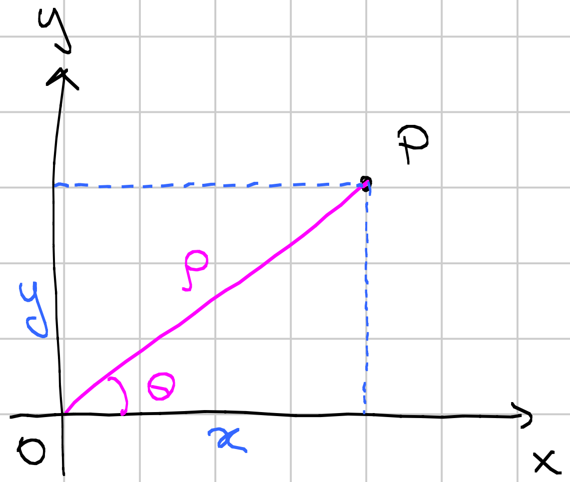
COORDINATE POLARI



Coord. CARTESIANE di P : (x, y)

Coord. POLARI di P : ρ, θ

ρ = distanza OP

θ = angolo formato dal semiasse positivo delle x e OP

Eccezione: P = O (origine) allora $\rho = 0$, θ non è definito

ρ assume solo valori ≥ 0

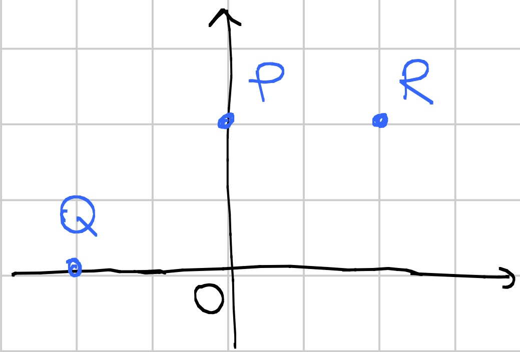
θ è un angolo, dunque definito a meno di multipli di 2π

Esempi

$P = (0,1)$ $\quad \rho = 1, \ \theta = \frac{\pi}{2}$

$Q = (-1,0)$ $\quad \rho = 1, \ \theta = \pi$

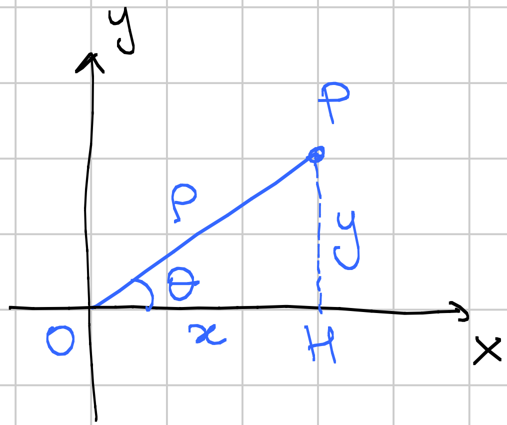
$R = (1,1)$ $\quad \rho = \sqrt{2}, \ \theta = \pi/4$



Formule di passaggio

Noti ρ e θ, calcolare x e y

$$\boxed{x = \rho \cos\theta \qquad y = \rho \sin\theta}$$



Noti x e y, calcolare ρ e θ

$$\boxed{\rho = \sqrt{x^2+y^2}}$$

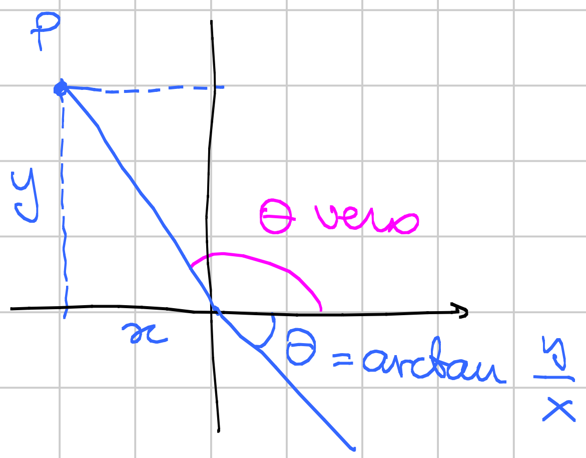
Per θ la situazione è + complessa

$\tan\theta = \frac{y}{x}$, quindi $\boxed{\theta = \arctan\frac{y}{x}}$

NO!!! arctan fornisce solo valori tra $-\frac{\pi}{2}$ e $\frac{\pi}{2}$

La formula $\boxed{\theta = \arctan \frac{y}{x}}$ è corretta nel I e IV quadrante con $x \neq 0$.

Se siamo nel $\underline{\text{II}}$ o $\underline{\text{III}}$ quadrante (I e IV) se uso la formula non ottengo θ, ma il suo supplementare, quindi

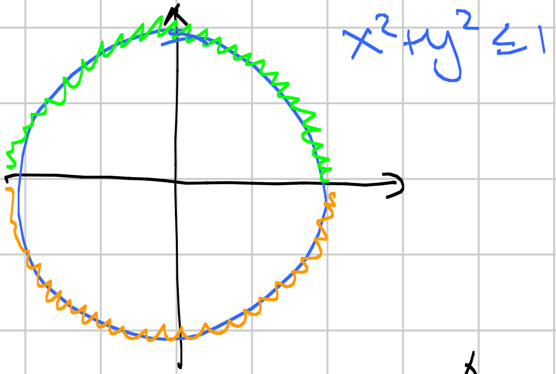


$$\boxed{\theta_{vero} = \pi - \arctan \frac{y}{x}}$$

II e III quadr.

— o — o —

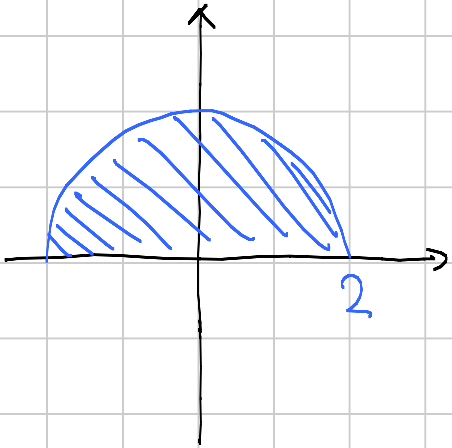
Descrizione di insiemi mediante coordinate polari



$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

$$A = \{(x,y) \in \mathbb{R}^2 : x \in [-1,1], -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$$

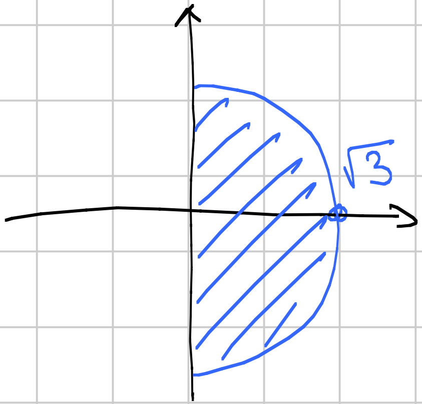
$$A = \{(\rho\cos\theta, \rho\sin\theta) : \rho \in [0,1], \theta \in [0, 2\pi]\}$$



$$B = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \le 4,\ y \ge 0 \}$$

$$= \{ (x,y) \in \mathbb{R}^2 : x \in [-2,2],\ 0 \le y \le \sqrt{4-x^2} \}$$

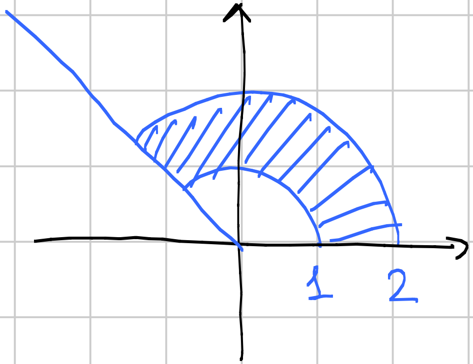
$$= \{ (\rho\cos\theta, \rho\sin\theta) : \rho \in [0,2],\ \theta \in [0,\pi] \}$$



$$C = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \le 3,\ x \ge 0 \}$$

$$= \{ (x,y) \in \mathbb{R}^2 : x \in [0,\sqrt{3}],\ -\sqrt{3-x^2} \le y \le \sqrt{3-x^2} \}$$

$$= \{ (\rho\cos\theta, \rho\sin\theta) : \rho \in [0,\sqrt{3}],\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \}$$



$$D = \{ (x,y) \in \mathbb{R}^2 : \underbrace{1 \le x^2+y^2 \le 4}_{\text{CORONA}},\ \underbrace{y \ge 0}_{\text{SOPRA ASSE } x},\ \underbrace{y \ge -x}_{\text{SOPRA RETTA } y=-x} \}$$

$$= \{ (\rho\cos\theta, \rho\sin\theta) : \rho \in [1,2],\ \theta \in \left[0, \frac{3\pi}{4}\right] \}$$

Integrali doppi in coordinate polari.

Esempio 1 $\displaystyle\iint_A (x^2+y^2)\,dx\,dy$ $\quad A =$ come nell'esempio

$$= \underbrace{\int_0^1 d\rho \int_0^{2\pi} d\theta}_{\text{sfrutto la descrizione di A in coord. polari}} \quad \underbrace{\rho^2}_{\text{funzione } x^2+y^2 \text{ in coord. polari.}} \quad \boxed{\rho}$$

$\boxed{\rho}$ → pagamento: occorre moltiplicare tutto per ρ

sfrutto la descrizione di A in coord. polari

funzione x^2+y^2 in coord. polari.
In generale pongo $x=\rho\cos\theta$, $y=\rho\sin\theta$

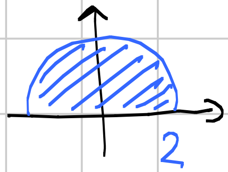
$$x^2+y^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta$$
$$= \rho^2(\cos^2\theta+\sin^2\theta) = \rho^2$$

$$= \int_0^1 d\rho \int_0^{2\pi} \rho^3\, d\theta = \int_0^1 \rho^3\, d\rho \underbrace{\int_0^{2\pi} d\theta}_{2\pi} = 2\pi \int_0^1 \rho^3\, d\rho = 2\pi\left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}$$

$$= 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

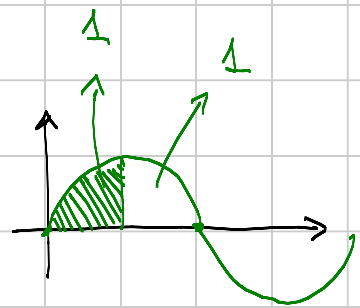
Esempio 2

$$\iint_B y \, dx \, dy = \qquad B =$$



$$= \underbrace{\int_0^2 d\rho \int_0^\pi d\theta}_{\text{DESCRIZIONE INSIEME}} \; \underbrace{\rho \sin\theta}_{\text{y in coord. polari}} \cdot \underbrace{\rho}_{\uparrow \text{ pagamento}}$$

$$= \int_0^2 d\rho \int_0^\pi \rho^2 \sin\theta \, d\theta = \int_0^2 \rho^2 d\rho \underbrace{\int_0^\pi \sin\theta \, d\theta}_{2}$$

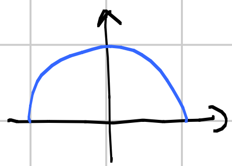


$$= 2 \int_0^2 \rho^2 d\rho = 2 \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=2} = \frac{16}{3}$$

Esempio 2.5

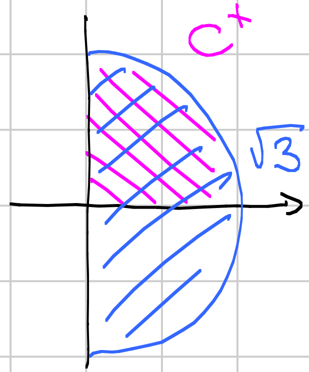
$$\iint_B x \, dx \, dy = 0$$

↑ per simmetria



Esempio 3

$$\iint_C |y|\,dx\,dy = 2\iint_{C^+} |y|\,dx\,dy =$$



$$= 2\iint_{C^+} y\,dx\,dy$$

$$= 2\underbrace{\int_0^{\sqrt{3}} d\rho \int_0^{\pi/2} d\theta}_{\text{DESCRIZIONE } C^+} \; \underbrace{\rho \sin\theta}_{y} \; \underbrace{\rho}_{\text{pag.}} = 2\int_0^{\sqrt{3}} \rho^2\,d\rho \underbrace{\int_0^{\pi/2} \sin\theta\,d\theta}_{1}$$

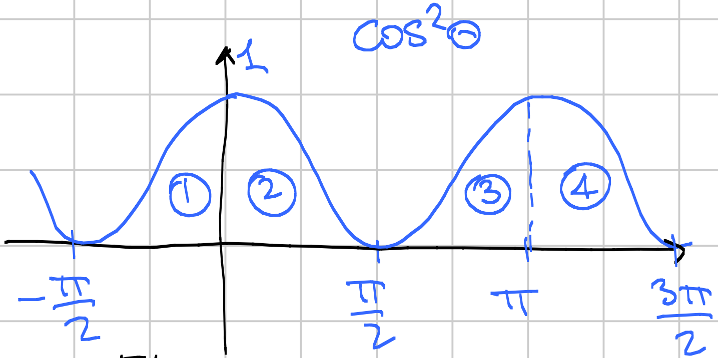
$$= 2\int_0^{\sqrt{3}} \rho^2\,d\rho = 2\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\sqrt{3}} = 2\,\frac{\cancel{3}\sqrt{3}}{\cancel{3}} = 2\sqrt{3}$$

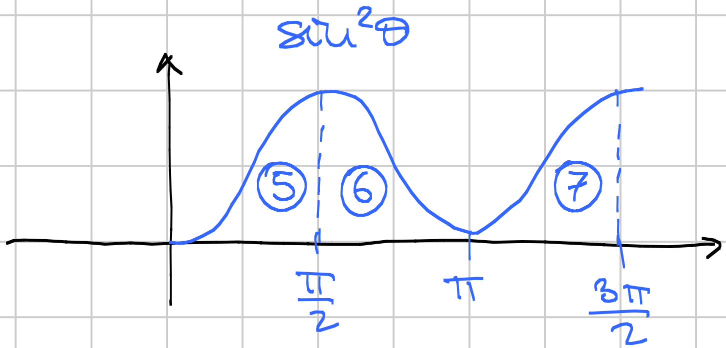
Esempio 4

$$\iint_C x^2\,dx\,dy = \underbrace{\int_0^{\sqrt{3}} d\rho \int_{-\frac{\pi}{2}}^{\pi/2} d\theta}_{\text{DESCRIZIONE DI C}} \quad \underbrace{\rho^2\cos^2\theta}_{x^2} \quad \underbrace{\rho}_{\text{PAG.}}$$

$$= \int_0^{\sqrt{3}} \rho^3\,d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

si potrebbe fare la primitiva di $\cos^2\theta$





$$② = \int_0^{\pi/2} \cos^2\theta\,d\theta \qquad ⑤ = \int_0^{\pi/2} \sin^2\theta\,d\theta$$

per simmetria

$$② + ⑤ = \int_0^{\pi/2} (\cos^2\theta + \sin^2\theta)\,d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2} \Rightarrow ② = ⑤ = \frac{\pi}{4}$$

Ciascuno dei pezzi ①, ②, ③, ..., ⑦ fa $\frac{\pi}{4}$

Conclusione: l'integrale di $\cos^2\theta$ o di $\sin^2\theta$ tra 2 estremi che sono multipli di $\frac{\pi}{2}$ è sempre uguale a metà della lungh. dell'intervallo

Tornando all'esempio

$$\int_0^{\sqrt{3}} \rho^3 d\rho \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta}_{} = \frac{\pi}{2} \int_0^{\sqrt{3}} \rho^3 d\rho = \frac{\pi}{2} \left[\frac{\rho^4}{4} \right]_{\rho=0}^{\rho=\sqrt{3}}$$

Misura intervallo $= \pi$

$\Rightarrow$ integr. $= \frac{\pi}{2}$

$$= \frac{3\pi}{8}.$$