Esempio (sistema lineare con molteplicità)

$$\frac{2x+1}{x^4-x^3} = \frac{2x+1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{d}{dx}\left(\frac{Cx+D}{x^2}\right)$$

equazioni = # incognite = grado denominatore

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C\cdot x^2 - 2x\,(Cx+D)}{x^4}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx - 2Cx - 2D}{x^3}$$

$$= \frac{Ax^2(x-1) + Bx^3 + (-Cx-2D)(x-1)}{x^3(x-1)}$$

$$= \frac{Ax^3 - Ax^2 + Bx^3 - Cx^2 + Cx - 2Dx + 2D}{x^3(x-1)}$$

$A+B=0$ coeff. x^3

$-A-C=0$ coeff. x^2

$C-2D=2$ coeff. x

$2D=1$ termine noto

$D = \frac{1}{2}$ $\quad C = 2D + 2 = 3$ $\quad A = -C = -3$ $\quad B = -A = 3$

$$\frac{2x+1}{x^4 - x^3} = -\frac{3}{x} + \frac{3}{x-1} + \frac{d}{dx}\left(\frac{3x + \frac{1}{2}}{x^2}\right)$$

$$\int \frac{2x+1}{x^4-x^3}\,dx = -3\int \frac{1}{x}\,dx + 3\int \frac{1}{x-1}\,dx + \int \frac{d}{dx}\left(\frac{3x+\frac{1}{2}}{x^2}\right)dx$$

$$= -3\log|x| + 3\log|x-1| + \frac{3x+\frac{1}{2}}{x^2}$$

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$\boxed{\text{FASE 4: INTEGRAZIONE}}$ Si tratta di integrare gli addendi prodotti in FASE 3.

Ci si ritrova con 3 tipi di pezzi

* $\frac{d}{dx}(\quad)$ ⇝ banale da integrare

* $\frac{A}{ax+b}$ $\qquad$ * $\frac{Ax+B}{\underbrace{ax^2+bx+c}_{\text{con } \Delta < 0}}$

$$\int \frac{A}{ax+b}\,dx = A\int \frac{1}{ax+b}\,dx = A\,\frac{1}{a}\int \frac{a}{ax+b}\,dx$$

$$= \frac{A}{a}\log|ax+b|$$

Restano i fattori del tipo $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}\,dx$

<u>Esempi</u>

$$\int \frac{1}{x^2+1}\,dx = \arctan x$$

volendo $y = x^2+1$
e sostituzione

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\log(x^2+1)$$

$$\int \frac{3x+4}{x^2+1}\,dx = 3\int \frac{x}{x^2+1}\,dx + 4\int \frac{1}{x^2+1}\,dx$$

$$= \frac{3}{2}\log(x^2+1) + 4\arctan x$$

$$\int \frac{2x+1}{x^2+x+1}\,dx = \log(x^2+x+1)$$

Num = derivata di denominatore

Pongo $x^2+x+1 = y$ $\quad \frac{dy}{dx} = 2x+1 \rightarrow dy = (2x+1)\,dx$

$$= \int \frac{dy}{y} = \log y = \log(x^2+x+1)$$

essendo $\Delta < 0$ il pol. di 2° grado è sempre maggiore di zero

$$\int \frac{dx}{1+2x^2} = \int \frac{dx}{1+(\sqrt{2}x)^2} = \frac{1}{\sqrt{2}} \arctan(\sqrt{2}x)$$

$$\int \frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}} \arctan(\sqrt{a}x) \qquad (\text{se } a>0)$$

$$\int \frac{dx}{1-2x^2}$$ il denominatore si può scomporre

$$\frac{1}{1-2x^2} = \frac{1}{(1-\sqrt{2}x)(1+\sqrt{2}x)} = \frac{A}{1-\sqrt{2}x} + \frac{B}{1+\sqrt{2}x}$$

sistema lineare, ..., vengono 2 logaritmi.

$$\int \frac{dx}{2+2x+x^2} = \int \frac{dx}{1+(x+1)^2} = \arctan(x+1)$$

$$\int \frac{dx}{2+x^2} = \frac{1}{2}\int \frac{dx}{1+\frac{1}{2}x^2} = \frac{1}{2}\,\frac{1}{\sqrt{\frac{1}{2}}}\arctan\left(\sqrt{\frac{1}{2}}\,x\right)$$

esercizio precedente con $a = \frac{1}{2}$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)$$

In generale $\boxed{\int \frac{dx}{b+x^2} = \frac{1}{\sqrt{b}}\arctan\left(\frac{x}{\sqrt{b}}\right)}$ se $b>0$

Se $b<0$, si scompone

$$\int \frac{3x+4}{x^2+2x+2}\,dx$$

Se num fosse $2x+2$ sarebbe log.
" " " $3x+3$ sarebbe ancora log

$$\int \frac{3x+3}{x^2+2x+2}\,dx = \frac{3}{2}\int \frac{2x+2}{x^2+2x+2}\,dx$$

$$= \frac{3}{2}\log(x^2+2x+2)$$

$$\int \frac{3x+4}{x^2+2x+2}\,dx = \int \frac{3x+3}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$= \frac{3}{2}\log(x^2+2x+2) + \arctan(x+1)$$

In generale i pezzi $\dfrac{Ax+B}{ax^2+bx+c}$ producono un pezzo con il logaritmo che "sistema la x" e un pezzo con arctan che "sistema il termine noto"

$$\int \frac{dx}{(x^2+1)^2}$$

FASE 3

$$\boxed{\frac{1}{(x^2+1)^2}} = \frac{Ax+B}{x^2+1} + \frac{d}{dx}\left(\frac{Cx+D}{x^2+1}\right)$$

$$= \frac{Ax+B}{x^2+1} + \frac{C(x^2+1)-2x(Cx+D)}{(x^2+1)^2}$$

$$= \frac{(Ax+B)(x^2+1)+Cx^2+C-2Cx^2-2Dx}{(x^2+1)^2}$$

$$= \boxed{\frac{\cancel{Ax^3}+\cancel{Ax}+\cancel{Bx^2}+B+\cancel{Cx^2}+C-\cancel{2Cx^2}-\cancel{2Dx}}{(x^2+1)^2}}$$

$A=0$ coeff. x^3

$B-C=0$ " x^2

$A-2D=0$ " x

$B+C=1$ term. noto

$A=0$

$B=\frac{1}{2}$

$D=0$

$C=\frac{1}{2}$

$B=C$

$2B=1$

$$\frac{1}{(x^2+1)^2} = \frac{1}{2}\frac{1}{x^2+1} + \frac{d}{dx}\left(\frac{1/2\ x}{x^2+1}\right)$$

$$\int \frac{1}{(x^2+1)^2} = \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \int \frac{d}{dx}\left(\quad\right)$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{x^2+1}$$

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<u>Trucchetto</u> $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

Moltiplico per $(x-2)$: $\displaystyle\frac{x}{x-3} = A + \frac{B}{(x-3)}(x-2)$

Pongo $x=2$ $\displaystyle\frac{2}{-1} = A \rightarrow A = -2$

Per ricavare B moltiplico per $x-3$

$$\frac{x}{x-2} = A\,\frac{x-3}{x-2} + B \qquad \text{sostituisco } x=3$$

$$\frac{3}{+1} = B \longrightarrow B = +3$$

$$\frac{x}{x^2-5x+6} = -\frac{2}{x-2} + \frac{3}{x-3}$$

$$\int \frac{x}{x^2-5x+6} = -2\log|x-2| + 3\log|x-3| = \log\frac{|x-3|^3}{|x-2|^2}$$

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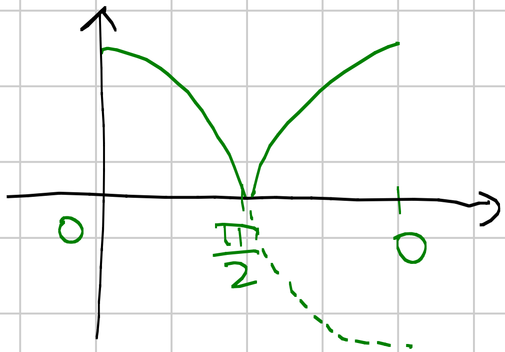
Esempio $\displaystyle\int_0^{\pi} |\cos x|\,dx$



$$= \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$$

$$= \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} (-\cos x) \, dx$$

$\cos x \geq 0$ in $[0, \pi/2]$

$\cos x \leq 0$ in $[\frac{\pi}{2}, \pi]$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}} - \left[\sin x\right]_{\frac{\pi}{2}}^{\pi} = \sin\frac{\pi}{2} - \sin 0 - \left(\sin\pi - \sin\frac{\pi}{2}\right)$$

$$= 1 + 1 = 2$$