MAT I TLC

$\boxed{\text{ORA28}}$

Titolo nota 24/10/2006

Formula di Taylor → composizione di funzioni

↘ centro in un p.to $x_0 \neq 0$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + o\left((2x)^3\right) =$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3) \qquad = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + o(x^3)$$

$$o\left((2x)^3\right) = (2x)^3\,\omega(2x) = x^3 \cdot \underbrace{8\,\omega(2x)}_{\omega_2(x)\ \to\ 0}$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + o(x^6)$$

$$+ o(x^7)$$

$$— o — o —$$

$$e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{6} + o((\sin x)^3)$$

$$\sin t = t - \frac{t^3}{6} + o(t^3)$$

$$= 1 + \left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 +$$

$$+ \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^3)\right)^3 + o((\sin x)^3)$$

$$= 1 + x - \cancel{\frac{x^3}{6}} + \frac{1}{2}x^2 + \cancel{\frac{1}{6}x^3} + o\left((\sin x)^3\right) + o(x^3)$$

il resto del $(\ldots)^2$
se lo è mangiato
$o(x^3)$

MANGIATO?

$o\left((\sin x)^3\right) \Leftrightarrow o(x^3)$

STESSA FORZA

$$= 1 + x + \frac{x^2}{2} + o(x^3)$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \underbrace{t^3\omega(t)}$$

$\omega(t) \to 0$ per $t \to 0$

$$o\left((\sin x)^3\right) = (\sin x)^3 \cdot \omega(\sin x) =$$

$$= x^3 \underbrace{\boxed{\left(\frac{\sin x}{x}\right)^3 \omega(\sin x)}}_{\omega_2(x)}$$

$\omega_2(x) \to 1 \cdot 0 = 0$

$$\cos(\sin x) =$$

$$\boxed{n=4}$$

$$\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} + o(t^4)$$

$$= 1 - \frac{1}{2}(\sin x)^2 + \frac{1}{24}(\sin x)^4 + o((\sin x)^4)$$

$$= 1 - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 + \frac{1}{24}\left(x - \frac{x^3}{6} + o(x^4)\right)^4 + o(x^4)$$

$$= 1 - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{24}x^4 + o(x^4)$$

$$2 \cdot x\left(-\frac{x^3}{6}\right)$$

DOPPIO PRODOTTO

$$= 1 - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^5)$$

$f(x) = \cos(\sin x)$ è una funzione PARI

$$f(-x) = \cos(\sin(-x)) \underset{\substack{\uparrow \\ \sin \\ \text{DISP.}}}{=} \cos(-\sin x) \underset{\substack{\uparrow \\ \cos \\ \text{PARI}}}{=} \cos(\sin x) = f(x)$$

Essendo f pari, nel suo sviluppo compaiono solo potenze PARI, quindi non c'è il termine in x^5

— o — o —

$$f(x) = \sin(\cos x) = \qquad \boxed{n=4}$$

$$\sin t = t - \frac{t^3}{6} + o(t^4)$$

$$= \cos x - \frac{(\cos x)^3}{6} + o\left((\cos x)^4\right)$$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) - \frac{1}{6}\left(\quad \right)^3 + \boxed{o(x^4)} \;???$$

$$o\left((\cos x)^4\right) = (\cos x)^4 \, \omega(\cos x) = x^4 \underbrace{\boxed{\left(\frac{\cos x}{x}\right)^4 \omega(\cos x)}}_{\omega_2(x)}$$

$$\lim_{x\to 0} \omega_2(x) = \lim_{x\to 0} \underbrace{\boxed{\left(\frac{\cos x}{x}\right)^4}}_{\downarrow \; +\infty} \cdot \underbrace{\boxed{\omega(\cos x)}}_{\downarrow \; \omega(1)}$$

Con $\sin(\cos x)$ non va bene perché la funzione interna, nel nostro caso $\cos x$, non tende a 0 per $x \to 0$.

$\sin t = t - \frac{t^3}{6} + o(t^4)$ buona approssimazione per $t \sim 0$

Se metto $t = \cos x$ e poi faccio $x \to 0$, ho che $t \to 1$.

$$\cos t = 1 - \frac{t^2}{2} + o(t^3)$$

buona per $t \sim 0$

Se metto $t = \sin x$ e faccio $x \to 0$ ho che $t \to 0$, quindi ok.

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) - \frac{1}{6}\left(\quad \right)^3 +$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{6}\left(1 + \cdots\right) + \frac{1}{5!}\underbrace{\left(\quad \right)^5}_{1+\cdots} + o(x^4)$$

tutte $\left(\quad \right)^k$ producono delle costanti, quindi non finisco mai nemmeno di calcolare il termine noto.

Esempio $\boxed{x \to 0}$

SVILUPPI DI ORDINE 1

$$\frac{e^{2x}+\cos(3x)+\sqrt{1+x}-3}{\arctan x+\arcsin x} = \frac{\cancel{1}+2x+\cancel{1}+\cancel{1}+\frac{x}{2}-\cancel{3}+o(x)}{2x+o(x)}$$

$$e^{2x} = 1+2x+o(x)$$

$$\cos(3x) = 1+o(x)$$

$$\sqrt{1+x} = (1+x)^{1/2} = 1+\frac{x}{2}+o(x)$$

$$\arctan x = x+o(x)$$

$$\arcsin x = x+o(x)$$

$$= \frac{\frac{5}{2}x+o(x)}{2x+o(x)}$$

$$= \frac{\cancel{x}\left(\frac{5}{2}+\underbrace{\boxed{\frac{o(x)}{x}}}_{\to 0}\right)}{\cancel{x}\left(2+\underbrace{\boxed{\frac{o(x)}{x}}}_{\to 0}\right)} \to \frac{5}{4}$$

$$\lim_{x\to 0}\frac{\log^4(1+x)}{\cos x^2-1} = \lim_{x\to 0}\frac{x^4+o(x^4)}{\cancel{1}-\frac{x^4}{2}+o(x^4)-\cancel{1}} = -2$$

$$\log(1+x) = x + o(x) \quad ; \quad [\log(1+x)]^4 = (x + o(x))^4$$
$$= x^4 + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\cos x^2 = 1 - \frac{x^4}{2} + o(x^4)$$

— o — o —

$\boxed{x \to 0}$

$$x\sqrt{x} = x^{\frac{3}{2}}$$

$$\frac{x\sqrt{x} + \cos x - 1}{\sin x^5 + \sin^5 x + \sqrt{\arctan(2x^3)}} =$$

$$= \frac{x\sqrt{x} + \cancel{1} + o(x^{3/2}) - \cancel{1}}{\sqrt{2x^3} \qquad o(x^{3/2})}$$

$$= \frac{x\sqrt{x} + o(x^{3/2})}{\sqrt{2}\, x\sqrt{x} + o(x^{3/2})} \longrightarrow \frac{1}{\sqrt{2}}.$$

$$\cos x = 1 + o(x^{\frac{3}{2}})$$
$$\sin x^5 = o(x^{3/2})$$
$$(\sin x)^5 = o(x^{3/2})$$

MAT I TLC

ORA 29

RICEVIMENTO : GIOVEDÌ
da [FINE LEZIONE $\to +\infty$)
dove si è fatto lezione

$$1-\cos^2 x^3 = x^6 + o(x^6)$$

$$1-\cos^3 x^2 = \frac{3}{2}x^4 + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3)$$

$$\cos x^3 = 1 - \frac{x^6}{2} + o\left(x^9\right)$$

$$(\cos x^3)^2 = 1 - \underset{\substack{\uparrow \\ \text{DOPPIO} \\ \text{PRODOTTO}}}{x^6} + o(x^6)$$

$$\cos x^2 = 1 - \frac{x^4}{2} + o\left(x^7\right)$$

$$(\cos x^2)^3 = \left(1 - \frac{x^4}{2} + o(x^7)\right)^3$$

$$= 1 - \underset{\substack{\uparrow \\ \text{TRIPLO PRODOTTO}}}{\frac{3}{2}x^4} + o(x^4)$$

TRIPLO PRODOTTO $3\cdot(1)^2\left(-\frac{x^4}{2}\right)$

$$\lim_{x \to 0} \frac{1 - \cos^2 x^3}{1 - \cos^3 x^2} = \frac{x^6 + o(x^6)}{\frac{3}{2} x^4 + o(x^4)} =$$

$$= \frac{\cancel{x^6}^{\,2}\left(1 + \underbrace{\frac{o(x^6)}{x^6}}_{\to 0}\right)}{\cancel{x^4}\left(\frac{3}{2} + \underbrace{\frac{o(x^4)}{x^4}}_{\to 0}\right)} \to 0 \cdot \frac{1}{\frac{3}{2}} = 0.$$

SVILUPPI CON CENTRO $x_0 \neq 0$.

$$f(x) = P_n(x) + o(x^n) \qquad \text{per } x \to 0$$

$$f(x) = \qquad\qquad\qquad \text{per } x \to x_0$$

Cambio variabile: $h = x - x_0$ Quando $x \to x_0$ ho che $h \to 0$

$x = x_0 + h$

$$f(x) = f(x_0+h) = \text{funzione di } h \text{ che chiamo } g(h)$$

Applico solito Taylor a g. Ottengo

$$g(h) = g(0) + g'(0)h + \frac{g''(0)}{2!}h^2 + \dots + \frac{g^{(n)}(0)}{n!}h^n + o(h^n)$$

Ora torno indietro:

$$g(h) = f(x_0+h), \quad g'(h) = f'(x_0+h), \quad g''(h) = f''(x_0+h)$$

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \dots + \frac{f^{(n)}(x_0)}{n!}h^n + o(h^n)$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots +$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o\left((x-x_0)^n\right)$$

Quando $x_0 = 0$ tutte le deriv. erano calcolate in 0, ora sono calcolate in x_0.
Inoltre $x \sim 0$ e nello sviluppo comparivano potenze di x
Ora $x - x_0 \sim 0$ e " " compaiono " " $x - x_0$

—— o —— o ——

$e^{\cos x}$

Non si può fare la sostituzione brutale
$\cos x \to 1$, quindi mi serve uno sviluppo di e^t
per $t \sim 1$

$$e^{1+R} = \text{dovrei calcolare le derivate di } e^t \text{ per } t=1$$

$$= e \cdot e^{R} = e\left(1 + R + \frac{R^2}{2} + \frac{R^3}{6} + o(R^3)\right) \quad \text{per } R \to 0$$

$$= e + eR + \frac{e}{2}R^2 + \frac{e}{6}R^3 + o(R^3)$$

$$e^{\cos x} = e^{1 + \boxed{\cos x - 1}}$$

è un $R \to 0$

$$\cos x - 1 = \cancel{1} - \frac{x^2}{2} + o(x^3) \cancel{-1}$$

$$= e + e(\cos x - 1) + \frac{e}{2}(\cos x - 1)^2 + \frac{e}{6}(\cos x - 1)^3 + o(\ )$$

$$= e + e\left(-\frac{x^2}{2} + o(x^3)\right) + \frac{e}{2}\left(-\frac{x^2}{2} + o(x^3)\right)^2 + \frac{e}{6}(\ )^3 + o(\ )$$

$$= e - \frac{e}{2}x^2 \qquad + o(x^3)$$

$$\boxed{e^{\cos x} = e - \frac{e}{2}x^2 + o(x^3)}$$

$$\lim_{x\to 0} \frac{e^{\sin x} - \cos x - x}{\arctan x \cdot \sin(\sin x)} =$$

$\arctan x = x + o(x)$

$\sin(\sin x) = x + o(x)$

$\sin t = t + o(t)$; $\sin(\sin x) = \sin x + o(\sin x)$

$= x + o(x)$

Denom.: $(x + o(x)) \cdot (x + o(x)) = x^2 + o(x^2)$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + o(x^3) \quad \text{(FATTO PRIMA)}$$

$$= \lim_{x\to 0} \frac{\cancel{1} + \cancel{x} + \frac{x^2}{2} - \cancel{1} + \frac{x^2}{2} - \cancel{x} + o(x^2)}{x^2 + o(x^2)} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{x^2 + o(x^2)} = 1.$$

$$\lim_{x\to 0} \frac{\arctan(x+x^3) - \sin x}{\log(1+\sin^2 x)\tan x} = \frac{5}{6}$$

$\tan x = x + o(x)$; $\sin x = x + o(x)$; $\sin^2 x = x^2 + o(x^2)$

$\tan x \sim x$; $\sin^2 x \sim x^2$; $\log(1+\sin^2 x) \sim \log(1+x^2) \sim x^2$

Denominatore $\sim x^3$. Sviluppo numeratore all'ordine 3

$\sin x = x - \frac{x^3}{6}$; $\arctan(x+x^3) \sim x + x^3$

NOOOOO

$\arctan t \sim t$

$$\arctan(x+x^3) - \sin x \sim \cancel{x} + x^3 - \cancel{x} + \frac{x^3}{6} = \frac{7}{6}x^3$$

$$\arctan t = t + o(t) \; ; \quad \arctan(x+x^3) = x + x^3 + o(x+x^3)$$

$$= x + x^3 + o(x) =$$

$$= x + o(x)$$

NO BUONO

$$o(x+x^3) = (x+x^3)\,\omega(x) = x \underbrace{(1+x^2)\,\omega(x)}_{\omega_2(x)}$$

NON POSSO RACCOGLIERE x^3

$$= x^3\left(\frac{1}{x^2} + x\right)\omega(x)$$

$\searrow$ DISASTRO

VA BENE: $\arctan t = t - \dfrac{t^3}{3} + o(t^3)$

$$\arctan(x+x^3) = (x+x^3) - \frac{(x+x^3)^3}{3} + o\left((x-x^3)^3\right)$$

$$= x + x^3 - \frac{x^3}{3} + o(x^3) = x + \frac{2}{3}x^3 + o(x^3)$$

Numeratore =

$$= \arctan(x + x^3) - \sin x = \cancel{x} + \frac{2}{3}x^3 - \cancel{x} + \frac{x^3}{6} + o(x^3)$$

$$= \frac{5}{6}x^3 + o(x^3)$$

$$\text{Lim} = \frac{5}{6}$$