Linear hyperbolic equations equations with time-dependent propagation speed and strong damping

(Unabridged printout of the lectures of a PhD course)

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Contents

FREIBERG 2017 — LECTURE 1

Note Title 22/05/2017

FUNCTIONAL SPACES, CONTINUITY MODULI

Fix notation

H is a real Hilbert space ($H := L^2(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$)
H is separable

Def. A orthonormal system is a subset $\{e_k\}_{k\in\mathbb{N}} \subseteq H$ such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Def. Given any $u \in H$, set $u_k := \langle u, e_k \rangle$ and

$$\sum_{k\in\mathbb{N}} u_k^2 = \|u\|^2 < +\infty$$

Advantage: we can identify vectors $u \in H$ with sequence $\{u_k\}$

$$\sum_{k\geq 0} u_k^2 < +\infty$$

Multiplication operators $A : D(A) \longrightarrow H$ if there exist a sequence $\{\lambda_k\} \subseteq \mathbb{R}$ (most of the time $\lambda_k \geq 0$) $\{e_k\}$ orth. system

$$u = \sum_{k\geq 0} u_k e_k \quad \leadsto \quad Au = \sum_{k\geq 0} \lambda_k^2 u_k e_k$$

well defined when

$$u \in D(A) := \left\{ u \in H : \sum_{k\geq 0} \lambda_k^4 u_k^2 < +\infty \right\}$$

Examples $H := L^2((0,\pi))$ $Au = -u_{xx}$ with
DBC
NBC
PBC

In this A is a mult. operator with

$\lambda_k = k$ $\quad k \geq 1$ if DBC $\quad$ (coercive operator)
$\quad k \geq 0$ if NBC o PBC

Similar situation if $\Omega \subseteq \mathbb{R}^n$, $A = -\Delta$, Ω open, bounded
$\partial\Omega$ smooth enough.

Multiplication operator in general

A be any self-adjoint nonnegative operator in H
$\langle Au, v\rangle = \langle u, Av\rangle$ $\quad \Rightarrow \langle Au, u\rangle \geq 0$

Result We can identify H with $L^2(M, \mu)$
measure space $\quad$ measure on M

$H \ni u \quad \rightsquigarrow \quad u(\xi) \in L^2(M,\mu)$

$\Big\downarrow \qquad\qquad \Big\downarrow$

$Au \qquad \lambda^2(\xi)\, u(\xi) \in L^2(M,\mu)$

$$D(A) \rightsquigarrow \left\{ u(\xi) \in L^2(M,\mu) : \int_M \lambda^4(\xi)\, u^2(\xi)\, d\mu < +\infty \right\}$$

Standard example $\quad H := L^2(\mathbb{R}^n)$ $\quad A = -\Delta$

$u(\xi)$ is the Fourier transform

$\boxed{\text{Functional spaces}}$ $\{e_k\}$ orth system
$\{\lambda_k^2\}$ seq. of eigen. of A

$$u = \sum_{k\geq 0} \underset{\substack{\uparrow \\ \text{comp. of } u}}{u_k} e_k \qquad Au = \sum_{k\geq 0} \lambda_k^2 u_k e_k$$

Def. (Spaces with respect to the operator A)

1. Sobolev spaces Given any $\alpha \geq 0$ ($\alpha \in \mathbb{R}$)

$$D(A^\alpha) := \left\{ \sum_{k\geq 0} u_k e_k \;:\; \sum_{k\geq 0} (1+\lambda_k)^{4\alpha} u_k^2 < +\infty \right\}$$

$$A^\alpha u := \sum_{k\geq 0} \lambda_k^{2\alpha} u_k e_k$$

2. Distributions Given any $\alpha \geq 0$ ($\alpha \in \mathbb{R}$)

$$D(A^{-\alpha}) := \left\{ \sum_{k\geq 0} u_k e_k \;:\; \sum_{k\geq 0} (1+\lambda_k)^{-4\alpha} u_k^2 < +\infty \right\}$$

3. Gevrey spaces Given any $\alpha \geq 0$
Given $r > 0$ (radius)
Given $\varphi : [0, +\infty) \to (0, +\infty)$

$$\mathcal{G}_{\varphi, r, \alpha}(A) := \left\{ \sum_{k\geq 0} u_k e_k \;:\; \sum_{k\geq 0} (1+\lambda_k)^{4\alpha} \underbrace{e^{2r\varphi(\lambda_k)}}_{\text{huge weigth}} u_k^2 < +\infty \right\}$$

Examples
- $\varphi(\lambda) = \lambda$, $\alpha = 0$ $\rightsquigarrow$ analytic functions with power series converging with radius $= r$.
- $\varphi(\lambda) = \lambda^{\frac{1}{s}}$, $\alpha = 0$ $\rightsquigarrow$ Standard Gevrey space. $(s > 1)$ $s \in (0,1)$

④ Gevrey ultra distributions

$\alpha \geq 0$

$R > 0$

$\psi : [0, +\infty) \to (0, +\infty)$

$$\mathcal{G}_{-\psi, R, \alpha}(A) := \left\{ \sum_{k \geq 0} u_k e_k : \sum_{k \geq 0} (1+\lambda_k)^{4\alpha} e^{-2R\psi(\lambda_k)} u_k^2 < +\infty \right\}$$

Basic inclusions For every admissible value of the parameters (assume also that $\varphi(\lambda) \to +\infty$ as $\lambda \to +\infty$, $\psi(\lambda) \to +\infty$ as $\lambda \to +\infty$)

It turns out that

$$\boxed{\mathcal{G}_{\varphi, r, \alpha}(A) \subseteq D(A^{\alpha}) \subseteq H \subseteq D(A^{-\alpha}) \subseteq \mathcal{G}_{-\psi, r, -\alpha}(A)}$$

All inclusions are strict if $\{\lambda_k\}$ is unbounded

$$\mathcal{G}_{\varphi, r_2, \alpha}(A) \subseteq \mathcal{G}_{\varphi, r_1, \alpha}(A) \qquad r_2 > r_1$$

$$\mathcal{G}_{-\psi, R_2, \alpha}(A) \supseteq \mathcal{G}_{-\psi, R_1, \alpha}(A) \qquad R_2 > R_1$$

General philosophy $\displaystyle\sum_{k \geq 0} w_k u_k^2 < +\infty$

weigth dep. on k.

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Def. (Continuity modulus) Let (X, d_X) and (Y, d_Y) be metric spaces (for example $\mathbb{R}$ with standard metric).
Let $\omega : [0, +\infty) \to [0, +\infty)$ be a function
Let $c : X \to Y$ be any function.

We say that c is ω – continuous in $\mathfrak{X}$ if

$$\boxed{d_Y(f(b), f(a)) \leq \omega(d_{\mathfrak{X}}(b,a)) \qquad \forall (a,b) \in \mathfrak{X}^2}$$

Examples

- $f: \mathbb{R} \to \mathbb{R}$ $\qquad \omega(\sigma) = L|\sigma|$

$$|f(b) - f(a)| \leq L\,|b-a| \qquad \forall (a,b) \in \mathbb{R}^2$$

Lipschitz continuity

- $\omega(\sigma) = H\,|\sigma|^\alpha \qquad \alpha \in (0,1)$

$$|f(b) - f(a)| \leq H\,|b-a|^\alpha \qquad \forall (a,b) \in \mathbb{R}^2$$

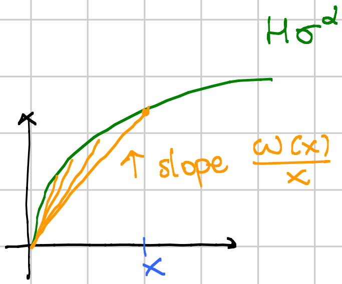
α- Hölder continuity

Always assume that

(ω1) $\quad \omega(0) = 0$

(ω2) $\quad \omega$ is $\nearrow$

(ω2) $\quad \dfrac{\omega(x)}{x}$ is $\searrow$



(easy if ω is concave)

Remark Any cont. function on a compact space has a continuity modulus.

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FREIBERG 2017 — LECTURE 2

Note Title 22/05/2017

Approximated energy estimates

Consider a family of ODEs

$$\boxed{\ddot{u}_\lambda + \boxed{\underset{\delta(t)\geq 0}{\delta(t)}}\,\dot{u}_\lambda + \underset{\text{parameter}}{\lambda^2}\, c(t)\, u_\lambda = 0}$$

Goal: estimating growth of $u(t)$ as $t \to +\infty$ depending on λ

ENERGY ESTIMATES

⓪ Stupid energy $\quad S(t) := \dot{u}^2(t) + u^2(t)$

$$S'(t) := 2\dot{u}\ddot{u} + 2u\dot{u}$$

$$= 2\dot{u}\left(-\delta(t)\dot{u} - \lambda^2 c(t) u\right) + 2u\dot{u}$$

$$= \underbrace{-2\delta(t)\dot{u}^2}_{\leq 0} + \underbrace{\left(-\lambda^2 c(t) + 1\right)}_{\leq 1 + \lambda^2 |c(t)|} \underbrace{2u\dot{u}}_{\leq \dot{u}^2 + u^2 = S(t)}$$

$$\leq \left(1 + \lambda^2 |c(t)|\right) S(t)$$

$$S(t) \leq S(0) \underbrace{\exp\left\{\int_0^t \left(1 + \lambda^2 |c(s)|\right) ds\right\}}_{\sim \exp(\lambda^2 t)}$$

① KOVALESKIAN ENERGY

$$\boxed{E(t) := \dot u^2 + \lambda^2 u^2}$$

$$E'(t) = 2\dot u \ddot u + 2\lambda^2 u \dot u$$
$$= 2\dot u\left(-\delta(t)\dot u - \lambda^2 c(t) u\right) + 2\lambda^2 u \dot u$$
$$= \underbrace{-2\delta(t)\dot u^2}_{\leq 0} + \lambda \underbrace{\left(1 - c(t)\right)}_{\leq 1+|c(t)|} \underbrace{2\lambda u \dot u}_{\leq \dot u^2 + \lambda^2 u^2 = E(t)}$$

$$\leq \lambda\left(1 + |c(t)|\right)E(t)$$

λ instead of λ^2

② HYPERBOLIC ENERGY

$$\boxed{F(t) := \dot u^2 + \lambda^2 \gamma(t) u^2}$$

class C^1, nonnegative (or strictly pos.)

$$F'(t) = 2\dot u\left(-\delta(t)\dot u - \lambda^2 c(t) u\right) + 2\lambda^2\gamma(t) u \dot u + \lambda^2\dot\gamma(t) u^2$$

$$= \underbrace{-2\delta(t)\dot u^2}_{\leq 0} + \lambda \underbrace{\frac{\left(\gamma(t) - c(t)\right)}{\sqrt{\gamma(t)}}}_{\leq \frac{|\gamma(t)-c(t)|}{\sqrt{\gamma(t)}}} \underbrace{2\sqrt{\gamma(t)}\lambda u \dot u}_{\leq \dot u^2 + \lambda^2\gamma(t)u^2 = F(t)} + \underbrace{\frac{\dot\gamma(t)}{\gamma(t)}}_{\leq \frac{|\dot\gamma(t)|}{\gamma(t)}} \underbrace{\gamma(t)\lambda^2 u^2}_{\leq F(t)}$$

$$\boxed{F'(t) \leq \left(\frac{|\gamma(t) - c(t)|}{\sqrt{\gamma(t)}}\lambda + \frac{|\dot\gamma(t)|}{\gamma(t)}\right) F(t)}$$

Standard applications

① $c(t) \in L^1((0,T))$ for every $T>0$. No sign condition

Kowaleskian energy $\rightsquigarrow$

$$E(t) \leq E(0) \exp\Big\{ \lambda \underbrace{\int_0^t (1+|c(s)|)\,ds}_{\text{convergent}} \Big\}$$

$$= E(0) \exp\Big\{ \lambda t + \lambda \int_0^t |c(s)|\,ds \Big\}$$

② $c(t)$ Lipschitz cont. and $c(t) \geq \mu_1 > 0$ (STRICT HYPERB)

Assume for a while that $c \in C^1$. Take $\gamma(t) = c(t)$ in the hyp. energy and obtain

$$F'(t) \leq \frac{|c'(t)|}{c(t)} F(t) \leq \frac{\overset{\text{Lip. const.}}{L}}{\mu_1} F(t) = M \cdot F(t)$$

$$F(t) \leq F(0) \exp\{ Mt \}$$

↑ No λ here!!!

③ $c \in L^1((0,T))$ $\quad c(t) \geq \mu_1 > 0$ (S.H.)

Take $\gamma(t)$ such that $\|\gamma(t) - c(t)\|_{L^1} \leq \varepsilon$, $\gamma \in C^1$ and use hyperb. energy

(assume that $\gamma(t) \geq \mu_1 > 0$

$$F(t) \leq F(0) \exp\Big\{ \lambda \int_0^t \frac{|\gamma(s) - c(s)|}{\sqrt{\gamma(s)}}\,ds + \int_0^t \frac{|\gamma'(s)|}{\gamma(s)}\,ds \Big\}$$

$$= F(0) \exp\Big\{ \frac{\lambda \varepsilon}{\mu_1} + M_\varepsilon t \Big\}$$

↑ indep. of λ

④ $c(t) \geq \mu_1 > 0$ (S.H.) + $c(t)$ ω – continuous.

For every $\varepsilon > 0$ take $\gamma(t) := c_\varepsilon(t) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} c(s)\,ds$

Easy estimates:

(1) $|c_\varepsilon'(t)| \leq \frac{\omega(\varepsilon)}{\varepsilon}$

(2) $|c(t) - c_\varepsilon(t)| \leq \omega(\varepsilon)$

Use the hyperbolic energy with $\gamma(t) := c_\varepsilon(t)$

$$\boxed{F'(t) \leq \left(\frac{|\gamma(t) - c(t)|}{\sqrt{\gamma(t)}} \lambda + \frac{|\gamma'(t)|}{\gamma(t)} \right) F(t)}$$

$$\leq \left(\frac{\omega(\varepsilon)}{\sqrt{\mu_1}} \lambda + \frac{\omega(\varepsilon)}{\varepsilon} \frac{1}{\mu_1} \right) F(t)$$

$\boxed{\text{KEY IDEA : choose } \varepsilon := \frac{1}{\lambda}}$ [De Giorgi, Colombini, Spagnolo SNS 1979]

$$F'(t) \leq k \left(\lambda\omega\left(\tfrac{1}{\lambda}\right) + \lambda\,\omega\left(\tfrac{1}{\lambda}\right) \right) F(t)$$

$$= 2k\,\lambda\omega\left(\tfrac{1}{\lambda}\right) F(t)$$

$$F(t) \leq F(0) \exp\left\{ 2k\,\lambda\omega\left(\tfrac{1}{\lambda}\right) t \right\}$$

Example Assume $c(t)$ is α–Hölder ... $\omega(x) = Hx^\alpha$

$$F(t) \leq F(0) \exp\left\{ \text{const } \lambda^{1-\alpha}\, t \right\}$$

better than λ

Proof of (1) and (2)

(1) $c_\varepsilon'(t) = \frac{1}{\varepsilon}\left(c(t+\varepsilon) - c(t)\right)$

$$|c_\varepsilon'(t)| = \frac{1}{\varepsilon}\,|c(t+\varepsilon) - c(t)| \leq \frac{\omega(\varepsilon)}{\varepsilon}$$

(2) $$|c_\varepsilon(t) - c(t)| = \left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} c(s)\,ds - c(t)\right|$$

$$= \left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \left(c(s) - c(t)\right) ds\right|$$

$$\leq \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \underbrace{|c(s) - c(t)|}_{\omega(\varepsilon)}\, ds$$

$$= \omega(\varepsilon).$$

FREIBERG 2017 — LECTURE 3

Note Title 23/05/2017

Equations with time-dependent coefficients: classical theory

(Rephrasing of De Giorgi – Colombini – Spagnolo [Ann. SNS 1979] in modern language)

$$\ddot{u} + \underbrace{\delta(t)}_{\geq 0}\dot{u} + c(t)Au = 0$$

$\delta \equiv 0$ is ok

$A : D(A) \to H$ with $D(A) \subseteq H$

As in Lecture 1 $\exists \{e_k\} \subseteq H$ orth. syst. $\exists \{\lambda_k\} \subseteq [0,+\infty)$ s.t.

$$Ae_k = \lambda_k^2 e_k$$

<u>Fact</u> Any $u : [0,T] \to H$ can be identified with the sequence of components

$$u_k(t) := \langle u(t), e_k \rangle$$

Original equ. is equivalent to system of ODEs

$$\boxed{\ddot{u}_k + \delta(t)\dot{u}_k + \lambda_k^2 c(t) u_k = 0}$$

"Space regularity" of u $\rightsquigarrow$ $u(t) \in D(A^\alpha)$

$\in D(A^{-\alpha})$

$\in \mathcal{G}_{\varphi,r,\alpha}(A)$

$\in \mathcal{G}_{-\psi,R,\alpha}(A)$

Sufficient condition for $u \in C^0([0,T], D(A^\alpha))$ is that

$$\sum_{k \geq 0} (1+\lambda_k)^{4\alpha} \sup\{|u_k(t)|^2 : t \in [0,T]\}$$

and $u_k(t)$ continuous for every $k \in \mathbb{N}$

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① $\boxed{\text{Easy case}}$ Assume $c(t) \in C^1([0,T])$

$c(t) \geq \mu_1 > 0$ (S.H.)

Then the problem

$$\ddot{u} + \underbrace{\boxed{\delta(t)}}_{\geq 0} \dot{u} + c(t) A u = 0$$

$$u(0) = u_0 \qquad \dot{u}(0) = u_1$$

is well-posed in $D(A^{1/2}) \times H$ (or more generally $D(A^{\alpha+1/2}) \times D(A^\alpha)$)

$$(u_0, u_1) \in D(A^{1/2}) \times H \Rightarrow$$

$$u \in C^0([0,T], D(A^{1/2})) \cap C^1([0,T], H)$$

$\boxed{\text{Proof.}}$ Lecture 2 $\Rightarrow \exists M$ indep. of k s.t.

$$\dot{u}_k(t)^2 + \lambda_k^2 u_k(t)^2 \leq \underbrace{(\dot{u}_{1k}^2 + \lambda_k^2 u_{0k}^2)}_{\substack{\text{the series Bounded if} \\ (u_0, u_1) \in D(A^{1/2}) \times H}} \exp(Mt)$$

$\Rightarrow$ supremum of LHS in $[0,T]$ gives a converging series.

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(2) $\boxed{\text{WORST CASE}}$ $c(t) \in L^1((0,T))$
NO SIGN CONDITION

Then the problem is well-posed in

$$\mathcal{G}_{-\psi,R,1/2}(A) \times \mathcal{G}_{-\psi,R,0}(A)$$

provided that $\psi(\sigma) = \sigma$ (analytic ultradistribution)

More precisely, assume that $(u_0,u_1) \in \mathcal{G}_{-\psi,R_0,1/2} \times \mathcal{G}_{-\psi,R_0,0}$
then there exists

$$R : [0,T] \to (0,+\infty)$$

with

$$R(0) = R_0$$

such that

$$u(t) \in \mathcal{G}_{-\psi,R(t),1/2} \times \mathcal{G}_{-\psi,R(t),0}$$

Proof. We proved that

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \le (u_{1k}^2 + \lambda_k^2 u_{0k}^2) \exp\left(\lambda_k t + \lambda_k \int_0^t |c(s)|\,ds\right)$$

$$= \underbrace{(\dots)\exp(-2\psi(\lambda_k)R_0)}_{\substack{\text{convergent series by}\\ \text{assumption on } (u_0,u_1)}} \underbrace{\exp\left(2\psi(\lambda_k)R_0 + \lambda_k t + \lambda_k C_1(t)\right)}_{\exp\left(\lambda_k \underbrace{(2R_0 + t + C_1(t))}_{R(t)}\right)}$$

If we want a convergent series we need to multiply by

$$\exp(-\lambda_k R(t))$$

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Remark Cauchy–Kow theory.

③ $c(t) \geq \mu_1 > 0 \qquad c(t) \in L^1([0,T])$

The pbm. is well posed in $\mathcal{G}_{\varphi,r,1/2} \times \mathcal{G}_{\varphi,r,0}$
with $\varphi(\lambda) = \lambda$

Proof. Lecture 2 $\rightsquigarrow$ $\exists\, M_1$ $\forall \varepsilon > 0$ $\exists\, M_\varepsilon$ s.t.

$$\underbrace{\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)}_{LHS_k} \leq (u_{1k}^2 + \lambda_k^2 u_{0k}^2) \exp\{M_1 \varepsilon \lambda_k t + M_\varepsilon\}$$

Choose $\varepsilon > 0$ small enough so that $M_1 \varepsilon T = r_1 < r_0$

radius of initial conditions

$$\sum LHS_k \exp((r_0 - r_1)\lambda_k) \leq \sum (u_{1k}^2 + \lambda_k^2 u_{0k}^2) \exp((r_0 - r_1)\lambda_k + M_1 \varepsilon \lambda_k t + M_\varepsilon)$$

$$= \overbrace{\sum (u_{1k}^2 + \lambda_k^2 u_{0k}^2) \exp(r_0 \lambda_k)}^{\text{convergent}} \exp(\underbrace{-r_1 \lambda_k + M_1 \varepsilon \lambda_k t}_{\leq 0} + M_\varepsilon)$$

④–⑤ $c(t) \geq \mu_1 > 0$
$c(t)$ is ω-continuous in $[0,T]$

Assume that $\dfrac{\lambda_k}{\varphi(\lambda_k)}\, \omega\left(\dfrac{1}{\lambda_k}\right)$ is bounded (wrt k)

Then the pbm is well posed

both in $\mathcal{G}_{\varphi,r,1/2} \times \mathcal{G}_{\varphi,r,0}$

and in $\mathcal{G}_{-\varphi,R,1/2} \times \mathcal{G}_{-\varphi,R,0}$

Proof. Lecture 2 $\rightsquigarrow$ $\exists M$ (indep of everything but for μ_1) such that

$$\underbrace{\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)}_{LHS_k} \leq (\dot{u}_{1k}^2 + \lambda_k^2 u_{0k}^2) \exp\{\lambda_k \omega(\tfrac{1}{\lambda_k}) M t\}$$

Consider now initial conditions in $\mathcal{G}_{\varphi, r_0, 1/2} \times \mathcal{G}_{\varphi, r_0, 0}$

$$\sum LHS_k \exp\{+2\varphi(\lambda_k)(r_0 - \underset{\substack{\uparrow \\ \text{decay of the radius}}}{R}t)\} \leq$$

$$\leq \underbrace{\sum (\dot{u}_{1k}^2 + \lambda_k^2 u_{0k}^2) \exp\{2\varphi(\lambda_k) r_0}_{\text{convergent}} \underbrace{- 2\varphi(\lambda_k) Rt + \lambda_k \omega(\tfrac{1}{\lambda_k}) M t\}}_{\text{Hope: choosing } R \text{ so that this is bounded from above}}$$

$$-2\varphi(\lambda_k) R + \lambda_k \omega\left(\frac{1}{\lambda_k}\right) M \leq 0$$

$$\underbrace{\frac{\lambda_k}{\varphi(\lambda_k)} \omega\left(\frac{1}{\lambda_k}\right)}_{\substack{\text{by assumption} \\ \text{this is bounded}}} M \leq 2R$$

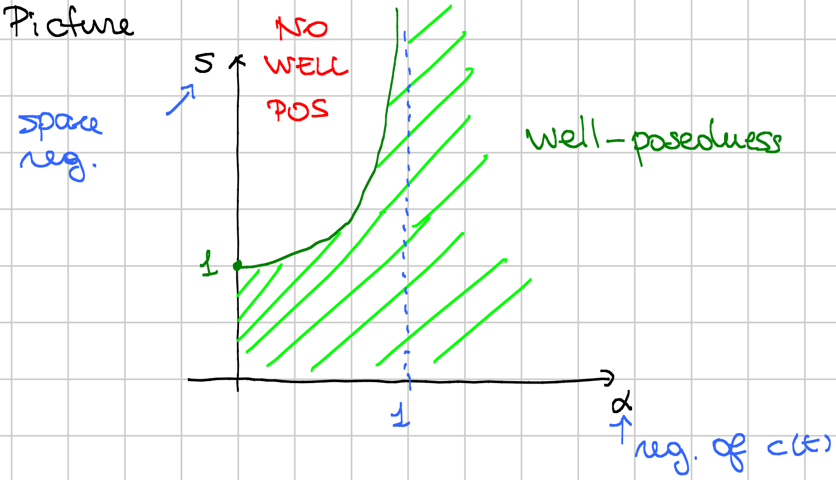
Remark Solution exists as long as $r_0 - Rt > 0$
$\iff t < \frac{r_0}{R}$

Proof of (5) is analogous.

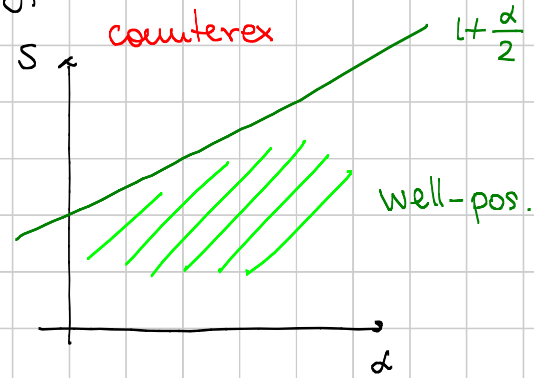
Special case $\omega(\sigma) = H\sigma^\alpha \rightsquigarrow \frac{\lambda_k}{\varphi(\lambda_k)} \frac{1}{\lambda_k^\alpha}$ is bounded

$\varphi(\lambda_k) = \lambda_k^{1-\alpha} \rightsquigarrow$ Gevrey space with $s = \frac{1}{1-\alpha}$

Picture



Weak hyper



FREIBERG 2017 — LECTURE 4

Note Title 23/05/2017

$$\boxed{\ddot{u} + 2\delta A^{\sigma}\dot{u} + Au = 0} \qquad \delta > 0, \ \sigma \geq 0$$

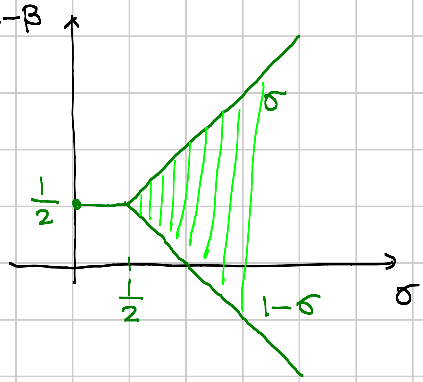
As always we reduce ourselves to

$$\boxed{\ddot{u}_k + 2\delta \lambda_k^{2\sigma} \dot{u}_k + \lambda_k^2 u_k = 0}$$

$\boxed{\text{Result}}$

① Well-posedness in $D(A^{\alpha}) \times D(A^{\beta})$ provided that

$\alpha - \beta \in$



- if $\sigma \in [0, \frac{1}{2}]$, then $\alpha - \beta = \frac{1}{2}$ (standard hyperb. gap)
- if $\sigma > \frac{1}{2}$, then $1 - \sigma \leq \alpha - \beta \leq \sigma$.

<u>Remark</u> Assume $\sigma \leq \frac{1}{2}$, assume $(u_0, u_1) \in D(A^{37}) \times D(A^{14})$
Then for every $t > 0$ it turn out that

$$(u(t), \dot{u}(t)) \in D(A^{14 + \frac{1}{2}}) \times D(A^{14})$$

Assume $\sigma = 4$ and (u_0, u_1) as before. then

$$(u(t), \dot{u}(t)) \in D(A^{18}) \times D(A^{14})$$

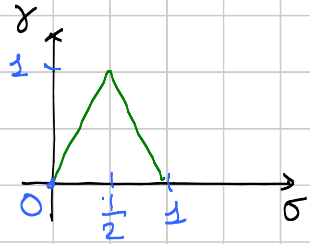
② $u^{(m)} \in D(A^{\beta-(m-1)\sigma})$ for every $m \geq 1$ if $\sigma \geq \frac{1}{2}$

m derivatives wrt time

③ Assume that $\sigma \in (0,1)$ then for every $\boxed{t>0}$

$$(u(t), \dot{u}(t)) \in \mathcal{G}_{\varphi,r,\alpha} \times \mathcal{G}_{\varphi,r,\beta}$$

with $\varphi(x) = x^{\gamma}$ with



④ Assume that $\sigma > 1$

Then $u^{(m)} \in D(A^{\alpha + m(\sigma-1)})$ $\boxed{\forall t > 0}$

higher derivatives are in better spaces

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"Proof" (sketch of)

$$\ddot{u}_k + 2\delta \lambda_k^{2\sigma} \dot{u}_k + \lambda_k^2 u_k = 0$$

Char equ : $x^2 + 2\delta\lambda_k^{2\sigma} x + \lambda_k^2 = 0$

$$\boxed{x_{1,2} := -\delta\lambda_k^{2\sigma} \pm \sqrt{\delta^2 \lambda_k^{4\sigma} - \lambda_k^2}}$$

Case $\sigma > \frac{1}{2}$ or $\sigma = \frac{1}{2}$ and $\delta > 1$ $\rightsquigarrow$ 2 real solutions (diff.)

$$x_1 \sim -2\delta\lambda_k^{2\sigma}$$

$$x_2 = \frac{\lambda_k^2}{x_1} \sim -\frac{1}{2\delta} \lambda_k^{2-2\sigma}$$

$u_k(t)$ is the sum of four terms ...

$$u_k(t) = c_1 e^{x_1 t} + c_2 e^{x_2 t}$$

⇝ compute c_1 and c_2 in terms of u_{0k} and u_{1k} and obtain 4 terms.

$$\pm u_{0k} \frac{x_2}{x_1-x_2} e^{+x_1 t} \qquad\qquad \pm u_{0k} \frac{x_1}{x_1-x_2} e^{+x_2 t}$$

$$\sim \underbrace{u_{0k}}_{\alpha} \frac{\lambda_k^{2-2\sigma}}{\lambda_k^{2\sigma}} e^{-\lambda_k^{2\sigma} t} \qquad\qquad \sim \underbrace{u_{0k}}_{\alpha} \frac{\lambda_k^{2\sigma}}{\lambda_k^{2\sigma}} e^{-\lambda_k^{2-2\sigma} t}$$

$$\underbrace{\lambda_k^{2-4\sigma}}_{\text{good}}$$

$$u_{0k} \frac{x_1 x_2}{x_1-x_2} e^{\cdots} \sim \underbrace{u_{0k}}_{\alpha} \frac{\lambda_k^{2\sigma}\,\lambda_k^{2-2\sigma}}{\lambda_k^{2\sigma}}$$

$$\lambda_k^{2-2\sigma}$$

exactly the same

$$\pm u_{1k} \frac{1}{x_1-x_2} e^{x_1 t} \qquad\qquad \pm u_{1k} \frac{1}{x_1-x_2} e^{x_2 t}$$

$$\sim \underbrace{u_{1k}}_{\beta} \frac{1}{\lambda_k^{2\sigma}} e^{-\lambda_k^{2\sigma} t} \qquad\qquad \sim \underbrace{u_{1k}}_{\beta} \frac{1}{\lambda_k^{2\sigma}} e^{-\lambda_k^{2-2\sigma} t}$$

$$\frac{1}{\lambda_k^{2\sigma}} \qquad\qquad \frac{1}{\lambda_k^{2\sigma}}$$

$$u_{1k} \frac{x_1}{x_1-x_2} \sim \underbrace{u_{1k}}_{\beta} \frac{\lambda_k^{2\sigma}}{\lambda_k^{2\sigma}} \qquad\qquad u_{1k} \frac{x_2}{x_1-x_2} \sim \underbrace{u_{1k}}_{\beta} \frac{\lambda_k^{2-2\sigma}}{\lambda_k^{2\sigma}}$$

$$\frac{1}{\lambda_k^{4\sigma-2}}$$

The space where u lies is the worst between

α and $\beta+\sigma$ ⇝ $\beta+\sigma \geq \alpha$ ⇝ $\boxed{\alpha-\beta \leq \sigma}$

The space where u_i lies is the worst between

$\alpha + \sigma - 1$ β $\underbrace{\beta + 2\sigma - 1}_{\text{too good}}$

$\rightsquigarrow \quad \beta \leq \alpha + \sigma - 1 \quad \rightsquigarrow \quad \boxed{\alpha - \beta \geq 1 - \sigma}$

FREIBERG 2017 – LECTURE 5

Note Title 24/05/2017

$$\boxed{u'' + 2\delta A^{\sigma} u' + c(t) A u = 0}$$

Main issue: competition between damping and oscillations of $c(t)$

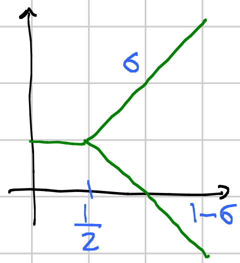
Supercritical case: either $\sigma > \frac{1}{2}$
or $\sigma = \frac{1}{2}$ and $\delta \gg 1$

$\boxed{\text{Main result}}$ Assume supercrit. case
Assume $\underset{\text{weak hyperb}}{0 \le} c(t) \underset{\text{boundedness}}{\le} \mu_2$

No further regularity of $c(t)$

Then dissipation wins, so that we have well posedness in

$D(A^{\alpha}) \times D(A^{\beta})$ with $1-\sigma \le \alpha - \beta \le \sigma$



Heuristics

→ Assume no dissipation at all. Then

$$E(t) \le E(0) \exp\left\{ \lambda \int_0^t (1+|c(s)|)\, ds \right\} \quad \text{(kov. estimate)}$$

$$\le E(0) \exp\left\{ \lambda \left(t + \mu_2 t \right) \right\}$$

(effect of oscillations)

→ Assume $c(t)$ is constant. Then for every component

$$E'(t) \sim -4\delta\lambda^{2\sigma}|\dot{u}|^2 \sim -4\delta\lambda^{2\sigma}E(t)$$

$$E(t) \leq E(0)\exp\{-4\delta\lambda^{2\sigma}t\}$$

(effect of dissipation)

→ Assume we have oscillations and dissipation.
Assume we can add the contributions:

$$E(t) \leq E(0)\exp\{\underbrace{-4\delta\lambda^{2\sigma}t}_{\text{dissip}} + \underbrace{\lambda t + \mu_2\lambda t}_{\text{low reg. of } c(t)}\}$$

Minus sign wins:
- if $\sigma > \frac{1}{2}$
- if $\sigma = \frac{1}{2}$ and δ is large enough.

— o — o —

Main tool: σ-correction of usual Kovalevskian energy

$$\boxed{E(t) := (\dot{u} + \delta\lambda^{2\sigma}u)^2 + \delta^2\lambda^{4\sigma}u^2 = \dot{u}^2 + 2\delta^2\lambda^{4\sigma}u^2 + 2\delta\lambda^{2\sigma}u\dot{u}}$$

Proof. Let us consider components $u_k(t) := \langle u(t), e_k \rangle$ which are solutions to

$$\ddot{u}_k + 2\delta\lambda_k^{2\sigma}\dot{u}_k + c(t)\lambda_k^2 u_k = 0$$

and consider $E_k(t) :=$ energy of u_k defined as above.

$$E' = 2\dot{u}\ddot{u} + 4\delta^2\lambda^{4\sigma}u\dot{u} + 2\delta\lambda^{2\sigma}\dot{u}^2 + 2\delta\lambda^{2\sigma}u\ddot{u}$$

$$= 2\dot{u}\,(\underbrace{-2\delta\lambda^{2\sigma}\dot{u}} - \lambda^2 c(t)u) + \cancel{4\delta^2\lambda^{4\sigma}u\dot{u}} + \underbrace{2\delta\lambda^{2\sigma}\dot{u}^2}$$

$$+ 2\delta\lambda^{2\sigma}u\,(\cancel{-2\delta\lambda^{2\sigma}\dot{u}} - \lambda^2 c(t)u)$$

$$= -2\delta\lambda^{2\sigma}\dot{u}^2 - 2\delta\lambda^{2+2\sigma}c(t)u^2 - 2\lambda^2 c(t)u\dot{u}$$

Hope: $E'(t) \leq 0$

$$\delta\lambda^{2\sigma}\underbrace{\dot{u}^2}_{x^2} + \delta\lambda^{2+2\sigma}c(t)\underbrace{u^2}_{y^2} + \lambda^2 c(t)\overbrace{u\dot{u}}^{xy} \geq 0$$

General fact in lin. algebra:

$$\boxed{\begin{array}{c} Ax^2 + By^2 + Cxy \geq 0 \\ \text{for every } (x,y) \in \mathbb{R}^2 \\ \Updownarrow \\ 4AB \geq C^2,\quad A \geq 0,\ B \geq 0 \end{array}}$$

$$E'(t) \leq 0 \iff 4\delta^2\lambda^{2+4\sigma}\cancel{c(t)} \geq \lambda^4 c(t)^{\cancel{2}}$$

$$\iff 4\delta^2 \geq \lambda^{2-4\sigma}c(t)$$

ok if $\sigma > \frac{1}{2}$ or $\sigma = \frac{1}{2}$ $4\delta^2 \geq \mu_2$

In these cases we obtain $E_k(t) \leq E_k(0)$

$$\Rightarrow \lambda_k^{4\sigma}u_k^2(t) \leq M\left(|u_{1k}|^2 + \lambda_k^{4\sigma}u_{0k}^2\right)$$

Good estimate for $u_k(t)$.

$$\lambda_k^{4\alpha} u_k^2(t) \le M\left(\lambda_k^{4\alpha} \frac{u_{1k}^2}{\lambda_k^{4\sigma}} + \lambda_k^{4\alpha} u_{0k}^2\right)$$

$$= M\left(\frac{\lambda_k^{4\beta} u_{1k}^2}{\underbrace{\lambda_k^{4\sigma+4\beta-4\alpha}}_{\text{exponent is } \ge 0}} + \lambda_k^{4\alpha} u_{0k}^2\right)$$

$\Leftrightarrow \quad 4\sigma + 4\beta - 4\alpha \ge 0$

$\Leftrightarrow \quad \alpha - \beta \le \sigma$

If $\alpha - \beta \le \sigma$ we obtain

$$\sum_{k\ge 0} \lambda_k^{4\alpha} u_k^2(t) \le M \underbrace{\sum_{k\ge 0}\left(\lambda_k^{4\beta} u_{1k}^2 + \lambda_k^{4\alpha} u_{0k}^2\right)}$$

convergent because $(u_0, u_1) \in$
$\in D(A^\alpha) \times D(A^\beta)$

$\Rightarrow u(t) \in D(A^\alpha)$.

— o — o —

Estimate for $\dot{u}_k$

$$\ddot{u}_k + 2\delta \lambda_k^{2\sigma} \dot{u}_k = -\lambda_k^2 c(t) u_k$$

See this as a first order linear equ. in $\dot{u}_k$ $\rightsquigarrow$

$$\dot{u}_k(t) = u_{1k} e^{-2\delta\lambda_k^{2\sigma} t} - e^{-2\delta\lambda_k^{2\sigma} t} \int_0^t \lambda_k^2 \underbrace{c(s)}_{\le \mu_2} u_k(s) e^{2\delta\lambda_k^{2\sigma} s} ds$$

$$|\dot{u}_k(t)| \le |u_{1k}| + \lambda_k^2 \mu_2 \max_{[0,t]} |u_k(s)|\, e^{-2\delta\lambda_k^{2\sigma} t} \int_0^t e^{2\delta\lambda_k^{2\sigma} s} ds$$

$$\le |u_{1k}| + \lambda_k^2 \mu_2 \max |u_k(s)| \frac{1}{2\delta \lambda_k^{2\sigma}}$$

$$\dot{u}_{k}^{2}(t) \leq M\left(|u_{1k}|^2 + \lambda_k^4 \max |u_k(s)|^2 \frac{1}{\lambda_k^{4\sigma}}\right)$$

use previous estimate

$$\leq M\left(|u_{1k}|^2 + \frac{\lambda_k^4}{\lambda_k^{4\sigma}}\left(\frac{u_{1k}^2}{\lambda_k^{4\sigma}} + u_{0k}^2\right)\right)$$

$$\leq M\left(|u_{1k}|^2 + \frac{\lambda_k^4}{\lambda_k^{4\sigma}} u_{0k}^2\right)$$

$$\lambda_k^{4\beta} \dot{u}_k^2(t) \leq M\left(\lambda_k^{4\beta} u_{1k}^2 + \boxed{\frac{\lambda_k^{4\beta}\lambda_k^4}{\lambda_k^{4\sigma}\lambda_k^{4\alpha}}} \lambda_k^{4\alpha} u_{0k}^2\right)$$

$$\frac{1}{\lambda_k^{4\sigma+4\alpha-4\beta-4}}$$

exponent $\geq 0 \iff \alpha - \beta - 1 + \sigma \geq 0$
$\iff \alpha - \beta \geq 1 - \sigma$

As before we conclude that $\dot{u}(t) \in D(A^\beta)$.

— o — o —

<u>Remark 1</u> We obtain also boundedness of high frequencies of u in

$$D(A^\alpha) \times D(A^\beta)$$

This is because $E_k'(t) \leq 0$ for k large

<u>Remark</u> If $\sigma \in [\frac{1}{2}, 1)$, then $u(t)$ has Gevrey regularity for $t > 0$
How much regularity? $\varphi(\lambda) = \lambda^{2(1-\sigma)}$

Main idea for Gevrey regularity: improve

$$E_k'(t) \leq 0$$

to

$$E_k'(t) \leq -R \underbrace{c(t)}_{\text{needed}} \lambda_k^{2(1-\sigma)} E_k(t)$$

$\rightsquigarrow$ $u(t) \in \mathcal{G}_{\varphi, R\,C(t), 1/2}$ with $C(t) := \int_0^t c(s)\,ds$

— o — o —

FREIBERG 2017 — LECTURE 6

Note Title 24/05/2017

$$\ddot{u} + 2\delta A^{\sigma} \dot{u} + c(t) A u = 0$$

Subcritical case : $\sigma \in [0, \frac{1}{2}]$

$\boxed{\text{Statement}}$ Assume strict hyperb $0 < \mu_1 \leq c(t) \leq \mu_2$

Assume $c(t)$ is ω – continuous

Assume $\sigma \leq \frac{1}{2}$

Assume

$$4\delta^2 \mu_1 \geq L^2 + 2\delta L \qquad (*)$$

where

$$L := \limsup_{\varepsilon \to 0^+} \frac{\omega(\varepsilon)}{\varepsilon^{1-2\sigma}}$$

Then the problem is well posed in $D(A^{1/2}) \times H$ (gap $= \frac{1}{2}$)

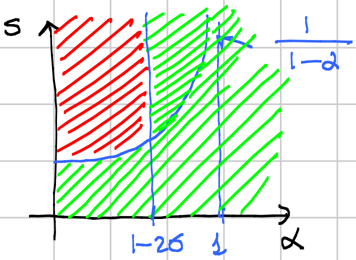
Special case Assume $\omega(\varepsilon) = M \varepsilon^{\alpha}$ for some $\alpha \in (0,1)$

$$L := \lim_{\varepsilon \to 0^+} \frac{M \varepsilon^{\alpha}}{\varepsilon^{1-2\sigma}}$$

- If $\alpha > 1 - 2\sigma$, then $L = 0$, and $(*)$ is satisfied
  $\sigma > \frac{1-\alpha}{2}$

- If $\alpha = 1 - 2\sigma$, then $L = M$ has to be small enough

- If $\alpha < 1 - 2\sigma$, then $L = +\infty$ and $(*)$ is not satisfied
  In that case there are counterexamples (see lecture 8)



Heuristics → Effect of dissipation

$$E(t) \leq E(0) \exp(-4\delta\lambda^{2\sigma} t)$$

→ Effect of low regularity (see lecture 2)

$$E(t) \leq E(0) \exp\left(\lambda\omega\left(\tfrac{1}{\lambda}\right) t\right)$$

→ Superposition

$$E(t) \leq E(0) \exp\left(-4\delta\lambda^{2\sigma} t + \lambda\omega\left(\tfrac{1}{\lambda}\right)t\right)$$

Negative wins $\iff 4\delta\lambda^{2\sigma} \gg \lambda\omega\left(\frac{1}{\lambda}\right)$

$\iff \lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right)$ is small for large λ

$\underset{\lambda = \frac{1}{\varepsilon}}{\iff} \dfrac{\omega(\varepsilon)}{\varepsilon^{1-2\sigma}} \ll 1$ for $\varepsilon \ll 1$.

— o — o —

$\boxed{\text{MAIN TOOL}}$ σ correction of hyperb. energy

$$F(t) := \underbrace{(\dot{u} + \delta\lambda^{2\sigma} u)^2 + \delta^2\lambda^{4\sigma} u^2}_{\text{hyp. energy}} + \lambda^2 \gamma(t)\, u^2$$

has to be choosed (will be $\gamma(t) = c_\varepsilon(t)$ for some ε)

Hope: compute $F_k(t)$ (= energy of k-th component) and prove that

$$F_k'(t) \leq 0$$

$\boxed{\text{Proof.}}$

$$F'(t) = -2\delta\lambda^{2\sigma}\dot{u}^2 - 2\delta\lambda^{2+2\sigma}c(t)u^2 - 2\lambda^2 c(t)u\dot{u}$$

$$+\lambda^2\dot{\gamma}(t)u^2 + 2\lambda^2\gamma(t)u\dot{u}$$

$$= -\underbrace{2\delta\lambda^{2\sigma}}_{A}\dot{u}^2 - 2\underbrace{\left(\delta\lambda^{2+2\sigma}c(t) - \tfrac{1}{2}\lambda^2\dot{\gamma}(t)\right)}_{B}u^2$$

$$-2\underbrace{\lambda^2\left(c(t)-\gamma(t)\right)}_{C}u\dot{u}$$

As before we have a quadratic form in $u\ \dot{u}$

$$4AB \geq C^2$$

$$4\delta\lambda^{2\sigma}\left(\delta\lambda^{2+2\sigma}c(t) - \tfrac{1}{2}\lambda^2\dot{\gamma}(t)\right) \geq \lambda^4\left(c(t)-\gamma(t)\right)^2$$

Choose $\gamma(t) := c_\varepsilon(t) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} c(s)\,ds$

(i) $|\dot{c}_\varepsilon(t)| \leq \frac{\omega(\varepsilon)}{\varepsilon}$ (ii) $|c(t)-c_\varepsilon(t)| \leq \omega(\varepsilon)$

It is enough to prove that

$$4\delta\lambda^{2\sigma}\Big(\delta\lambda^{2+2\sigma}\underset{\substack{\uparrow \\ \text{strict} \\ \text{hyperb.}}}{\mu_1} - \tfrac{1}{2}\lambda^2\frac{\omega(\varepsilon)}{\varepsilon}\Big) \geq \lambda^4\omega(\varepsilon)^2$$

Take $\varepsilon = \frac{1}{\lambda}$ and obtain

$$4\delta^2\mu_1\lambda^{2+4\sigma} \geq 2\delta\lambda^{2\sigma+3}\omega\left(\tfrac{1}{\lambda}\right) + \lambda^4\omega\left(\tfrac{1}{\lambda}\right)^2$$

$$4\delta^2\mu_1 \geq \underbrace{2\delta\lambda^{1-2\sigma}\omega\left(\tfrac{1}{\lambda}\right)}_{L} + \underbrace{\lambda^{2-4\sigma}\omega\left(\tfrac{1}{\lambda}\right)^2}_{L^2}$$

This is condition $(*)$

Therefore

$$F_k'(t) \leq 0 \qquad \text{for } k \text{ large (large frequencies)}$$

$$\Rightarrow \quad F_k(t) \leq F_k(0)$$

$$\Rightarrow \quad \lambda_k^2 \, u_k^2(t) \leq M_1 F_k(0) \leq M_2 \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2 \right)$$

$$\dot{u}_k^2(t) \leq M_3 F_k(0) \leq M_4 \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2 \right)$$

As before, this leads to $(u(t), \dot{u}(t)) \in D(A^{1/2}) \times H$

$$(u_0, u_1) \in D(A^{\alpha + \frac{1}{2}}) \times D(A^{\alpha}) \Rightarrow (u(t), \dot{u}(t)) \in \text{same space.}$$

—o—o—

Remark 1 Again boundedness of high frequencies

Remark 2 Again Gevrey regularity with $\varphi(\lambda) = \lambda^{2\sigma}$

Need to prove that

$$F_k'(t) \leq -\nu \lambda_k^{2\sigma} F_k(t)$$

no $c(t)$ due to strict hyperbolicity

FREIBERG 2017 – LECTURE 7

Note Title 26/05/2017

Counterexamples

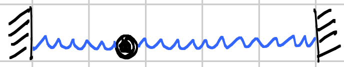
Basic estimate   ODE $\ddot{u} + \lambda^2 c(t) u = 0$

$$E(t) \le E(0) \exp\{ M \lambda t \}$$

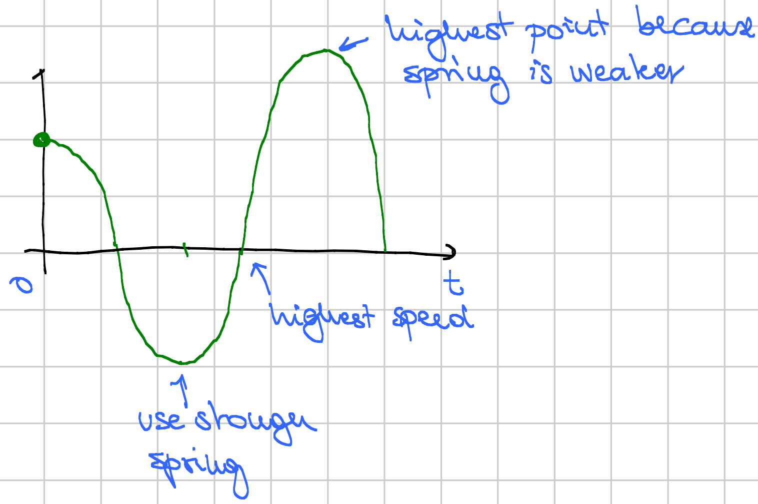
$$E = \dot{u}^2 + \lambda^2 u^2$$

$\boxed{\text{Basic tool}}$ Showing that previous estimate is optimal: finding $c(t)$ and $u(t)$ for which we have equality.

Heuristics $\ddot{u} + \lambda^2 u = 0$ $u(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t)$



If we change the springs with time, then energy can grow exponentially.



Basic idea:

- use weaker spring when moving in the opposite dir. with respect to the origin
- use stronger spring toward the origin.

Idea Look for a solution of type

$$u(t) = \sin(\lambda t)\, e^{a(t)}$$ final possibly increasing with time

$$\dot{u} = \lambda \cos(\lambda t)\, e^{a(t)} + \sin(\lambda t)\, \dot{a}(t)\, e^{a(t)}$$

$$\ddot{u} = -\lambda^2 \sin(\lambda t)\, e^{a(t)} + 2\lambda \cos(\lambda t)\, \dot{a}(t)\, e^{a(t)} + \sin(\lambda t)\, \ddot{a}(t)\, e^{a(t)} + \sin(\lambda t)\, \dot{a}(t)^2\, e^{a(t)}$$

$$\ddot{u} + c(t)\lambda^2 u(t) = 0 \iff$$

$$\underbrace{-\lambda^2 \sin(\lambda t)} + 2\lambda\cos(\lambda t)\,\dot{a}(t) + \underbrace{\sin(\lambda t)\,\ddot{a}(t)} + \underbrace{\sin(\lambda t)\,\dot{a}(t)^2} + \underbrace{c(t)\lambda^2 \sin(\lambda t)} = 0$$

$$\sin(\lambda t)\left\{-\lambda^2 + \ddot{a}(t) + \dot{a}(t)^2 + c(t)\lambda^2\right\} + 2\lambda\cos(\lambda t)\,\dot{a}(t) = 0$$

Reasonable choice: choose $\dot{a}(t) = \sin(\lambda t)\cdot \text{something}$
$= \sin(\lambda t)\cdot \sin(\lambda t)\, \alpha$ ← constant
$= \alpha \sin^2(\lambda t)$

$$\ddot{a}(t) = 2\alpha \sin(\lambda t)\cos(\lambda t)\cdot\lambda$$

$$\rightsquigarrow \; -\lambda^2 + \overset{4}{\cancel{2}}\alpha\lambda \sin(\lambda t)\cos(\lambda t) + \alpha^2 \sin^4(\lambda t) + c(t)\lambda^2 + \cancel{2\alpha\lambda\cos(\lambda t)\sin(\lambda t)} = 0$$

Choose $\alpha = \varepsilon\lambda$

$$\rightsquigarrow \; -\cancel{\lambda^2} + 4\varepsilon\cancel{\lambda^2}\sin(\lambda t)\cos(\lambda t) + \varepsilon^2\cancel{\lambda^2}\sin^4(\lambda t) + c(t)\cancel{\lambda^2} = 0$$

$$\rightsquigarrow \; c(t) = 1 - 4\varepsilon\sin(\lambda t)\cos(\lambda t) - \varepsilon^2\sin^4(\lambda t)$$

If ε is small enough, then $\frac{1}{2} \leq c(t) \leq \frac{3}{2}$

$\rightsquigarrow \dot{a}(t) = \varepsilon \lambda \sin^2(\lambda t)$

$\rightsquigarrow a(t) = \frac{1}{2} \varepsilon \lambda t - \frac{1}{4} \varepsilon \sin(2\lambda t)$

$$\rightsquigarrow u(t) = \sin(\lambda t) \exp\Big\{ \underbrace{\tfrac{1}{2}\varepsilon\lambda t}_{\substack{\text{grows as } \lambda t \\ \text{when } t \to +\infty}} - \tfrac{1}{4}\varepsilon \sin(2\lambda t) \Big\}$$

Remark $\rightarrow$ $u(t)$ oscillates with "period" $\frac{2\pi}{\lambda}$

$\rightarrow$ $c(t)$ oscillates with period $\frac{\pi}{\lambda}$ $\frac{1}{2}$

—o—o—

Consider now a damped equ

$$\boxed{\ddot{u} + 2\delta \lambda^{2\sigma} \dot{u} + \lambda^2 c(t) u = 0}$$

As before

$$E(t) \leq E(0) \exp\{ M \lambda t \}$$

Question: is this optimal?

Let us look for a solution of the form

$$u(t) = \sin(\lambda t)\, e^{a(t)}$$

Compute $\dot{u}$ and $\ddot{u}$ as before:

$$\dot u = \lambda \cos(\lambda t)\, e^{a(t)} + \sin(\lambda t)\, \dot a(t)\, e^{a(t)}$$

$$\ddot u = -\lambda^2 \sin(\lambda t)\, e^{a(t)} + 2\lambda \cos(\lambda t)\, \dot a(t)\, e^{a(t)} + \sin(\lambda t)\, \ddot a(t)\, e^{a(t)} + \sin(\lambda t)\, \dot a(t)^2 e^{a(t)}$$

$$\ddot u + 2\delta \lambda^{2\delta} \dot u + \lambda^2 c(t)\, u = 0 \iff$$

$$-\lambda^2 \sin(\,) + 2\lambda \cos(\lambda t)\, \dot a + \sin(\,)\, \ddot a + \sin(\,)\, \dot a^2 +$$

$$2\delta\lambda^{2\delta+1} \cos(\,) + 2\delta\lambda^{2\delta} \sin(\,)\, \dot a + \lambda^2 c(t) \sin(\,) = 0$$

$$\sin(\,)\left\{-\lambda^2 + \ddot a + \dot a^2 + 2\delta\lambda^{2\delta}\dot a + \lambda^2 c(t)\right\} +$$

$$+\cos(\,)\left\{2\lambda\dot a + 2\delta\lambda^{2\delta+1}\right\} = 0$$

Reasonable choice:

$$2\lambda \dot a(t) = -2\delta\lambda^{2\delta+1} + \sin(\lambda t)\cdot \varepsilon\lambda^2 \sin(\lambda t)$$

namely $\dot a = -\delta\lambda^{2\delta} + \frac{1}{2}\varepsilon\lambda \sin^2(\lambda t)$

$$\rightsquigarrow \ddot a = \varepsilon\lambda^2 \sin(\lambda t)\cos(\lambda t)$$

$$\rightsquigarrow \cancel{\sin}(\,)\Big\{\underbrace{-\cancel{\lambda^2}}_{} + \underbrace{\varepsilon\cancel{\lambda^2}\sin(\,)\cos(\,)}_{} + \delta^2\lambda^{4\delta-2} + \frac{1}{4}\varepsilon^2\cancel{\lambda^2}\sin^4(\,) +$$

$$-\delta\lambda^{2\delta+\cancel{1}-1}\varepsilon\sin^2(\,) - 2\delta^2\lambda^{4\delta-2} + \underbrace{\delta\varepsilon\lambda\sin^2(\,)}_{} + \cancel{\lambda^2} c(t)\Big\}$$

$$+\underbrace{\cos(\,)\ \varepsilon\cancel{\lambda^2}\cancel{\sin(\,)}\sin(\,)}_{} = 0$$

$$\rightsquigarrow c(t) = 1 - 2\varepsilon \sin(\lambda t)\cos(\lambda t) + \varepsilon\delta\lambda^{2\sigma-1}\sin^2(\lambda t)$$
$$+ \delta\lambda^{2\sigma-1}\varepsilon\sin^2(\lambda t) + \delta^2\lambda^{4\sigma-2}$$

If ε is small enough, then $\frac{1}{2} \leq c(t) \leq \frac{3}{2}$

$4\sigma - 2 \leq 0 \iff \sigma \leq \frac{1}{2}$

$$a(t) = \underbrace{(2\varepsilon\lambda - \delta\lambda^{2\sigma})t}_{\uparrow} - \varepsilon\sin(2\lambda t)$$

if $2\sigma < 1$ $(\sigma < \frac{1}{2})$

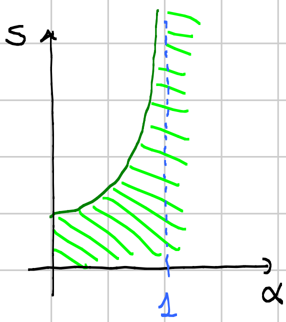
or $2\sigma = 1$ and $\delta \ll 1$

then $a(t)$ grows as λt

FREIBERG 2017 — LECTURE 8

Note Title 26/05/2017

$$\boxed{\ddot{u} + c(t)\,Au = 0}$$



Well posedness. Assume

- $0 < \mu_1 \leq c(t) \leq \mu_2$
- $c(t)$ is ω-continuous
- $(u_0, u_1) \in \mathcal{G}_{\varphi, r, 1/2} \times \mathcal{G}_{\varphi, r, 0}$

$$\frac{\lambda}{\varphi(\lambda)}\,\omega\left(\frac{1}{\lambda}\right) \quad \text{is bounded}$$

Counterexample

Assume $c(t)$ a above
Assume (u_0, u_1) as above BUT $\lim\limits_{\lambda \to +\infty} \frac{\lambda}{\varphi(\lambda)}\,\omega\left(\frac{1}{\lambda}\right) = +\infty$

Then it may happen that $(u(t), \dot{u}(t))$ is extremely irregular for $t > 0$, namely

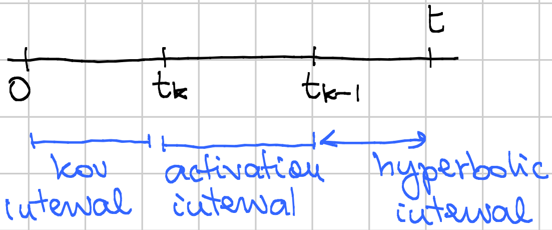
$$(u(t), \dot{u}(t)) \notin \mathcal{G}_{-\psi, r, 1/2} \times \mathcal{G}_{\dots, 0}$$

with $\lim\limits_{\lambda \to +\infty} \frac{\lambda}{\psi(\lambda)}\,\omega\left(\frac{1}{\lambda}\right) = +\infty$

$$\ddot{u}_k + c(t)\,\lambda_k^2\, u_k = 0$$

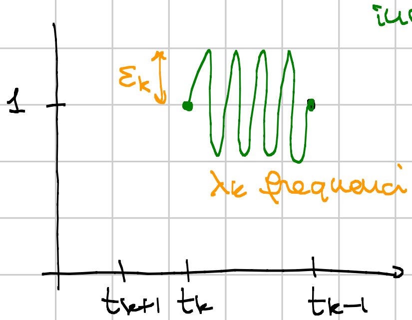
Want to find $c(t)$ in such a way that all components have growing energy.

$\boxed{\text{Strategy}}$



$c(t)$ is defined like this

$c(t)$ designed in order to increase the component u_k



In each interval

- λ_k is the frequency $(\lambda_k \to +\infty)$
- ε_k is the amplitude of oscillations $(\varepsilon_k \to 0)$

Hope: $c(t)$ globally is C^∞ – continuous

Define $u_k(t)$ as the solution which grows a lot in the interval $[t_k, t_{k-1}]$

$$0 \qquad t_k \qquad t_{k-1} \qquad t$$

$E_k(t_k) \rightsquigarrow$ to be choosen

$$E_k(t_{k-1}) \underset{\text{lecture 7}}{\sim} E_k(t_k) \cdot \exp\{\varepsilon_k \lambda_k (t_{k-1} - t_k)\}$$

$$\underset{\text{reasonable if } t_k \ll t_{k-1}}{\sim} E_k(t_k) \cdot \exp\{\varepsilon_k \lambda_k t_{k-1}\}$$

$$\boxed{E_k(0) \underset{\text{kovalevskian estimate}}{\leq} E_k(t_k) \exp\{\lambda_k t_k\}}$$

$$\boxed{E_k(t) \underset{\text{hyperbolic estimate}}{\geq} E_k(t_{k-1}) \exp\{-\varepsilon_{k-1} \lambda_{k-1} t\}}$$

Hyperbolic estimate: if $c(t)$ is of class C^1, then the variation of the energy in a interval is less than

$$\exp\left\{\int_{\text{interval}} \frac{|\dot{c}(s)|}{c(s)}\, ds\right\}$$

Final requirements

$\rightarrow$ very regular at time $t=0$

$$\sum_{k \geq 0} E_k(0) \exp\{2r\varphi(\lambda_k)\} < +\infty \qquad \forall r > 0$$

$$\boxed{\sum_{k \geq 0} E_k(0) \exp\{2k\varphi(\lambda_k)\} < +\infty}$$

→ very irregular at time $t>0$

$$\sum_{k\geq 0} E_k(t) \exp\{-2R\psi(\lambda_k)\} = +\infty \qquad \forall R>0$$

$$\boxed{\sum_{k\geq 0} E_k(t) \exp\{-2k\psi(\lambda_k)\} = +\infty}$$

→ $c(t)$ to be ω-continuous ... equivalent to

$$\frac{\varepsilon_k}{\omega(\frac{1}{\lambda_k})} \quad \text{bounded}$$

Conditions are

$$\sum E_k(t_k) \exp\{\lambda_k t_k + 2k\varphi(\lambda_k)\} < +\infty$$

$$\sum E_k(t_k) \exp\{-\varepsilon_{k-1}\lambda_{k-1}t + \varepsilon_k\lambda_k t_{k-1} - 2k\psi(\lambda_k)\} = +\infty$$

$$\frac{\varepsilon_k}{\omega(\frac{1}{\lambda_k})} \quad \text{bounded}$$

Set $\quad t_k = \dfrac{2\pi}{\lambda_k} \qquad \varepsilon_k = \omega\left(\dfrac{1}{\lambda_k}\right)$

The first series becomes

$$\sum \underbrace{E_k(t_k)\exp\{2\pi + 2k\varphi(\lambda_k)\}}_{\frac{1}{k^2}} = \sum \exp\{-2\log k\}$$

$$E_k(t_k) = \exp\{-2\log k - 2k\varphi(\lambda_k) - 2\pi\}$$

Guarantees the convergence of the first series

Need the divergence of

$$\sum E_k(t_k)\, \exp\left\{-\varepsilon_{k-1}\lambda_{k-1}t + \varepsilon_k \lambda_k t_{k-1} - 2k\,\psi(\lambda_k)\right\}$$

$$= \sum \exp\Big\{-2\log k - 2k\,\varphi(\lambda_k) - 2\pi - \varepsilon_{k-1}\lambda_{k-1}t - 2k\,\psi(\lambda_k) + \underbrace{+\,\varepsilon_k \lambda_k t_{k-1}}_{\text{this term has to win !!!}}\Big\} = +\infty$$

$$\varepsilon_k \lambda_k t_{k-1} = \omega\left(\tfrac{1}{\lambda_k}\right)\lambda_k \frac{2\pi}{\lambda_{k-1}}$$

$$\omega\left(\tfrac{1}{\lambda_k}\right)\lambda_k \frac{2\pi}{\lambda_{k-1}} \gg \log k$$

$$\boxed{\omega\left(\tfrac{1}{\lambda_k}\right)\lambda_k \frac{2\pi}{\lambda_{k-1}} \gg k\,\varphi(\lambda_k)}$$

$$\omega\left(\tfrac{1}{\lambda_k}\right)\lambda_k \frac{2\pi}{\lambda_{k-1}} \gg k\,\psi(\lambda_k)$$

$$\omega\left(\tfrac{1}{\lambda_k}\right)\lambda_k \frac{2\pi}{\lambda_{k-1}} \gg t\,\lambda_{k-1}\,\omega\left(\tfrac{1}{\lambda_{k-1}}\right)$$

$$\boxed{\frac{\lambda_k}{\varphi(\lambda_k)}\,\omega\left(\tfrac{1}{\lambda_k}\right)} \gg k\,\lambda_{k-1}$$

Unbounded $\Rightarrow$ given λ_{k-1} I can choose λ_k