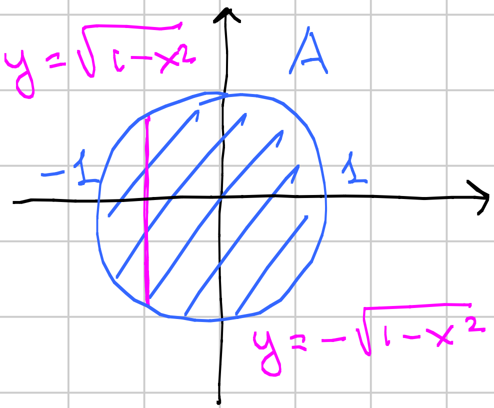
ANALISI 2 CIV

ORA 29

Titolo nota 15/11/2006

Utilizzo coordinate polari / cambi di variabile.

DESCRIZIONE DI INSIEMI IN COORD. POLARI



A è normale rispetto asse x

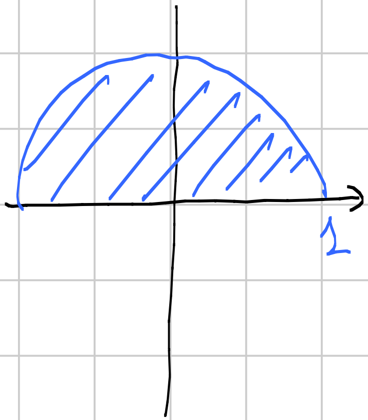
$$A=\left\{(x,y)\in\mathbb{R}^2 : x\in[-1,1],\ -\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\right\}$$

$$\iint_A f(x,y)\,dx\,dy=\int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}f(x,y)\,dy$$

SCOMODO

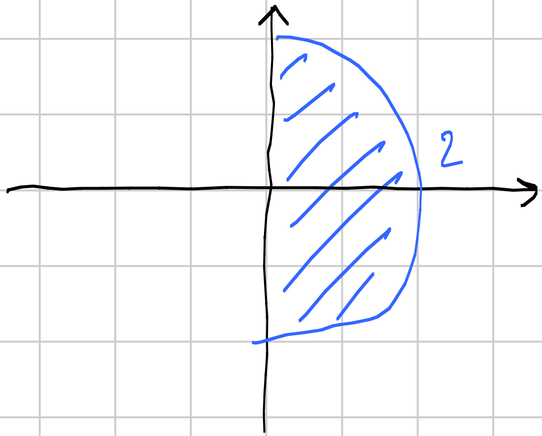
In coordinate polari l'insieme A si descrive come

$$A = \{ (\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2 : \rho \in [0,1], \theta \in [0, 2\pi] \}$$



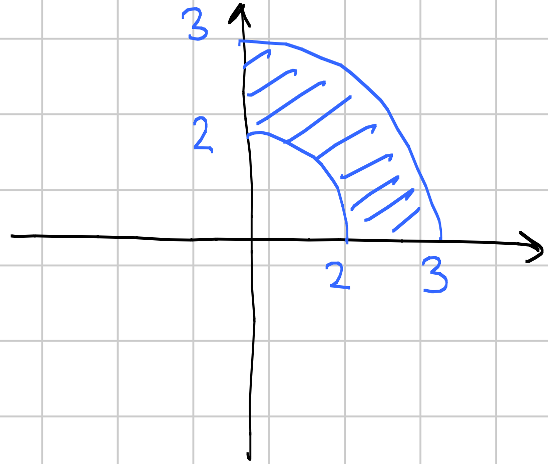
$\rho \in [0,1]$

$\theta \in [0,\pi]$



$\rho \in [0,2]$

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



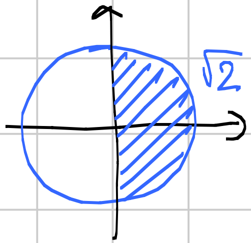
$\rho \in [2,3]$

$\theta \in \left[0, \frac{\pi}{2}\right]$

SI $\theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$

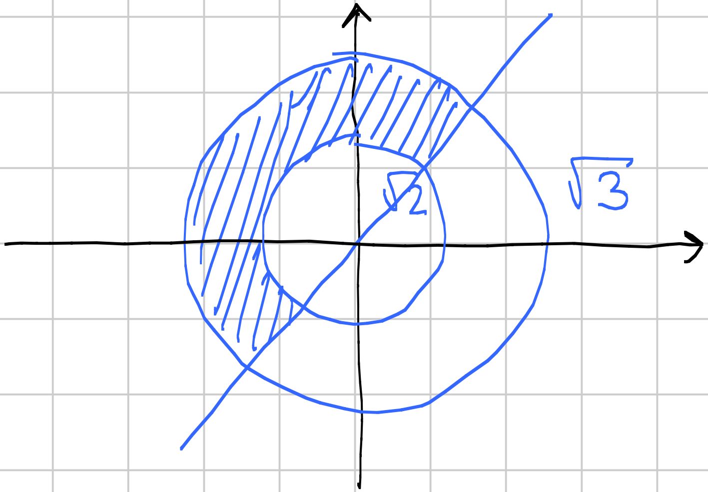
$\boxed{\theta \in \left[\frac{3\pi}{2}, \frac{\pi}{2}\right]}$ NO!

$$A = \{ (x,y) \in \mathbb{R}^2 : \; x \geq 0, \; x^2+y^2 \leq 2 \}$$



$$\rho \in [0, \sqrt{2}], \; \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$B = \{ (x,y) \in \mathbb{R}^2 : \; 2 \leq x^2+y^2 \leq 3, \; y \geq x \}$$



$$\rho \in [\sqrt{2}, \sqrt{3}]$$

$$\theta \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Calcolo di integrali doppi mediante coord. polari

$$\boxed{\iint_A f(x,y)\,dx\,dy = \iint_B g(\rho,\theta)\cdot \boxed{\rho}\,d\rho\,d\theta}$$

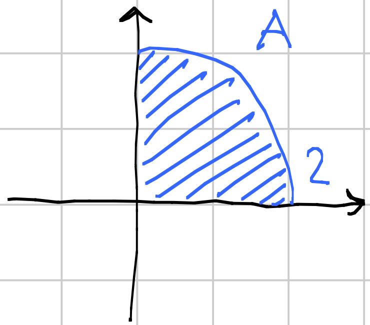
B = l' insieme A descritto mediante coord. polari

$g(\rho,\theta)$ = funzione $f(x,y)$ descritta in coord. polari

$$g(\rho,\theta) = f(\rho\cos\theta, \rho\sin\theta)$$

ρ è il "pagamento"

Esempio 1

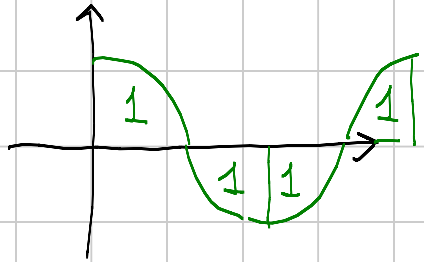


$$\iint_A x \, dx\, dy =$$

$$= \underbrace{\int_0^2 d\rho \int_0^{\pi/2} d\theta}_{\text{Descrizione di A in coord. polari}} \quad \underbrace{\rho\cos\theta}_{\text{x in coord. polari}} \quad \rho \quad =$$

ρ ↑ pagamento

A ; $\rho \in [0,2]$, $\theta \in [0, \frac{\pi}{2}]$



$$= \int_0^2 d\rho \int_0^{\pi/2} d\theta \; \rho^2 \cos\theta = \int_0^2 \rho^2 d\rho \underbrace{\int_0^{\pi/2} \cos\theta \, d\theta}_{=1} = \int_0^2 \rho^2 d\rho = \left[\frac{\rho^3}{3}\right]_0^2$$

$$= \boxed{\frac{8}{3}}$$

Esempio 2 A = cerchio con centro in $(0,0)$ e raggio 3

$$\iint_A (x^2+y^2)\,dx\,dy =$$

In coord. polari $A = \begin{array}{l}\rho \in [0,3] \\ \theta \in [0,2\pi]\end{array}$

$$= \underbrace{\int_0^3 d\rho \int_0^{2\pi} d\theta}_{\text{insieme in coord. polari}} \quad \underbrace{\rho^2}_{\substack{x^2+y^2 \\ \text{in coord. polari}}} \quad \rho$$

↑ pag.

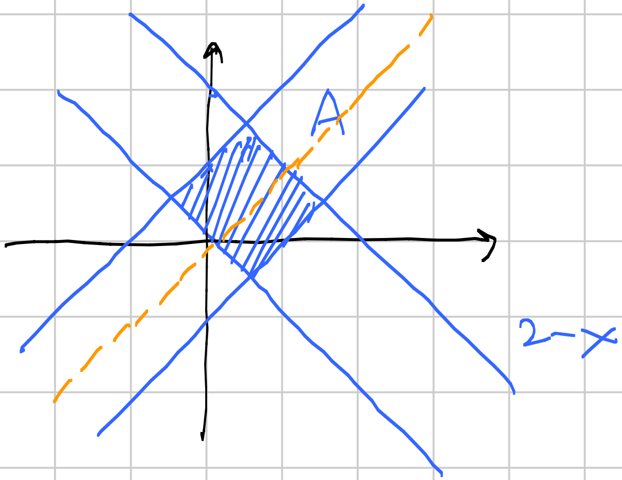
$$= \int_0^3 d\rho \int_0^{2\pi} d\theta \cdot \rho^3$$

$$= \int_0^3 \rho^3 d\rho \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$= 2\pi \int_0^3 \rho^3 d\rho = 2\pi \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=3} = \boxed{\pi \frac{81}{2}}$$

CAMBIO DI VARIABILI IN GENERALE

$$A = \{ (x,y) \in \mathbb{R}^2 : 0 \le x+y \le 2, \ -1 \le x-y \le 1 \}$$



$y \le 2-x$

$y \ge -x$

$x-y \le 1 ; \quad y \ge x-1$

$x-y \ge -1 \quad y \le x+1$

$$0 \le \underset{\shortparallel}{x+y} \le 2 , \quad -1 \le \underset{\shortparallel}{x-y} \le 1$$

$$0 \le u \le 2 \qquad -1 \le v \le 1$$

$$\iint_A (x^2-y^2)\,dx\,dy = \underbrace{\int_0^2 du \int_{-1}^1 dv}_{\text{insieme A descritto nelle variabili u e v}} \ \underbrace{u\,v}_{\substack{x^2-y^2= \\ (x+y)(x-y)}} \ \underbrace{J(u,v)}_{\text{pagamento}}$$

$J(u,v)$ è una funzione di u,v detta DETERMINANTE JACOBIANO e si calcola in 3 tappe

1. RICAVARE
2. MATRICE
3. DETERMINANTE

1. $\boxed{\text{RICAVARE}}$

$$\begin{cases} x+y = u \\ x-y = v \end{cases}$$

$\boxed{\text{Ricavo } x \text{ e } y \text{ in funzione di } u \text{ e } v}$

Sommo: $2x = u+v \Rightarrow x = \dfrac{u+v}{2}$

Sottraggo: $2y = u-v \Rightarrow y = \dfrac{u-v}{2}$

② $\boxed{\text{MATRICE}}$

Alla fine dalla ① mi ritrovo

$$x = \varphi(u,v) \qquad y = \psi(u,v)$$

MATRICE JACOBIANA : $\begin{pmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$

Nell' esempio $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

③ $\boxed{\text{DETERMINANTE}}$

$$J(u,v) = |\det(\text{matrice jacobiana})|$$

Nell' esempio

$$J(u,v) = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

$$\iint_A (x^2-y^2)\,dx\,dy = \underbrace{\int_0^2 du \int_{-1}^1 dv}_{\text{insieme A descritto nelle variabili u e v}} \underbrace{u\,v}_{x^2-y^2=(x+y)(x-y)} \underbrace{\overset{\frac{1}{2}}{\overset{\shortparallel}{J(u,v)}}}_{\text{pagamento}}$$

$$= \frac{1}{2}\int_0^2 du \int_{-1}^1 dv\; u\,v$$

$$= \frac{1}{2}\int_0^2 u\,du \underbrace{\boxed{\int_{-1}^1 v\,dv}}_{0} = 0$$

ANALISI II CN (NUCL) $\boxed{\text{ORA 30}}$

Esempio 1 Usiamo la formula generale per $J(u,v)$ nel caso delle coordinate polari

① Ricavare x e y in funzione di ρ e θ

$$x = \rho\cos\theta \qquad y = \rho\sin\theta$$

② Matrice

$$\begin{pmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$$

③ Determinante

$$J = |\mathrm{Det}| = |\rho\cos^2\theta + \rho\sin^2\theta| = \rho$$

Casi facili di cambio di variabili

* Traslazioni $\quad u = x + a \quad v = y + b \quad$ a, b numeri

Ricavare: $x = u - a$, $y = v - b$

Matrice: $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$ identità

$J(u, v) = 1$

* Dilatazioni sugli assi $\quad u = ax \quad v = by \quad$ a, b numeri > 0

Ricavare $\quad x = \frac{u}{a} \quad y = \frac{v}{b}$

Matrice: $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} =$ matrice diagonale

$$J(u,v) = |\mathrm{Det}| = \frac{1}{ab}$$

* <u>Trasformazioni affini</u> $u = \alpha x + \beta y$, $v = \gamma x + \delta y$

Ricavare: $x = au + bv$ $y = cu + dv$

(a, b, c, d si ricavano in funzione di $\alpha, \beta, \gamma, \delta$)

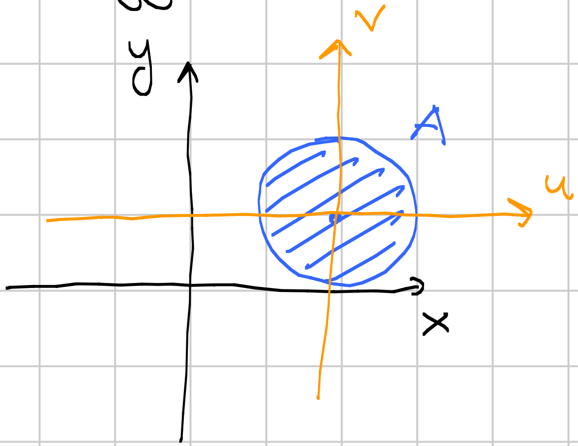
Matrice: $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Matrice indip. da u e v

$$J(u,v) = |ad - bc|$$

Esempio 1

$$\iint_A xy \, dx\, dy$$

$A =$ cerchio con centro in $(2,1)$ e raggio 1



L'equazione di A è

$$(\underset{u}{\underset{\shortparallel}{x-2}})^2 + (\underset{v}{\underset{\shortparallel}{y-1}})^2 \leq 1$$

Nelle coord. (u,v) l'insieme A diventa $u^2+v^2 \leq 1$

$$\iint_A xy \, dx\, dy = \iint_{u^2+v^2\leq 1} \underbrace{(u+2)(v+1)}_{x \,\cdot\, y} \underset{J(u,v)}{\underset{\uparrow}{1}} \, du\, dv$$

$\uparrow$ insieme A nelle coord. u e v

$$= \iint_{u^2+v^2\le 1} (u+2)(v+1)\,du\,dv$$

⊙ uso coord. polari

$u=\rho\cos\theta$

$v=\rho\sin\theta$

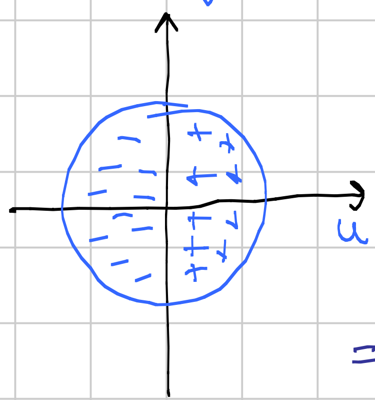
$$= \int_0^1 d\rho \int_0^{2\pi} d\theta\ \underbrace{(\rho\cos\theta+2)}_{u+2}\ \underbrace{(\rho\sin\theta+1)}_{v+1}\ \rho$$

pagamento

⊙ $$\iint (u+2)(v+1) = \underbrace{\iint uv}_{=0} + \underbrace{\iint u}_{=0} + 2\underbrace{\iint v}_{=0} + 2\iint 1$$

svolgo il prodotto

l'integrale della somma è la somma degli integrali



segno di u nel cerchio

$= 2$ Area (cerchio)

$= 2\pi$

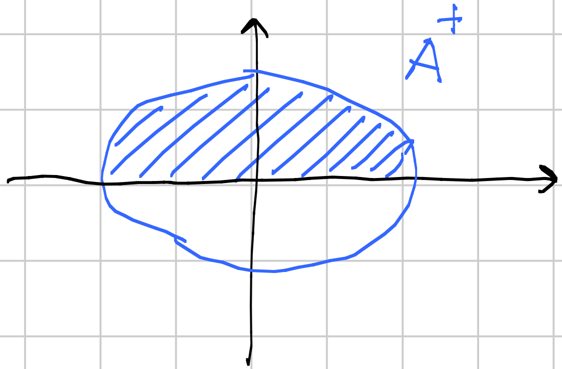
$$\boxed{\iint_A 1\,dx\,dy = \text{Area}(A)}$$

In 1 variabile

$$\int_a^b 1\,dx = b-a$$

= lungh. del segmento

<u>Esempio 2</u> $A = \{(x,y) \in \mathbb{R}^2 : 2x^2+3y^2 \leq 5\}$



$$\iint_A x\,dx\,dy = 0$$

$$\iint_{A^+} x\,dx\,dy = 0 \quad ; \quad \iint_{A^+} y\,dx\,dy$$

di sicuro positivo

$$\iint_{A^+} y\,dx\,dy =$$

$$2x^2+3y^2 \leq 5$$

$$(\underbrace{\sqrt{2}x}_{u})^2 + (\underbrace{\sqrt{3}y}_{v})^2 \leq 5$$

$$= \iint_{\underbrace{u^2+v^2\leq 5}_{\text{insieme in } u \text{ e } v},\ v\geq 0} \underbrace{\frac{v}{\sqrt{3}}}_{y}\ \underbrace{\frac{1}{\sqrt{6}}}_{J(u,v)}\, du\,dv =$$

$v \geq 0$ perché siamo in A^+

Ricavare $\quad u=\sqrt{2}\,x \quad v=\sqrt{3}\,y \quad x=\frac{1}{\sqrt{2}}u \quad y=\frac{1}{\sqrt{3}}v$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad J(u,v) = \frac{1}{\sqrt{6}}$$

$$= \frac{1}{3\sqrt{2}} \iint_{\substack{u^2+v^2 \le 5 \\ v \ge 0}} v \, du \, dv = \frac{1}{3\sqrt{2}} \int_0^{\sqrt{5}} d\rho \int_0^{\pi} d\theta \, \rho \sin\theta \, \rho$$

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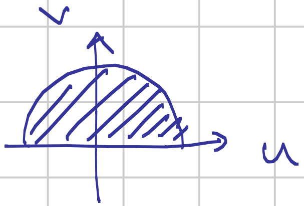


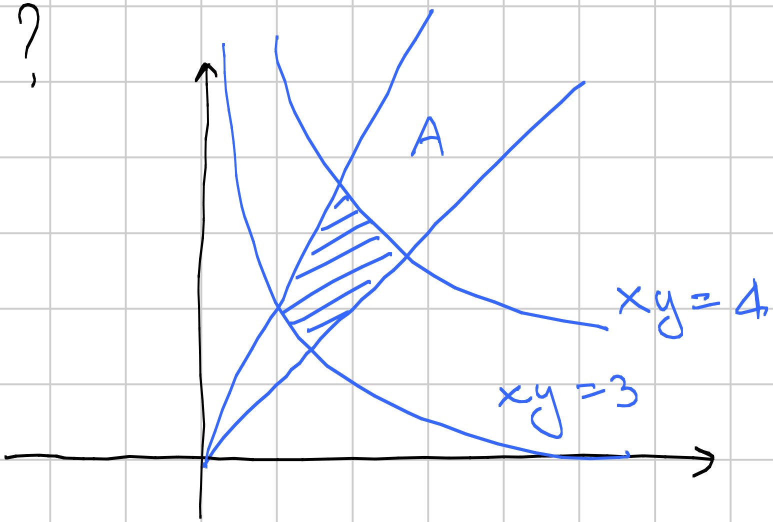
$$= \ldots$$

—o—o—

Esempio 3

$$A = \{ (x,y) \in \mathbb{R}^2 : x \ge 0, \ y \ge 0, \ x \le y \le 2x, \ 3 \le xy \le 4 \}$$

$$\text{Area}(A) = \, ?$$



$$x \le y \le 2x \qquad\qquad 3 \le xy \le 4$$

$$1 \le \underbrace{\frac{y}{x}}_{u} \le 2 \qquad\qquad \underbrace{xy}_{v}$$

$$\text{Area}(A) = \iint_A 1\,dx\,dy = \underbrace{\int_1^2 du \int_3^4 dv}_{\text{insieme in } u \text{ e } v}\ \underbrace{1}\ \cdot \underbrace{\frac{1}{2u}}_{J(u,v)} = \dots$$

① Ricavare $\quad u = \frac{y}{x} \qquad v = xy$

Moltiplico : $y^2 = u \cdot v \;;\quad y = \sqrt{uv} = \sqrt{u}\sqrt{v}$

Divido $\quad x^2 = \frac{v}{u} \qquad x = \frac{\sqrt{v}}{\sqrt{u}}$

(2) Matrice $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} =$

$$\begin{pmatrix} -\frac{\sqrt{v}}{2u\sqrt{u}} & \frac{1}{2\sqrt{u}\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{pmatrix}$$

(3) $|\mathrm{Det}| = \left| -\frac{1}{4u} - \frac{1}{4u} \right| = \frac{1}{2u}$

[Provare a calcolare $\mathrm{Det}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = 2u$]