1. Let us consider the functional

$$F(u) = \int_0^{\pi} \left(\dot{u}^2 - u \sin x\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_0^{\pi} u(x)\, dx = 0$.

(b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u'' = u^7 - x^7, \qquad u(0) = 7, \quad u'(7) = 7.$$

See CdV 2019_4 Ex 1 and Ex 2.

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3. Let us consider, for every real number $\ell > 0$, the square $Q_\ell := (0,\ell) \times (0,\ell)$.

Determine for which values of $\ell$ there exist two constants $A_\ell$ and $B_\ell$ such that

$$\int_{Q_\ell} |u(x,y)|^{2019}\, dx\, dy \le A_\ell \left| \int_{Q_\ell} \left(\cos y \cdot u_x(x,y)^2 + \cos x \cdot u_y(x,y)^2\right) dx\, dy \right|^{B_\ell}$$

for every $u \in C^1_c(Q_\ell)$.

$\boxed{\ell \in (0,\pi)}$

- If $\ell \in (0,\pi)$, then $\cos x \geq \alpha > 0$ and $\cos y \geq \alpha > 0$ $\forall (x,y) \in Q_\ell$.

  Therefore from the continuous embedding

$$H^1 \to L^{2019}$$

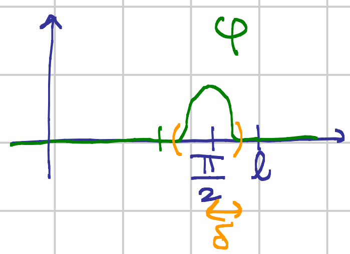
  it follows that

Poincaré

$$\|u\|_{L^{2019}} \underset{\text{Sobolev}}{\leq} c_1 \left(\|u\|_{L^2} + \|\nabla u\|_{L^2}\right) \leq c_2 \|\nabla u\|_{L^2}$$

$$\leq c_3 \left\{ \int_{Q_\ell} \cos y \cdot u_x^2 + \cos x \cdot u_y^2 \right\}^{1/2}$$

$\cos x \geq \alpha > 0$
$\cos y \geq \alpha > 0$




- If $\ell > \frac{\pi}{2}$, consider $\varphi \in C^\infty_c\left(\left(\frac{\pi}{2}-\delta, \frac{\pi}{2}+\delta\right)\right)$ symmetric wrt $\frac{\pi}{2}$ (with $\delta$ small enough) and then set

$$u(x,y) = \varphi(x) \cdot \varphi(y).$$

  One can verify that RHS $= 0$ and LHS $> 0$.

- If $\ell = \frac{\pi}{2}$ the problem is equivalent to the one with

$$\text{LHS} = \int_{Q_\ell} \left(y\, u_x^2 + x\, u_y^2\right)$$

Now take any $u \in C^\infty_c(Q_\ell)$ and set $u_n(x,y) = n^a u(nx, ny)$.

Then

$$\int_{Q_\ell} |u_n|^{2019} \sim n^{2019a - 2} \qquad \int_{Q_\ell} \left(y\, u_{nx}^2 + x\, u_{ny}^2\right) \sim n^{2a-1}$$

If $a = \frac{1}{2}$, the inequality cannot be true

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4. For every $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^2) \qquad \forall x \in (0,1).$$

Determine for which real numbers $p \geq 1$ the restriction of $T$ defines

(a) a *continuous* operator $L^p((0,1)) \to L^2((0,1))$,
(b) a *continuous* operator $L^2((0,1)) \to L^p((0,1))$,
(c) a *compact* operator $H^1((0,1)) \to L^p((0,1))$.

converges $\iff \frac{p}{p-2} \cdot \frac{1}{2} < 1$

$\iff p > 4$

(a) $\boxed{p > 4}$ Indeed

- if $p>4$ it turns out that

$$\int_0^1 f(x^2)^2 dx \underset{x^2=y}{=} \int_0^1 f(y)^2 \frac{1}{2y^{1/2}} dy \le \frac{1}{2} \left\{ \int_0^1 |f(y)|^p \right\}^{\frac{2}{p}} \left\{ \int_0^1 \frac{1}{y^{\frac{1}{2}\frac{p}{p-2}}} \right\}^{\frac{p-2}{p}}$$

$\frac{2}{p}$ $\quad$ $\frac{p-2}{p}$

- if $p \le 4$ then $f(x) = \dfrac{1}{|x|^{1/4} |\log(x/2)|^{1/2}} \in L^p$ but $f(x^2) \notin L^2$

(b) $\boxed{\text{NEVER}}$ Indeed

$$f(x) = \frac{1}{\sqrt{x}\, |\log(x/2)|} \in L^2 \quad \text{but} \quad f(x^2) \notin L^p \quad \forall p \ge 1$$

(c) $\boxed{p \ge 1}$ Let $\{f_n\}$ be a sequence that is bounded in $H^1$.

Then $\{f_n\}$ satisfies the assumptions of Ascoli–Arzelà thm. Therefore $\{f_n\}$ is relatively compact in $C^0([0,1])$ and a fortiori in $L^p((0,1))$ for every $p \ge 1$.

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