Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 05 July 2019

1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u} + x^3 u\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) + u(1) = 3$.

   (b) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) - u(1) = 3$.

2. For every $f \in L^2((0,1))$, let us consider the Dirichlet problem

$$u'' = u^3 + \sin u + f(x), \qquad u(0) = u(1) = 0.$$

   (a) Prove that the problem admits a unique solution.

   (b) Discuss the regularity of this solution.

   (c) Let $S : L^2((0,1)) \to L^2((0,1))$ be the operator that associates to each function f the corresponding solution u. Determine whether S is a compact operator.

3. Let d be a positive integer, and let B_d denote the unit ball in $\mathbb{R}^d$ with center in the origin. For every real number $m > 0$, let us set

$$I_d(m) := \inf \left\{ \int_{B_d} \left(u^{19} + \arctan(u^2)\right)\, dx : u \in C^1_c(B_d),\ \int_{B_d} \|\nabla u(x)\|^7\, dx \leq m \right\}.$$

   (a) In dimension $d = 3$, determine whether there exists $m > 0$ such that $I_3(m) = 0$.

   (b) Determine for which values of d it turns out that $I_d(m)$ is a real number for every $m > 0$.

4. For every $f : (1, +\infty) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^4) \qquad \forall x \in (1, +\infty).$$

   Determine for which real numbers $p \geq 1$ the restriction of T defines

   (a) a *continuous* operator $L^p((1, +\infty) \to L^2((1, +\infty))$,

   (b) a *continuous* operator $L^2((1, +\infty)) \to L^p((1, +\infty))$,

   (c) a *compact* operator $H^1((1, +\infty)) \to L^p((1, +\infty))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.