Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 11 June 2019

1. Let us consider the functional

$$F(u) = \int_0^\pi \left(\dot{u}^2 - u\sin x\right)\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_0^\pi u(x)\,dx = 0$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u''(x) = u^7 - x^7, \qquad u(0) = 7, \quad u'(7) = 7.$$

3. Let us consider, for every real number $\ell > 0$, the square $Q_\ell := (0,\ell)\times(0,\ell)$.

   Determine for which values of ℓ there exist two constants A_ℓ and B_ℓ such that

$$\int_{Q_\ell} |u(x,y)|^{2019}\,dx\,dy \le A_\ell \left| \int_{Q_\ell} \left(\cos y\cdot u_x(x,y)^2 + \cos x\cdot u_y(x,y)^2\right)\,dx\,dy \right|^{B_\ell}$$

   for every $u \in C^1_c(Q_\ell)$.

4. For every $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^2).$$

   Determine for which real numbers $p \ge 1$ the restriction of T defines

   (a) a *continuous* operator $L^p((0,1)) \to L^2((0,1))$,

   (b) a *continuous* operator $L^2((0,1)) \to L^p((0,1))$,

   (c) a *compact* operator $H^1((0,1)) \to L^p((0,1))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.