Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 23 February 2019

1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\ddot{u}^2 + \dot{u}^2\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) = u'(0) = 1$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Let us consider the boundary value problem

$$u''(x) = \frac{1 + e^{u(x)}}{1 + e^{u'(x)}}, \qquad u(0) = 3, \quad u(3) = 0.$$

   (a) Discuss existence, uniqueness and regularity of the solution.

   (b) Prove that $u'(0) < -1$.

3. Let Ω be a ball in $\mathbb{R}^3$. For every positive integer m, let us set

$$\sup \left\{ \int_{\Omega} |u - \arctan(xyz)|^m \, dx\, dy\, dz : u \in C^1(\Omega),\ \int_{\Omega} (u^2 + 2u_x^2 + 3u_y^2 + 4u_z^2)\, dx\, dy\, dz \leq 5 \right\}.$$

   (a) Prove that in the case $m = 3$ the supremum is actually a maximum, at least in the larger class $H^1(\Omega)$.

   (b) Determine all positive integers m such that the supremum is a real number.

4. For every $f : (0, 5) \to \mathbb{R}$, let us set

$$[Tf](x) := \sin(f(x)).$$

   Determine whether the restriction of T defines

   (a) a *continuous* mapping $L^8((0, 5)) \to L^1((0, 5))$,

   (b) a *continuous* mapping $L^1((0, 5)) \to L^8((0, 5))$,

   (c) a *compact* mapping $H^8((0, 5)) \to L^{2019}((0, 5))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.