1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 - 3u\dot{u} + xu\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u(1) = 0$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

See CdV_19_CS2, Ex. 1.

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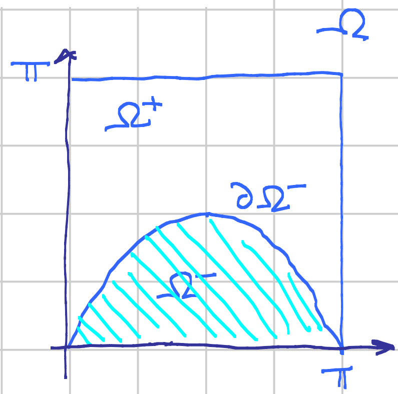
2. Let us consider the square $Q := (0,\pi)^2$ in the plane.

Find all exponents $p \geq 1$ for which there exists a constant C_p such that

$$\int_0^\pi [f(t,\sin t)]^2\,dt \leq C_p \left\{\int_Q \|\nabla f(x,y)\|^p\,dx\,dy\right\}^{2/p} \qquad \forall f \in C_c^\infty(Q).$$

The constant exists if and only if $\boxed{p > \frac{4}{3}}$



The key point is that in dimension $d=2$ the trace of a function in $W^{1,p}$ lies in L^2 if and only if $p > \frac{4}{3}$ (in general the trace lies in L^q with $q = \frac{(d-1)p}{d-p}$). In this case

$$\int_0^\pi f(t,\sin t)^2 \leq \int_0^\pi f(t,\sin t)^2 \sqrt{1+\cos^2 t}\,dt = \|f\|^2_{L^2(\partial\Omega^-)}$$

$$\underset{\text{reg. trace}}{\leq} C\,\|f\|^2_{1,\frac{4}{3},\Omega^-} \leq C\,\|f\|^2_{1,\frac{4}{3},\Omega}$$

$$\underset{p \geq \frac{4}{3}}{\leq} C_p\,\|f\|^2_{1,p,\Omega} \underset{\substack{\text{Poincaré}\\ \text{(thanks to DBC)}}}{\leq} C_p'\,\|\nabla f\|^2_{L^p(\Omega)}$$

which proves the required inequality for $p \geq \frac{4}{3}$.

When $p < \frac{4}{3}$, it is easy to check that

$$f(x,y) := \frac{1}{(x^2+y^2)^{a/2}} \qquad \text{with} \qquad \overbrace{\frac{1}{2} < a < \frac{2}{p} - 1}^{\text{this range is } \neq \emptyset \text{ if } p < \frac{4}{3}}$$

belongs to $W^{1,p}(\Omega)$, but $f(t,\sin t)^2 \sim \frac{1}{t^{2a}}$ is not integrable.

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"Alternative" approach: instead of using the theory of traces one could argue as in the proof of that result

$$f(t,\sin t)^2 = \int_0^{\sin t} [f(t,y)^2]_y\,dy \leq 2\int_0^\pi |f(t,y)|\cdot|f_y(t,\sin t)|$$

Then integrate wrt t, apply Hölder ineq. keeping into account that $f \in L^{p^*}$.

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3. Let $\Omega \subseteq \mathbb{R}^2$ be the unit ball with center in $(4, 5)$. For every real number λ, let us set

$$I(\lambda) = \inf \left\{ \int_\Omega \left(\arctan y \cdot u_x^2 + \arctan x \cdot u_y^2 - \lambda \frac{u^4}{1+u^2} \right) dx\, dy : u \in C_c^\infty(\Omega) \right\}.$$

(a) Determine whether there exists $\lambda > 0$ such that $I(\lambda)$ is a real number.

(b) Determine whether there exists $\lambda > 0$ such that $I(\lambda) = -\infty$.

(a) $\boxed{\text{YES}}$ The infimum is 0 when $\lambda > 0$ is small enough. Indeed

positive!!

$$\int_\Omega \arctan y \cdot u_x^2 + \arctan x \cdot u_y^2 \geq c_1 \int_\Omega (u_x^2 + u_y^2)$$

$$\underset{\text{Poincaré}}{\geq} c_2 \int_\Omega u^2 \underset{s^2 \geq \frac{s^4}{1+s^2}}{\geq} c_2 \int_\Omega \frac{u^4}{1+u^2} \underset{\lambda \text{ small } (\lambda \leq c_2)}{\geq} \lambda \int_\Omega \frac{u^4}{1+u^2}$$

(b) $\boxed{\text{YES}}$ The infimum is $-\infty$ when λ is large enough.

To this end, we observe that there exists $A \in \mathbb{R}$ such that

$$\frac{s^4}{1+s^2} \geq \frac{1}{2} s^2 - A \qquad \forall s \in \mathbb{R}$$

Let us choose any $u_0 \in C_c^\infty(\Omega)$ not identically equal to 0, and let $\lambda > 0$ be such that

$$\int_\Omega \arctan y \cdot u_{0x}^2 + \arctan x \cdot u_{0y}^2 - \frac{\lambda}{2} u_0^2 = M < 0$$

Then setting $u(x,y) := n\, u_0(x,y)$ we deduce that

$$\int_\Omega \arctan y \cdot u_x^2 + \arctan x \cdot u_y^2 - \lambda \frac{u^4}{1+u^2} \leq n^2 M + \lambda A.$$

Letting $n \to +\infty$ and recalling that $M < 0$ we deduce that $I(\lambda) = -\infty$.

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<u>Remark</u> The previous argument shows also that

either $I(\lambda) = 0$ (true for $\lambda \leq \lambda_*$) or $I(\lambda) = -\infty$ (true for $\lambda > \lambda_*$)

It is interesting to show that λ_* falls in the first regime.

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4. For every sequence $\{x_n\}_{n\geq 1}$, let us set

$$T(x_1, x_2, x_3, \ldots, x_n, \ldots) = \left(\frac{x_1}{\sqrt{1}}, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{3}}, \ldots, \frac{x_n}{\sqrt{n}}, \ldots\right).$$

(a) Determine whether the restriction of T defines a continuous operator for each of the following choices of the sequence space:

$$\ell^2 \to \ell^2, \qquad \ell^2 \to \ell^1, \qquad \ell^3 \to \ell^2.$$

When the answer is positive, determine the norm of the operator.

(b) Determine for which values of p the restriction of T defines a continuous operator $\ell^p \to \ell^1$.

(a) $\boxed{\text{Ok with norm} = 1}$ $\boxed{\text{NO}}$ $\boxed{\text{ok with norm } \left\{\sum \frac{1}{n^3}\right\}^{1/6}}$

(a-left) $\displaystyle \sum \frac{x_n^2}{n} \leq \sum x_n^2$ with equality if $\{x_n\} = 1, 0, 0, 0, \ldots$

(a-center) $\displaystyle x_n = \frac{1}{\sqrt{n}\log(n+7)} \in \ell^2$, but $\displaystyle T(\{x_n\}) = \frac{1}{n\log(n+7)} \notin \ell^1$

(a-right)

$$\sum \frac{x_n^2}{n} = \sum \underset{\frac{2}{3}}{x_n^2} \cdot \underset{\frac{1}{3}}{\frac{1}{n}} \leq \left\{\sum |x_n|^3\right\}^{2/3} \underbrace{\left\{\sum \frac{1}{n^3}\right\}^{1/3}}_{\text{convergent}}$$

with equality if $\{x_n\} = \left\{\frac{1}{n}\right\}$

(b) If and only if $p < 2$. The counterexample for $p \geq 2$ is (a-center)

On the other hand, for $p < 2$ it turns out that

$$\sum \frac{x_n}{\sqrt{n}} = \sum \underset{\frac{1}{p}}{x_n} \cdot \underset{\frac{p-1}{p}}{\frac{1}{\sqrt{n}}} \leq \left\{\sum |x_n|^p\right\}^{1/p} \left\{\underbrace{\sum \left(\frac{1}{n}\right)^{\frac{p}{2(p-1)}}}_{\text{convergent if } p<2}\right\}^{\frac{p-1}{p}}$$

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