1. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{1 + u^3 + x^2}{1 + \dot{u}^2}, \qquad u(0) = u'(3) = 3.$$

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See CDV_18_CS1, Ex.2

2. Let V denote the set of sequences $\{x_n\}_{n\geq 1}$ of real numbers such that

$$\sum_{n=1}^{\infty} n|x_n| < +\infty,$$

with norm defined by the series above.

(a) Characterize the dual space of V.

(b) Determine all aligned functionals of the sequence with $x_1 = 9$, $x_2 = 8$, $x_3 = 7$, and $x_n = 0$ for every $n \geq 4$.

(c) Determine all aligned functionals of the sequence with $x_n = (-1)^n \cdot n^{-4}$.

(a) The dual V' is the space of sequences $\{a_n\}_{n\geq 1}$ s.t.

$$\|\{a_n\}\|_W = \sup\left\{ \frac{|a_n|}{n} : n \geq 1 \right\} < +\infty$$

(this is the norm in V')

Let W denote this space. We need to prove that $J : W \to V'$ defined by the usual pairing

$$[J(\{a_n\})](\{x_n\}) \sum_{n=1}^{\infty} a_n x_n$$

is a linear bijective isometry.

Linearity is trivial. Isometry (and injectivity) follow from the estimate

$$\left|\sum a_n x_n\right| \leq \sum \frac{|a_n|}{n} \cdot n|x_n| \leq \sup_{n\in\mathbb{N}} \frac{|a_n|}{n} \cdot \sum n|x_n|$$

which proves that

$$\|J(\{a_n\})\|_{V'} \leq \|\{a_n\}\|_W$$

For the opposite inequality, we fix $\varepsilon > 0$, we choose n_0 s.t. $\|\{a_n\}\|_W \geq \frac{|a_{n_0}|}{n_0} - \varepsilon$, and then we choose $\{x_n\}$ with $x_{n_0} = n_0$ and $x_n = 0$ otherwise.

Surjectivity: given $L \in V'$ we set $a_n := L(e_n)$, $0, \dots, 0, 1, 0, \dots$ (1 in position n)

and we observe that $\mathrm{Span}(\{e_n\})$ is dense (to be proved!) and

$$\frac{|a_n|}{n} = \left|L\left(\frac{e_n}{n}\right)\right| \leq \|L\|_{V'} \underbrace{\left\|\frac{e_n}{n}\right\|_V}_{=1} \leq \|L\|_{V'}$$

(b) The aligned functionals of $9, 8, 7, 0, 0, \dots$ are all functionals represented by $\{a_n\}$ with $\boxed{a_1 = 1, a_2 = 2, a_3 = 3, \text{ and } |a_n| \leq n \ \forall n \geq 4}$

(c) The unique aligned functional of $(-1)^n \frac{1}{n^4}$ is represented by $\boxed{a_n = (-1)^n n}$

Remark An alternative characterization of V' is ℓ^∞ with action

$$[J(\{a_n\})](\{x_n\}) := \sum_{n=1}^{\infty} n\, a_n x_n .$$

3. Let $\Omega = (-1,1)^2$ be a square in the plane.

(a) Determine whether

$$\sup\left\{\int_\Omega u_{xy}^2\,dx\,dy : u \in C_c^2(\Omega),\ \int_\Omega u_{xx}^2\,dx\,dy \le 7,\ \int_\Omega u_{yy}^2\,dx\,dy \le 8\right\}$$

is finite or infinite.

(b) Determine whether

$$\sup\left\{\int_\Omega u_{yy}^2\,dx\,dy : u \in C_c^2(\Omega),\ \int_\Omega u_{xx}^2\,dx\,dy \le 7,\ \int_\Omega u_{xy}^2\,dx\,dy \le 8\right\}$$

is finite or infinite.

(a) FINITE If $u_{xx} \in L^2$ and u_{yy}, then $\Delta u \in L^2$. Due to DBCs it follows that $u \in H^2$ and $\|D^2 u\|_{L^2} \le c\,\|\Delta u\|_{L^2}$, and in particular

Depending on Ω

$\|u_{xy}\|_{L^2} \le c\,\|\Delta u\|_{L^2}$

(b) INFINITE Let us consider any $\varphi \in C_c^2(\Omega)$ not identically 0, and let us extend it by 0 to the whole $\mathbb{R}^2$. Then let us set

$$u_n(x,y) = \frac{1}{n}\,u(x,ny) \qquad \text{(note that } u_n \in C_c^2(\Omega)\text{)}$$

Then it turns out that $(u_n)_{xx}$ and $(u_n)_{xy}$ are bounded in Ω, independently on n, while $\|(u_n)_{yy}\|^2_{L^2(\Omega)} = O(n)$.
This is enough to conclude.
Details are left to the interested reader.

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Alternative for point (a). Imitate the proof of H^2 regularity!
Due to the homogeneous DBC it turns out that

$$\int_\Omega u_{xy}^2 = \int_\Omega u_{xy}\cdot u_{xy} = \int_\Omega u_{xx}\cdot u_{yy} \le \|u_{xx}\|\cdot\|u_{yy}\|.$$

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4. Let us consider the open set $\Omega = (0, \pi)^2$. Determine if there exists a constant C such that

$$\int_\Omega \sin(xy) \cdot u^2 \, dx \, dy \le C \int_\Omega \left(e^{xy} \cdot u_x^2 + u_y^2 + \cos x \cdot u_x \cdot u_y\right) \, dx \, dy$$

for every $u \in C_c^1(\Omega)$.

The constant $\boxed{\text{exists.}}$ Indeed it turns out that

$$\int_\Omega \sin(xy)\, u^2 \le \int_\Omega u^2$$

Poincaré due to BCs

$$\le C_1 \int_\Omega |u_x|^2 + |u_y|^2$$

$$\le 2C_1 \int_\Omega \frac{1}{2}|u_x|^2 + \frac{1}{2}|u_y|^2$$

$\left(|ab| \le \frac{1}{2}a^2 + \frac{1}{2}b^2\right)$ $$\le 2C_1 \int_\Omega |u_x|^2 + |u_y|^2 - |u_x| \cdot |u_y|$$

$$\le 2C_1 \int_\Omega \underset{\ge 1 \text{ in } \Omega}{e^{xy}} |u_x|^2 + |u_y|^2 + \underset{\ge -|u_x|\cdot|u_y|}{\boxed{\cos x \; u_x \cdot u_y}}$$

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