BY GIMUSI 16/10/17

Limiti 2

Argomenti: limiti di funzioni di più variabili **Difficoltà:** ⋆⋆⋆⋆

Prerequisiti: tecniche per il calcolo di limiti in un punto per funzioni di più variabili

In ogni riga è assegnata una funzione, di cui si chiede di calcolare liminf e limsup per $(x, y) \to (0, 0)$. Nelle varie colonne, la funzione si intende definita nel suo "naturale dominio" intersecato l'insieme definito dalle relazioni indicate in testa alla colonna stessa.

		a) $(x,y) \in \mathbb{R}^2$		b) $x > 0,\ y > 0$		c) $0 \le x \le y$		d) $x > 0,\ y \le x^2$	
	Funzione	liminf	limsup	liminf	limsup	liminf	limsup	liminf	limsup
1)	$\dfrac{x^2 y}{x^4 + y^2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
2)	$\dfrac{x y^2}{x^4 + y^2}$	0	0	0	0	0	0	0	0
3)	$\dfrac{x^3 y}{x^4 + y^2}$	0	0	0	0	0	0	0	0
4)	$\dfrac{xy}{\lvert x\rvert + y^2}$	0	0	0	0	0	0	0	0
5)	$\dfrac{x}{x^3 + y^3}$	$-\infty$	$+\infty$	0	$+\infty$	0	$+\infty$	$-\infty$	$+\infty$
6)	$\dfrac{xy}{x^3 + y^3}$	$-\infty$	$+\infty$	0	$+\infty$	0	$+\infty$	$-\infty$	$+\infty$
7)	$\dfrac{x^2 y}{x^3 + y^3}$	$-\infty$	$+\infty$	0	$\frac{2}{3}\sqrt[3]{1/2}$	0	$\frac{1}{2}$	$-\infty$	$+\infty$
8)	$\dfrac{y^4}{\lvert x\rvert^3 + \lvert y\rvert^3}$	0	0	0	0	0	0	0	0
9)	$\dfrac{y^4}{x^3 + y^3}$	$-\infty$	$+\infty$	0	0	0	0	$-\infty$	$+\infty$
10)	$\dfrac{x + 2y}{x + y}$	$-\infty$	$+\infty$	1	2	$\frac{3}{2}$	2	$-\infty$	$+\infty$
11)	$\dfrac{x^3}{x - y^2}$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	0	0	$-\infty$	0
12)	$\dfrac{\sqrt{x^2 + \lvert y\rvert}}{\lvert x\rvert + \lvert y\rvert}$	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	1	$+\infty$

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1) $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ $\qquad x^2 = u \geq 0 \quad y = v$

$$f(x,y) = g(u,v) = \frac{uv}{u^2+v^2} = \frac{1}{2}\sin 2\theta$$

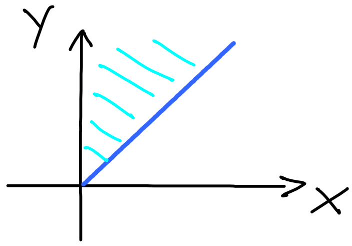
a) $(x,y) \in \mathbb{R}^2, \quad u \geq 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

LIMINF $= -\frac{1}{2}$ $\qquad$ LIMSUP $= \frac{1}{2}$

b) $x > 0, y > 0 \qquad 0 < \theta < \frac{\pi}{2}$

LIMINF $= 0$ $\qquad$ LIMSUP $= \frac{1}{2}$

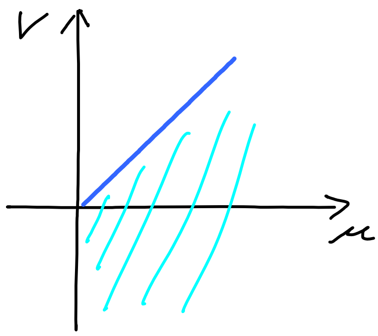
c) $0 \leq x \leq y$



$$0 \leq \frac{x^2 y}{x^4+y^2} \leq \frac{x^2 y}{x^4+x^2} = \frac{y}{x^2+1} \to 0$$

LIMINF $= 0$ $\qquad$ LIMSUP $= 0$

d) $x > 0 \quad y \leq x^2, \qquad u > 0 \quad v \leq u$



$$-\frac{1}{2} \leq \frac{uv}{u^2+v^2} \leq \frac{1}{2}$$

LIMINF $= -\frac{1}{2}$ $\qquad$ LIMSUP $= \frac{1}{2}$

2) $\lim_{(x,y)\to(0,0)} \frac{x y^2}{x^4+y^2}$

$y^2 = v \quad x = u \qquad \frac{uv}{u^2+v} = \frac{\rho \sin\theta \cos\theta}{\rho\cos^2\theta + \sin\theta}$

$0 \leq \frac{|x| y^2}{x^4+y^2} \leq |x| \to 0$ $\qquad$ LIMINF $=$ LIMSUP $= 0$

3) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^5+y^2}$

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$$0 \le \left|\frac{x^3 y}{x^5+y^2}\right| = |x| \left|\frac{x^2 y}{x^5+y^2}\right| \le |x| \frac{1}{2} \to 0$$

LIMINF = LIMSUP = 0

4) $\lim_{(x,y)\to(0,0)} \frac{xy}{|x|+y^2}$

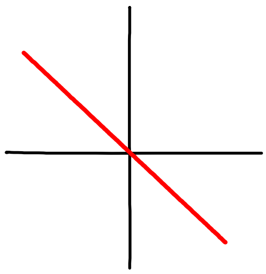
$x = u^2 \quad y = v \qquad f(x,y) = g(u,v) = \frac{\pm u^2 v}{u^2+v^2}$

$$0 \le \left|\frac{\pm u^2 v}{u^2+v^2}\right| \le |u| \frac{|uv|}{u^2+v^2} \le \frac{1}{2}|u| \to 0$$

LIMINF = LIMSUP = 0

5) $\lim_{(x,y)\to(0,0)} \frac{x}{x^3+y^3} \quad \left(= \frac{\cos\theta}{\rho^2(\cos^3\theta+\sin^3\theta)} \right)$

6) $(x,y) \in \mathbb{R}^2 \qquad x \neq -y$



$$f(\delta,\delta) = \frac{\delta}{2\delta^3} = \frac{1}{2\delta^2} \to +\infty$$

$$f(\delta^3,\delta^2) = \frac{\delta^3}{\delta^9+\delta^6} = \frac{1}{\delta^6+\delta^3} \to -\infty \qquad \delta \to 0^-$$

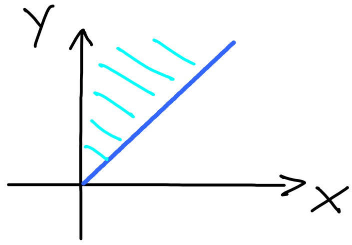
LIMINF = $-\infty$ LIMSUP = $+\infty$

b) $x>0,\ y>0 \qquad \dfrac{x}{x^3+y^3} \geq 0$

$$f(\sigma^4,\sigma) = \frac{\sigma^4}{\sigma^{12}+\sigma^3} = \frac{\sigma}{\sigma^9+1} \to 0 \qquad \rightsquigarrow \text{LIMINF} = 0$$

$$f(\sigma,\sigma) = \frac{1}{2\sigma^2} \to +\infty \qquad \rightsquigarrow \text{LIMSUP} = +\infty$$

c) $0 \leq x \leq y$

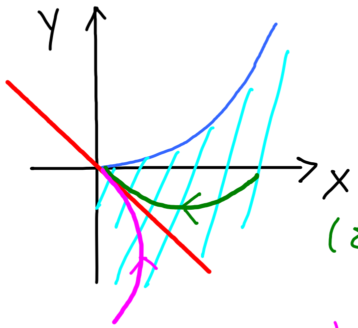


$\dfrac{x}{x^3+y^3} \geq 0$

$$f(0,\sigma) = 0 \qquad \rightsquigarrow \text{LIMINF} = 0$$

$$f(\sigma,\sigma) = \frac{1}{2\sigma^2} \to +\infty \qquad \rightsquigarrow \text{LIMSUP} = +\infty$$

d) $x>0 \quad y \leq x^2$



$$f(\sigma, \sigma^2-\sigma) = \frac{\sigma}{\cancel{\sigma^3}+\sigma^6-3\sigma^5+3\sigma^4-\cancel{\sigma^3}} = \frac{1}{\sigma^5-3\sigma^4+3\sigma^3} \to +\infty \quad \sigma \to 0^+$$

$$f(\sigma-\sigma^2, -\sigma) = \frac{\sigma-\sigma^2}{\cancel{\sigma^3}-3\sigma^4+3\sigma^5-\sigma^6-\cancel{\sigma^3}} = \frac{1-\sigma}{-3\sigma^3+3\sigma^4-\sigma^5} \to -\infty \quad \sigma \to 0^+$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^3+y^3}$

a) $(x,y)\in\mathbb{R}^2 \qquad x \neq -y$

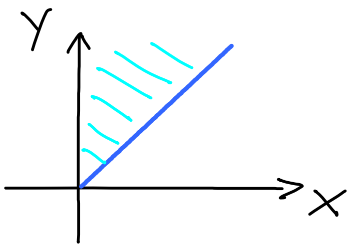
$$f(\delta,\delta) = \frac{\delta^2}{2\delta^3} = \frac{1}{2\delta} \to \pm\infty \rightsquigarrow \begin{cases} LIMSUP = +\infty \\ LIMINF = -\infty \end{cases}$$

b) $x>0,\ y>0 \qquad \dfrac{xy}{x^3+y^3} \geq 0$

$$f(\delta^4,\delta) = \frac{\delta^5}{\delta^{12}+\delta^3} = \frac{\delta^2}{\delta^9+1} \to 0 \rightsquigarrow LIMINF = 0$$

$$f(\delta,\delta) = \frac{1}{2\delta^2} \to +\infty \rightsquigarrow LIMSUP = +\infty$$

c) $0 \leq x \leq y$

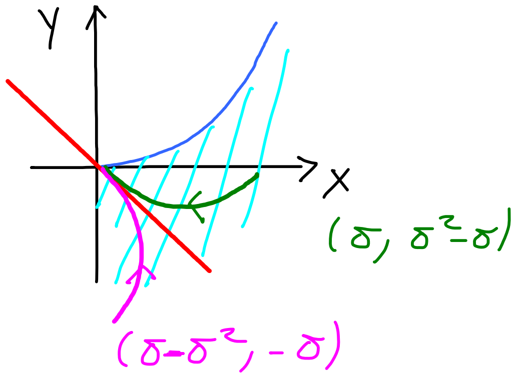


$\dfrac{x}{x^3+y^3} \geq 0$

$$f(0,\delta) = 0 \rightsquigarrow LIMINF = 0$$

$$f(\delta,\delta) = \frac{1}{2\delta^2} \to +\infty \rightsquigarrow LIMSUP = +\infty$$

d) $x>0 \quad y \leq x^2$



$$f(\delta,\delta^2-\delta) = \frac{\delta^3-\delta^2}{\cancel{\delta^3}+\delta^6-3\delta^5+3\delta^4-\cancel{\delta^3}} = \frac{\delta-1}{\delta^4-3\delta^3+3\delta^2} \to -\infty \quad \delta\to0^+$$

$$f(\delta-\delta^2,-\delta) = \frac{\delta^3-\delta^2}{\cancel{\delta^3}-3\delta^4+3\delta^5-\delta^6-\cancel{\delta^3}} = \frac{\delta-1}{-3\delta^2+3\delta^3-\delta^4} \to +\infty \quad \delta\to0^+$$

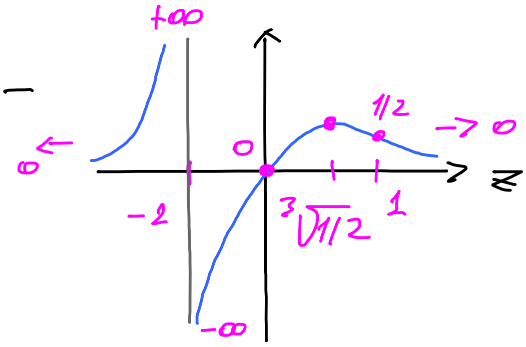
$LIMINF = -\infty \qquad LIMSUP = +\infty$

7) $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^3+y^3} \quad \left(= \frac{\cos^2\theta \sin\theta}{\cos^3\theta + \sin^3\theta} = \frac{tg\,\theta}{1+tg^3\theta}, \quad \cos\theta \neq 0, \ tg\,\theta \neq -1 \right)$$

$$z = tg\,\theta \in (-\infty, +\infty)$$

$$g(z) = \frac{z}{1+z^3} \qquad g'(z) = \frac{1+z^3-3z^3}{(1+z^3)^2} = \frac{1-2z^3}{(1+z^3)^2}$$

$g'(z)$ $\quad + + + \underset{}{\overset{-1}{\bullet}} + + + \overset{\sqrt[3]{1/2}}{\bullet} - - - - -$



$$g(\sqrt[3]{1/2}) = \frac{\sqrt[3]{1/2}}{1+\frac{1}{2}} = \frac{2}{3}\sqrt[3]{1/2}$$

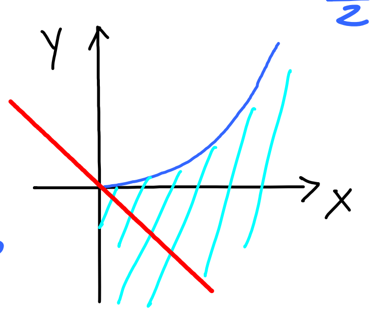
a) $(x,y) \in \mathbb{R}^2 \quad x \neq -y$ $\quad$ LIMINF $= -\infty$ $\quad$ LIMSUP $= +\infty$

b) $x>0,\ y>0$ $\quad$ LIMINF $= 0$ $\quad$ LIMSUP $= \frac{2}{3}\sqrt[3]{1/2}$

c) $0 \leq x \leq y$ $\quad$ LIMINF $= 0$ $\quad$ LIMSUP $= \frac{1}{2}$

d) $x>0 \quad y \leq x^2$

LIMINF $= -\infty$ $\quad$ LIMSUP $= +\infty$



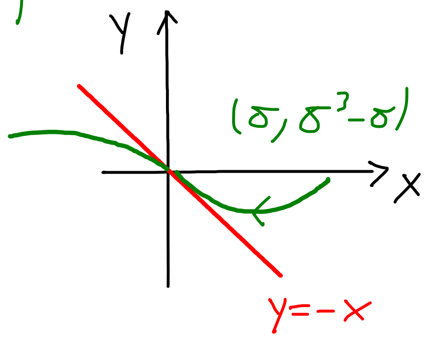
8) $$\lim_{(x,y)\to(0,0)} \frac{y^4}{|x|^3+|y|^3}$$

$$0 \leq \frac{y^4}{|x|^3+|y|^3} = \frac{\rho \cos^4\theta}{\underbrace{|\cos\theta|^3+|\sin\theta|^3}_{\geq m > 0}} \leq \frac{\rho \cdot 1}{m} \to 0$$

LIMINF = LIMSUP = 0

9) $\lim_{(x,y)\to(0,0)} \frac{y^5}{x^3+y^3} \quad \left(= \frac{\rho \cos^5\theta}{\cos\theta^3 + \sin\theta^3} \right)$

a) $(x,y)\in\mathbb{R}^2 \quad x \pm -y$



$$f(\delta, \delta^3-\delta) = \frac{\delta^5 - 5\delta^6 + 6\delta^8 - 5\delta^{10} + \delta^{12}}{\cancel{\delta^3} + \delta^9 - 3\delta^7 + 3\delta^5 - \cancel{\delta^3}} =$$

$$= \frac{1 - 5\delta^2 + 6\delta^5 - 5\delta^6 + \delta^8}{\delta^5 - 3\delta^3 + 3\delta} \to \begin{cases} +\infty & \delta \to 0^+ \\ -\infty & \delta \to 0^- \end{cases}$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

b) $x>0, y>0$

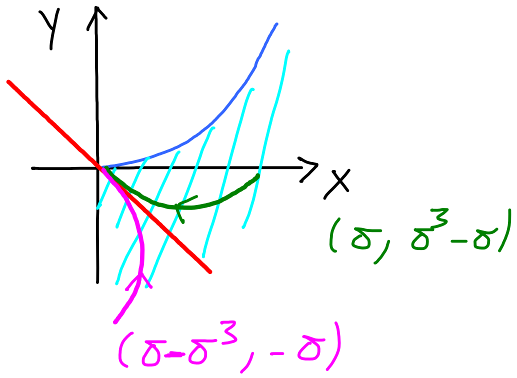
$$0 \le \frac{y^5}{x^3+y^3} = \frac{\rho\cos^5\theta}{\underbrace{\cos^3\theta + \sin^3\theta}_{\ge m > 0}} \le \frac{\rho \cdot 1}{m} \to 0$$

LIMINF $=$ LIMSUP $= 0$

c) $0 \le x \le y \quad 0 \le \frac{y^5}{x^3+y^3} = \frac{\rho\cos^5\theta}{\underbrace{\cos^3\theta + \sin^3\theta}_{\ge m > 0}} \le \frac{\rho \cdot \sqrt{2}/2}{m} \to 0$

LIMINF $=$ LIMSUP $= 0$

d) $x>0 \quad y \le x^2$



$$f(\delta, \delta^3-\delta) \xrightarrow{a)} +\infty \quad \delta \to 0^+$$

$$f(\delta-\delta^3, -\delta) = \frac{\delta^5}{\cancel{\delta^3} - 3\delta^5 + 3\delta^7 - \delta^9 - \cancel{\delta^3}} = \frac{1}{-3\delta + 3\delta^3 - \delta^5} \to -\infty \quad \delta \to 0^+$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

10) $\lim_{(x,y)\to(0,0)} \dfrac{x+2y}{x+y}$ $\left(= 1 + \dfrac{y}{x+y} = 1 + \dfrac{\sin\theta}{\cos\theta+\sin\theta} = 1 + \dfrac{\operatorname{tg}\theta}{1+\operatorname{tg}\theta} \right)$

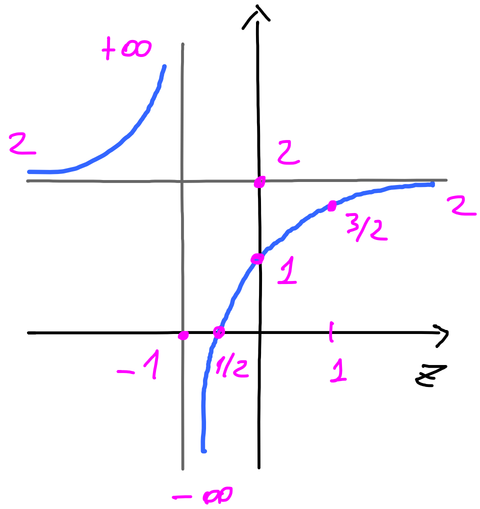
$\cos\theta \neq 0 \quad \operatorname{tg}\theta \neq -1$

$z = \operatorname{tg}\theta \in (-\infty, +\infty)$

$g(z) = 1 + \dfrac{z}{1+z} \qquad g'(z) = \dfrac{1+z-z}{(1+z)^2} = \dfrac{1}{(1+z)^2} > 0$

a) $(x,y) \in \mathbb{R}^2 \qquad x \neq -y$

LIMINF $= -\infty$ LIMSUP $= +\infty$



b) $x > 0, \ y > 0$

LIMINF $= 1$ LIMSUP $= 2$

c) $0 \leq x \leq y$

LIMINF $= \frac{3}{2}$ LIMSUP $= 2$

d) $x > 0 \quad y \leq x^2$

LIMINF $= -\infty$ LIMSUP $= +\infty$

11) $$\lim_{(x,y)\to(0,0)} \frac{x^3}{y-x^2}$$

a) $(x,y)\in\mathbb{R}^2 \quad y \neq x^2$

$$f(\delta, \delta^4+\delta^2) = \frac{\delta^3}{\delta^4+\delta^2-\delta^2} = \frac{1}{\delta} \to \pm\infty$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

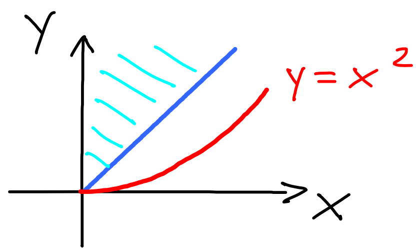
b) $x>0, y>0$

$$f(\delta, \delta^4+\delta^2) = \frac{\delta^3}{\delta^4+\delta^2-\delta^2} = \frac{1}{\delta} \to +\infty \qquad \delta\to 0^+$$

$$f(\delta, -\delta^4+\delta^2) = \frac{\delta^3}{-\delta^4+\delta^2-\delta^2} = -\frac{1}{\delta} \to -\infty \qquad \delta\to 0^+$$

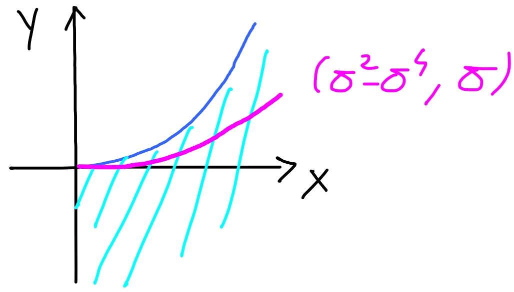
LIMINF $= -\infty$ LIMSUP $= +\infty$

c) $0 \le x \le y \qquad \dfrac{x^3}{y-x^2} \ge 0$



$$0 \le \frac{x^3}{y-x^2} \le \frac{x^3}{x-x^2} = \frac{x^2}{1-x} \to 0$$

LIMINF $=$ LIMSUP $= 0$



d) $x>0 \quad y \le x^2 \qquad \dfrac{x^3}{y-x^2} \le 0$

$$f(\delta, \delta^2-\delta^4) = -\frac{1}{\delta} \to -\infty \qquad \delta\to 0^+$$

$$f(\delta, 0) = -\delta \to 0$$

LIMINF $= -\infty$ LIMSUP $= 0$

12) $\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+|y|}}{|x|+|y|}$

$$\frac{\sqrt{x^2+|y|}}{|x|+|y|} = \frac{\sqrt{\cos^2\theta+|\sin\theta|/\rho}}{\underbrace{|\cos\theta|+|\sin\theta|}_{=m>0}} \to \begin{cases} +\infty & \sin\theta \neq 0 \\ 1 & \sin\theta = 0 \end{cases}$$

a) $(x,y)\in\mathbb{R}^2$ LIMINF = 1 LIMSUP = $+\infty$

b) $x>0, y>0$ LIMINF = LIMSUP = $+\infty$

c) $0\le x\le y$ LIMINF = LIMSUP = $+\infty$

d) $x>0 \quad y\le x^2$ LIMINF = 1 LIMSUP = $+\infty$