BY GIMUSI 28/12/15

Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Prova in Itinere di Analisi Matematica II

Pisa, ?? ?? ????

1. Sia
$$V := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 2,\ z \geq 0\} \cap \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 2\}.$$
Calcolare l'area della sua superficie.

2. Siano $S = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^3 = 9,\ 0 \leq z \leq 2\}$ orientata prendendo in $(2,2,1)$ la normale che punta verso le y negative e
$$F(x,y,z) = (x+y^2, -2y+z, x^2+z).$$
Calcolare il flusso di F attraverso S.

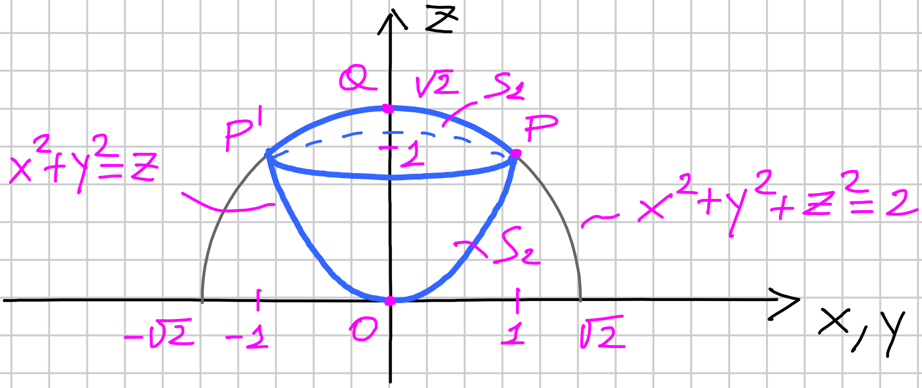
3. Sia
$$D := \{(x,y) \in \mathbb{R}^2 :\ x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\}.$$
Stabilire se convergono
$$\int_D \frac{x}{x^4+y^4}dx\,dy, \qquad \int_D \frac{x}{x^2+y^4}dx\,dy.$$

Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato. Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

1. Sia

$$V := \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 2,\ z \ge 0\} \cap \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2 \le z \le 2\}.$$

Calcolare l'area della sua superficie.



$$P: \begin{cases} x^2+y^2+z^2=2 \\ x^2+y^2=z \end{cases} \leadsto z^2+z-2=0 \qquad (z+2)(z-1)=0$$

$$P(1,1) \qquad P'(1,-1)$$

<u>CALOTTA SFERICA</u> "P'QP"

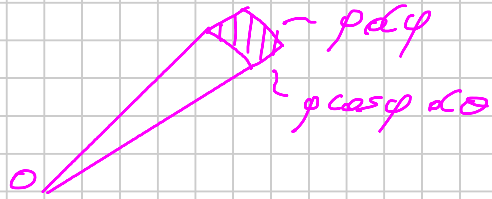
$$\begin{cases} (x, y, \sqrt{2-x^2-y^2}) \\ (x,y) \in \Omega \qquad \Omega = \{(x,y) : x^2+y^2 \le 1\} \end{cases}$$

$$\begin{cases} f_x = \frac{1}{\cancel{2}} \frac{-\cancel{2}x}{\sqrt{2-x^2-y^2}} = \frac{-x}{\sqrt{2-x^2-y^2}} \\ f_y = \frac{-y}{\sqrt{2-x^2-y^2}} \end{cases}$$

$$A_1 = \int_\Omega \sqrt{1+f_x^2+f_y^2}\, dx\,dy = \int_\Omega \sqrt{\frac{2-\cancel{x^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}}{2-x^2-y^2}}\, dx\,dy =$$

$$= \int_0^1 \int_0^{2\pi} \sqrt{\frac{2}{2-\rho^2}}\, \rho\, d\rho\, d\theta = 2\pi\sqrt{2} \int_0^1 \frac{\rho}{\sqrt{2-\rho^2}}\, d\rho$$

$$= 2\sqrt{2}\,\pi \left[-\sqrt{2-\rho^2}\right]_0^1 = 2\sqrt{2}\,\pi\,(\sqrt{2}-1) = (5-2\sqrt{2})\,\pi$$



VERIFICA IN COORD. POLARI

$$A_1 = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \rho^2 \cos\varphi \, d\theta \, d\varphi =$$

$$= 4\pi \left[\sin\varphi \right]_{\pi/4}^{\pi/2} = 4\pi \left(1 - \sqrt{2}/2 \right) = (4 - 2\sqrt{2})\pi$$

CALOTTA "P'OP" $\equiv$ SUP. DI ROTAZIONE $\quad \varphi(z) = \sqrt{z}$

$$A_2 = 2\pi \int_0^1 \varphi(z) \sqrt{1 + \dot{\varphi}(z)^2} \, dz = \qquad \dot{\varphi}(z) = \frac{1}{2\sqrt{z}}$$

$$= 2\pi \int_0^1 \sqrt{z} \sqrt{\frac{1 + 4z}{4z}} \, dz =$$

$$= 2\pi \int_0^1 \sqrt{\frac{1}{4} + z} \, dz = \qquad z + \frac{1}{4} = u$$

$$= 2\pi \int_{1/4}^{5/4} \sqrt{u} \, du = 2\pi \left[\frac{2}{3} u^{3/2} \right]_{1/4}^{5/4} = \frac{4}{3}\pi \left(\frac{5}{8}\sqrt{5} - \frac{1}{8} \right) =$$

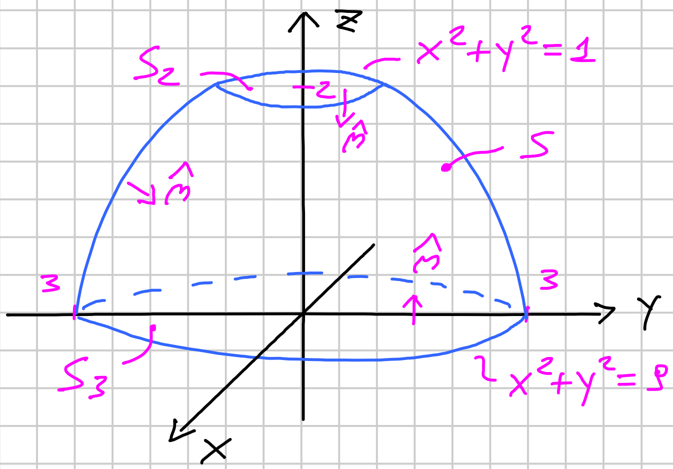
$$= \left(\frac{5}{6}\sqrt{5} - \frac{1}{6} \right)\pi$$

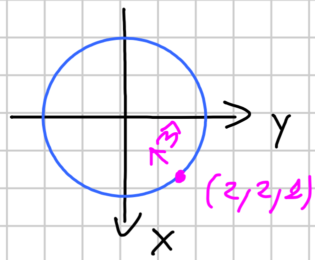
$$\rightsquigarrow \quad A = A_1 + A_2 = \left(\frac{5}{6}\sqrt{5} - 2\sqrt{2} + \frac{23}{6} \right)\pi$$

2. Siano $S = \{(x,y,z) \in \mathbb{R}^3 : \; x^2+y^2+z^3 = 9, \; 0 \le z \le 2\}$ orientata prendendo in $(2,2,1)$ la normale che punta verso le y negative e

$$F(x,y,z) = (x+y^2, -2y+z, x^2+z).$$

Calcolare il flusso di F attraverso S.





$$\int_S \langle \bar{F}, \hat{n} \rangle d\sigma = \int_{S \cup S_2 \cup S_3} \langle \bar{F}, \hat{n} \rangle d\sigma - \int_{S_2 \cup S_3} \langle \bar{F}, \hat{n} \rangle d\sigma$$

$$\Omega = \{(x,y,z) : \; x^2+y^2+z^3 \le 9, \; 0 \le z \le 2\}$$

SEGNO MENO XCHÉ $\hat{n}$ PUNTA VERSO L'INTERNO

$$\int_{S \cup S_2 \cup S_3} \langle \bar{F}, \hat{n} \rangle d\sigma = - \int_{\Omega} \operatorname{div} \bar{F} \, dx dy dz = - \int_{\Omega} (1-2+1) \, dx dy dz = 0$$

$$\rightsquigarrow \int_S \langle \bar{F}, \hat{n} \rangle d\sigma = - \int_{S_2} \langle \bar{F}, \hat{n} \rangle d\sigma - \int_{S_3} \langle \bar{F}, \hat{n} \rangle d\sigma$$

$$\int_{S_2} \langle \bar{F}, \hat{n} \rangle d\sigma = \int_{S_2} x^2 d\sigma = \int_0^{2\pi} \int_0^3 \rho^2 \cos^2\theta \, \rho \, d\rho \, d\theta =$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^3 \rho^3 d\rho = \left[\theta/2 + \frac{\sin 2\theta}{4} \right]_0^{2\pi} \left[\rho^4/4 \right]_0^3 =$$

$$= \frac{81}{4} \pi$$

$$\int_{S_3} \langle \overline{F}, \hat{n} \rangle \, d\sigma = \int_{S_3} (-x^2 - 2) \, d\sigma = -\int_{S_3} x^2 \, d\sigma - 2 \int_{S_3} d\sigma =$$

$$= -\int_0^{2\pi} \cos^2\theta \, d\theta \int_0^1 \rho^3 \, d\rho - 2\pi = -\frac{\pi}{5} - 2\pi = -\frac{9}{5}\pi$$

$$\rightsquigarrow \int_S \langle \overline{F}, \hat{n} \rangle \, d\sigma = -\int_{S_2} \langle \overline{F}, \hat{n} \rangle \, d\sigma - \int_{S_3} \langle \overline{F}, \hat{n} \rangle \, d\sigma =$$

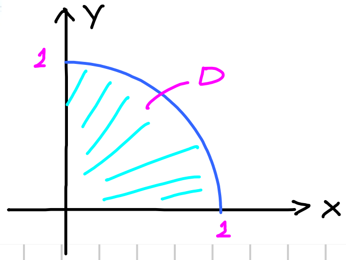
$$= -\frac{81}{5}\pi + \frac{9}{5}\pi = -18\pi$$

3. Sia

$$D := \{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}.$$

Stabilire se convergono

$$\int_D \frac{x}{x^4 + y^4}\, dx\, dy, \qquad \int_D \frac{x}{x^2 + y^4}\, dx\, dy.$$

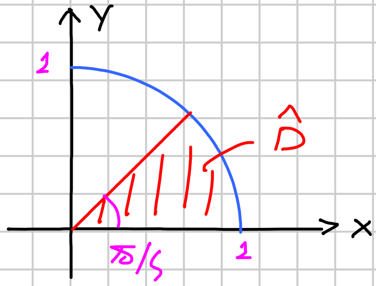


$$\int_D \frac{x}{x^4+y^4}\, dx\, dy = \int_D \frac{\rho\cos\theta}{\rho^4(\cos^4\theta+\sin^4\theta)}\, \rho\, d\rho\, d\theta =$$

$$= \int_D \frac{1}{\rho^2} \frac{\cos\theta}{\cos^4\theta+\sin^4\theta}\, d\rho\, d\theta \geq \int_{\hat D} \frac{1}{\rho^2} \frac{\cos\theta}{\cos^4\theta+\sin^4\theta}\, d\rho\, d\theta \geq$$

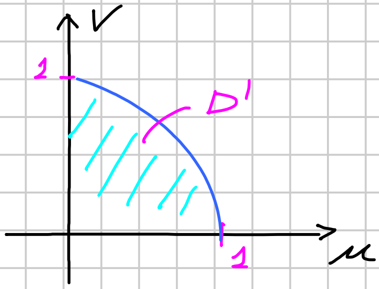
$0 < m \leq \cos^4\theta + \sin^4\theta \leq M$

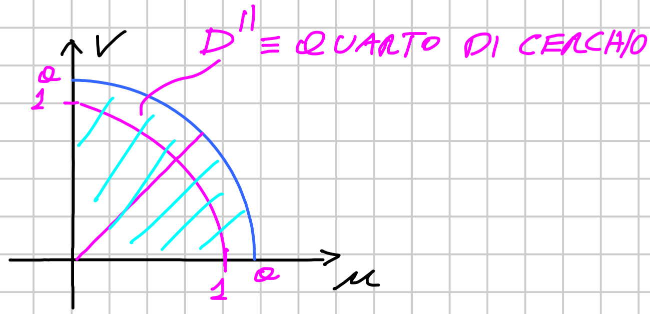
$$\geq \int_{\hat D} \frac{1}{\rho^2} \frac{\sqrt{2}/2}{M}\, d\rho\, d\theta = +\infty$$



$$\int_D \frac{x}{x^2+y^4}\, dx\, dy = \qquad \begin{cases} x = u^2 \\ y = v \end{cases} \qquad \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u$$

$$= \int_{D'} \frac{u^2}{u^4+v^4} \cdot 2u\, du\, dv = \int_{D'} \frac{2u^3}{u^4+v^4}\, du\, dv =$$





$$\leq \int_{D''} \frac{2\rho^3\cos^3\theta}{\rho^4(\cos^4\theta+\sin^4\theta)}\, \rho\, d\rho\, d\theta = \int_{D''} \frac{2\cos^3\theta}{\cos^4\theta+\sin^4\theta}\, d\rho\, d\theta < +\infty$$

INT. PROPRIO

$$\rightsquigarrow \begin{cases} \displaystyle\int_D \frac{x}{x^4+y^4}\, dx\, dy & \text{DIVERGE} \\ \displaystyle\int_D \frac{x}{x^2+y^4}\, dx\, dy & \text{CONVERGE} \end{cases}$$